mth 235 L20

Plan:
* Review Exam 2.
* Sections: 3.1 - 3.6, 5.2, 5.4, 5.5
* 6 problems (Shorter than E1)
* No books, No notes, No calculators
* Homework 7, Sects 6.1, 6.2
  Due on Friday, March 5.
* Today: Review Chptr 5.

* Section 5.5.

Eqs. with regular-singular points at x_0.

- Look for power series solutions

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{(r+n)}}$$

- Recall:

$$y' = \sum_{n=0}^{\infty} (r+n)\, a_n\, x^{(r+n-1)} \neq \sum_{n=1}^{\infty} (r+n)\, a_n\, (x-x_0)^{(r+n-1)}$$

$$\boxed{r \neq 0}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{(r+n-2)} \neq \sum_{n=2}^{\infty} (r+n)(r+n-1)\, a_n\, (x-x_0)^{(r+n-2)}$$

- Find the indicial eq. for r.

and the recurrence relation for a_n and r.

- Let r_1, r_2 be solutions of the indicial eq.

(1) If $r_1 - r_2 \neq$ integer, then both r_1, r_2 define linearly independent solutions y_1, y_2.

(2) If $r_1 - r_2 =$ integer, then only the larger of r_1, r_2 defines a solution y.

- Introduce the larger of r_1, r_2 into the recurrence relation and solve for a_n.

<u>Example</u> : [Find the first 3 terms in a power series sol. at r.s.p. $x_0 = 0$ of the eq.

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{9}\right) y = 0.]$$

<u>Sol</u>.

$x_0 = 0$ is a r. S. P.

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}}$$

Find r and a_n.

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r-1)}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$$

$$x y' = x \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r-1)}$$

$$\boxed{x y' = \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)}}$$

$$x^2 y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$$

$$\boxed{x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}}$$

$$\left(x^2 - \frac{1}{9}\right) y = \left(x^2 - \frac{1}{9}\right) \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$= x^2 \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$+ \sum_{n=0}^{\infty} -\frac{a_n}{9} x^{(n+r)}$$

$$\boxed{\left(x^2 - \frac{1}{9}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} -\frac{a_n}{9} x^{(n+r)}}$$

relabel $n+r+2 \to m+r$ on first term.

$m = n+2$

$n = m-2.$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)}$$

$n+2 = m$

$n = m-2$

$$= \sum_{m=2}^{\infty} a_{m-2} x^{m+r}$$

$$\boxed{x^2 y = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}}$$

$$\Bigg[\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}$$

$$+ \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{n-2} x^{(n+r)}$$

$$+ \sum_{n=0}^{\infty} -\frac{a_n}{9} x^{(n+r)} = 0 \Bigg]$$

$$\left[\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{(n+r)} = r(r-1)\, a_0\, x^r + (r+1)\, r\, a_1\, x^{r+1} + \sum_{n=2}^{\infty} (n+r)(n+r-1)\, a_n\, x^{(n+r)}\right]$$

$$\left[\sum_{n=0}^{\infty} (n+r)\, a_n\, x^{(n+r)} = r\, a_0\, x^r + (r+1)\, a_1\, x^{(r+1)} + \sum_{n=2}^{\infty} (n+r)\, a_n\, x^{(n+r)}\right]$$

$$\left[\sum_{n=0}^{\infty} -\frac{a_n}{9}\, x^{(n+r)} = -\frac{a_0}{9}\, x^r - \frac{a_1}{9}\, x^{r+1} + \sum_{n=2}^{\infty} \frac{-a_n}{9}\, x^{(n+r)}\right]$$

$$\Big[\; r(r-1)\, a_0\, x^r + (r+1)\, r\, a_1\, x^{r+1}$$

$$+ r\, a_0\, x^r + (r+1)\, a_1\, x^{r+1}$$

$$- \frac{a_0}{9}\, x^r - \frac{a_1}{9}\, x^{r+1}$$

$$+ \sum_{n=2}^{\infty} \Big[(n+r)(n+r-1)\, a_n + (n+r)\, a_n$$

$$+ a_{n-2} - \frac{a_n}{9} \Big]\, x^{(n+r)} = 0 \;\Big]$$

$$\Big[\; \left(r(r-1) + r - \frac{1}{9}\right) a_0\, x^r$$

$$\left((r+1)\, r + (r+1) - \frac{1}{9}\right) a_1\, x^{r+1}$$

$$+ \sum_{n=2}^{\infty} \Big[(n+r)\left[(n+r-1) + 1\right] a_n - \frac{a_n}{9} + a_{n-2} \Big] x^{(n+r)}$$

