mth 235 L19

Plan:

* Solving differential eqs. using Laplace Transforms.
* Homogeneous and non-homog. eqs.
* First, second, higher order eqs.
* Useful tool: Partial fraction decomposition.

(6.2)

* Solving differential eqs. using $\mathcal{L}[\,]$

- constant coefficients diff. eqs.
- Homogeneous, Non-Homogeneous eqs.
- First, second, Higher order diff. eqs.

* Idea

$\mathcal{L}\left[\begin{matrix}\text{diff. eq.}\\ y(t)\end{matrix}\right] \longrightarrow$ Algebraic eq. $\mathcal{L}[y]$

$\downarrow$

Transform back to $y(t)$ $\longleftarrow$ Solve for $\mathcal{L}[y]$

Recall:

(1) $\mathcal{L}[a f(t) + b\, g(t)] = a\, \mathcal{L}[f(t)] + b\, \mathcal{L}[g(t)]$

(2) $\mathcal{L}[y^{(n)}] = s^n \mathcal{L}[y] - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{(n-1)}(0).$

$n \geq 0$

* **Example** : [Use Laplace Transform to find Sol. IVP

$$y'' - y' - 2y = 0 \qquad \begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix}$$

]

Sol:

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] = 0$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0$$

$$(s^2\,\mathcal{L}[y] - s\,y(0) - y'(0))$$
$$- (s\,\mathcal{L}[y] - y(0)) \qquad - 2\,\mathcal{L}[y] = 0$$

$$(s^2 - s - 2)\,\mathcal{L}[y] + (-s + 1)\,y(0) - y'(0) = 0$$

$$\boxed{\mathcal{L}[y] = \frac{(s-1)\,y(0) - y'(0)}{(s^2 - s - 2)}}$$

Initial conditions: $y(0) = 1$, $y'(0) = 0$.

$$\boxed{\mathcal{L}[y] = \frac{s-1}{s^2 - s - 2}}$$

$\downarrow$

Not in table.

Find roots of denominator.

$$s^2 - s - 2 = 0$$

$$s = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \Rightarrow \boxed{s_1 = 2}, \ \boxed{s_2 = -1}$$

$$\boxed{\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}}$$

Recall: Partial Fractions Decomposition of a rational function.

$$\boxed{\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{(s-2)} + \frac{b}{(s+1)}}$$

Find a, b.

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1}$$

$$\boxed{(s-1) = a(s+1) + b(s-2)}$$

$$(s-1) = (a+b)s + (a-2b) \Rightarrow \boxed{\begin{array}{c} a+b=1 \\ \hline a-2b=-1 \end{array}}$$

$$b = 1-a \Rightarrow a - 2(1-a) = -1$$

$$3a - 2 = -1$$

$$3a = 1 \Rightarrow \boxed{a = \frac{1}{3}} \quad \boxed{b = \frac{2}{3}}$$

$$\boxed{\mathcal{L}[Y] = \frac{1}{3}\frac{1}{(s-2)} + \frac{2}{3}\frac{1}{(s+1)}}$$

In the table.

Table: $\mathcal{L}[e^{2t}] = \dfrac{1}{(s-2)}$

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

$$\mathcal{L}[y] = \frac{1}{3}\mathcal{L}[e^{2t}] + \frac{2}{3}\mathcal{L}[e^{-t}]$$

$$\mathcal{L}[y] = \mathcal{L}\left[\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\right]$$

$$\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}$$

Example Use Laplace Transforms to find y sol. IVP

$$y^{(4)} - 4y = 0$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$$

Sol:

$$\mathcal{L}[y^{(4)}] - 4\,\mathcal{L}[y] = 0$$

$$\boxed{s^4\,\mathcal{L}[y] - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - 4\,\mathcal{L}[y] = 0}$$

Initial conditions:

$$s^4\,\mathcal{L}[y] - s^3 - (-2)s - 4\,\mathcal{L}[y] = 0$$

$$(s^4 - 4)\,\mathcal{L}[y] - s^3 + 2s = 0$$

$$\boxed{\mathcal{L}[y] = \frac{s^3 - 2s}{s^4 - 4}}$$

$$\mathcal{L}[y] = \frac{s\,(s^2-2)}{(s^2-2)\,(s^2+2)}$$

$$\boxed{\mathcal{L}[y] = \frac{s}{s^2+2}}$$ $\Rightarrow$ in table.