$$= 0 \;\Big]$$

$$\boxed{r^2 - \frac{1}{9} = 0}$$

$$\boxed{(r+1)^2 - \frac{1}{9} = 0}$$

$$\boxed{\left((n+r)^2 - \frac{1}{9}\right) a_n + a_{n-2} = 0}$$

$$r^2 - \frac{1}{9} = 0 \quad \Rightarrow \quad r_1 = \frac{1}{3} \quad , \quad r_2 = -\frac{1}{3}$$

$$(r+1)^2 - \frac{1}{9} = 0 \quad \Rightarrow \quad r_3 = \frac{1}{3} - 1 \quad , \quad r_4 = -\frac{1}{3} - 1$$

Largest sol. $\boxed{r_1 = \frac{1}{3}}$

corresponds to the solution proportional to a_0. and $a_1 = 0$

$$a_0 \to a_2 \to a_4 \to \cdots \to a_{2n} \qquad y_1$$

$$a_1 \to a_3 \to a_5 \to \cdots \to a_{2n+1} \qquad y_2$$

$a_1 = 0,$ $\boxed{r_1 = \frac{1}{3}}$

$$\boxed{\left[\left(n + \frac{1}{3}\right)^2 - \frac{1}{9}\right] a_n + a_{n-2} = 0}$$

$$\boxed{a_n = -\frac{a_{n-2}}{\left[\left(n+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]}}$$

$n=2$

$$\boxed{a_2 = -\frac{a_0}{\left[\left(2+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]}}$$

$n=4$

$$a_4 = -\frac{a_2}{\left[\left(4+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]}$$

$$\boxed{a_4 = \frac{1}{\left[\left(4+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]} \frac{1}{\left[\left(2+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]} a_0}$$

$$\boxed{y(x) = a_0 \, x^{\frac{1}{3}} \left[1 - \frac{1}{\left[\left(2+\frac{1}{3}\right)^2 - \frac{1}{9}\right]} x^2 + \frac{1}{\left[\left(4+\frac{1}{3}\right) - \frac{1}{9}\right]} \frac{1}{\left[\left(2+\frac{1}{3}\right)^2 - \frac{1}{9}\right]} x^4 + \cdots \right]}$$

Example : Verify that $x_0 = 0$ is a r.s.p. of

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{9}\right) y = 0$$

Sol.

$P(x) = x^2 \quad \Rightarrow \quad x_0 = 0$ is Singular point

$Q(x) = x$

$R(x) = \left(x^2 - \frac{1}{9}\right)$

$$x \frac{Q(x)}{P(x)} = x \frac{x}{x^2} = 1 \xrightarrow[x \to 0]{} 1 \quad ✓$$

$$x^2 \frac{R(x)}{P(x)} = x^2 \frac{\left(x^2 - \frac{1}{9}\right)}{x^2} = x^2 - \frac{1}{9} \xrightarrow[x \to 0]{} -\frac{1}{9} \quad ✓$$

$x \frac{Q(x)}{P(x)} = 1 \quad , \quad x^2 \frac{R(x)}{P(x)} = \left(x^2 - \frac{1}{9}\right)$

have Taylor expansions at $x_0 = 0$.

$$\boxed{x_0 = 0 \text{ is a r.s.p.}}$$

* Sect. 5.4 : Euler eqs.

Recall: - Euler eq.

$$\boxed{(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0}$$

- Find r_1, r_2 sols. of

$$r(r-1) + \alpha r + \beta = 0$$

- If $r_1 \neq r_2$, then

$$\boxed{y_1(x) = |x-x_0|^{r_1}}\ , \quad \boxed{y_2(x) = |x-x_0|^{r_2}}$$

- If $r_1 = r_2$, then

$$\boxed{\begin{array}{l} y_1(x) = |x-x_0|^{r_1} \\ \hline y_2(x) = |x-x_0|^{r_1} \ln\left(|x-x_0|\right) \end{array}}$$

- If $r_1 \neq r_2$ and $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$,

then real-valued fundamental solutions are

$$\boxed{y_1 = |x - x_0|^{\lambda} \cos\left[\mu \ln(|x - x_0|)\right]}$$

$$\boxed{y_2 = |x - x_0|^{\lambda} \sin\left[\mu \ln(|x - x_0|)\right]}$$

Example: Find real-valued fundamental sols. near $x_0 = 2$ of

$$(x-2)^2 y'' + 5(x-2)y' + 8y = 0$$

Sols:

(Fast to solve)

$$r(r-1) + 5r + 8 = 0$$

$$r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2}$$

$$\boxed{r_1 = -2 + 2i} \qquad \boxed{r_2 = -2 - 2i}$$

$$\boxed{y_1(x) = (x-2)^{-2} \cos(2 \ln |x-2|)}$$

$$\boxed{y_2(x) = (x-2)^{-2} \sin(2 \ln |x-2|)}$$

* Sect. 5.2 : Power Series solutions near regular points.

Example : [Find the recurrence relation and a power series sol. at $x_0 = 0$ of

$$(4-x^2)\, y'' + 2y = 0$$]

Sol:

$x_0 = 0$ is a regular point, since $P(0) = 4$.

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^n}$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(4-x^2)\,y'' = (4-x^2)\sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2}$$

$$\boxed{\begin{aligned}(4-x^2)\,y'' &= \sum_{n=0}^{\infty} 4n(n-1)\,a_n\,x^{n-2}\\ &+ \sum_{n=0}^{\infty} -n(n-1)\,a_n\,x^n\end{aligned}}$$

$$\left[\begin{aligned}(4-x^2)\,y'' &= \sum_{n=2}^{\infty} 4n(n-1)\,a_n\,x^{n-2}\\ &+ \sum_{n=0}^{\infty} -n(n-1)\,a_n\,x^n\end{aligned}\right]$$

$n-2=m \qquad\qquad n=m+2$

$$\boxed{\begin{aligned}(4-x^2)\,y'' &= \sum_{m=0}^{\infty} 4(m+2)(m+1)\,a_{m+2}\,x^m\\ &+ \sum_{n=0}^{\infty} -n(n-1)\,a_n\,x^n\end{aligned}}$$

$$\left[\sum_{n=0}^{\infty} 4(n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} -n(n-1)\, a_n\, x^n + \sum_{n=0}^{\infty} 2a_n\, x^n = 0 \right]$$

$$\sum_{n=0}^{\infty} \left[4(n+2)(n+1)\, a_{n+2} + (-n(n-1)+2)\, a_n \right] x^n = 0$$

$$4(n+2)(n+1)\, a_{n+2} + (-n^2+n+2)\, a_n = 0$$

Notice : $-n^2+n+2 = -(n-2)(n+1)$

$$4(n+2)(n+1)\, a_{n+2} - (n-2)(n+1)\, a_n = 0$$

$$\boxed{4(n+2)\, a_{n+2} - (n-2)\, a_n = 0}$$

$a_0 \to a_2 \to a_4 \to \cdots \to a_{2n}$ $\quad y_1$

$a_1 \to a_3 \to a_5 \to \cdots \to a_{2n+1}$ $\quad y_2$

we choose even power solution.

$$\boxed{a_{n+2} = \frac{(n-2)}{4(n+2)} a_n}$$

$n=0$ $\quad a_2 = \frac{-2}{8} a_0 \Rightarrow \boxed{a_2 = -\frac{1}{4} a_0}$

$n=2$ $\quad a_4 = \frac{0}{4(4)} a_2 \Rightarrow \boxed{a_4 = 0}$

$n=4$ $\quad a_6 = \frac{2}{4(6)} a_4 \Rightarrow \boxed{a_6 = 0}$

$\vdots$

$n \geqslant 2.$ $\quad \boxed{a_{2n} = 0}$

The series terminates at $n=2$.

$$\boxed{y_1(x) = a_0 \left[1 - \frac{1}{4} x^2 \right]}$$