$$\mathcal{L}[y] = \frac{s}{s^2+(\sqrt{2})^2} = \mathcal{L}[\cos(\sqrt{2}\,t)]$$

$$\boxed{y(t) = \cos(\sqrt{2}\,t)}$$

<u>Example</u> : use Laplace Transform to solve IVP

$$y'' - 4y' + 4y = 0$$

$$y(0) = 1 \quad , \quad y'(0) = 1$$

<u>Sol:</u>

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0$$

$$\left[s^2 \mathcal{L}[y] - s\,y(0) - y'(0) \right]$$

$$- 4 \left[s\,\mathcal{L}[y] - y(0) \right] + 4\,\mathcal{L}[y] = 0$$

$$(s^2 - 4s + 4)\,\mathcal{L}[y] + (-s+4)\,y(0) - y'(0) = 0$$

Initial conditions

$$(s^2 - 4s + 4)\,\mathcal{L}[y] + (-s+4) - 1 = 0$$

$$\boxed{\mathcal{L}[y] = \frac{s-3}{s^2 - 4s + 4}}$$

→ Not in table.

roots of the denominator.

$$s^2 - 4s + 4 = 0$$

$$s = \frac{4 \pm \sqrt{16-16}}{2} = 2 \Rightarrow \boxed{s_1 = 2}$$

$$\boxed{\mathcal{L}[y] = \frac{s-3}{(s-2)^2}}$$

Idea:

$$\mathcal{L}[y] = \frac{(s-2+2-3)}{(s-2)^2}$$

$$\mathcal{L}[y] = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2}$$

$$\boxed{\mathcal{L}[y] = \frac{1}{(s-2)} - \frac{1}{(s-2)^2}}$$

Table: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ $(a=2)$

$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$ $(n=1, a=2)$

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

$$= \mathcal{L}[e^{2t}] - \mathcal{L}[t\,e^{2t}]$$

$$= \mathcal{L}[\,e^{2t} - t\,e^{2t}]$$

$$= \mathcal{L}[\,(1-t)\,e^{2t}]$$

$$\boxed{y(t) = (1-t)\,e^{2t}}$$

Example: use Laplace Transform to solve IVP.

$$y'' - 4y' + 4y = 3\sin(2t)$$

$$y(0) = 1 \qquad y'(0) = 1$$

Sol:

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)]$$

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = 3\,\mathcal{L}[\sin(2t)]$$

$$= 3\,\frac{2}{s^2 + 2^2}$$

$$\boxed{\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \frac{6}{s^2+4}}$$

$$[s^2\mathcal{L}[y] - s\,y(0) - y'(0)]$$

$$-4[s\mathcal{L}[y] - y(0)] + 4\mathcal{L}[y] = \frac{6}{s^2+4}$$

$$(s^2 - 4s + 4)\ \mathcal{L}[y] + (-s+4)\ y(0) - y'(0) = \frac{6}{s^2+4}$$

$$(s-2)^2\ \mathcal{L}[y] + (-s+4-1) = \frac{6}{s^2+4}$$

$$(s-2)^2\ \mathcal{L}[y] = (s-3) + \frac{6}{s^2+4}$$

$$\boxed{\mathcal{L}[y] = \frac{s-3}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}}$$

$\downarrow$

Previous example.

$\downarrow$

$$\boxed{\mathcal{L}[y] = \mathcal{L}[(1-t)e^{2t}] + \frac{6}{(s-2)^2(s^2+4)}}$$

$\downarrow$

Partial Fraction Decomposition.

$$\boxed{\frac{6}{(s^2+4)(s-2)^2} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}}$$

$$\frac{6}{(s^2+4)(s-2)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)\ (s-2)^2}$$

$$\boxed{6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}$$

$$6 = (as+b)(s^2-4s+4) + c(s^3+4s-2s^2-8)$$

$$+ d(s^2+4)$$

$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4)$$

$$+ c(s^3 + 4s - 2s^2 - 8) + d(s^2+4)$$

$$\boxed{\begin{aligned} 6 = {} & (a+c)s^3 + (-4a+b-2c+d)s^2 \\ & + (4a-4b+4c)s + (4b-8c+4d) \end{aligned}}$$

Four eqs., one for each power of s

$$\begin{bmatrix} a+c=0 \\ -4a+b-2c+d=0 \\ 4a-4b+4c=0 \\ 4b-8c+4d=6 \end{bmatrix} \qquad \begin{bmatrix} a+c=0 \\ -4a+b-2c+d=0 \\ a-b+c=0 \\ b-2c+d=\frac{3}{2} \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

(1) and (2) $\Rightarrow$ $\boxed{b=0}$; (1) $\Rightarrow$ $\boxed{c=-a}$

(2) $\Rightarrow$ $-4a+2a+d=0$

$$\boxed{-2a+d=0}$$

(4) $\Rightarrow$ $\boxed{2a+d=\frac{3}{2}}$

$\oplus$ $\quad 2d=\frac{3}{2} \Rightarrow \boxed{d=\frac{3}{4}}$

$a=\frac{d}{2} \Rightarrow \boxed{a=\frac{3}{8}} \quad \boxed{c=-\frac{3}{8}}$

$$\boxed{\frac{6}{(s^2+4)(s-2)^2} = \frac{3}{8}\frac{s}{(s^2+4)} - \frac{3}{8}\frac{1}{(s-2)} + \frac{3}{4}\frac{1}{(s-2)^2}}$$

$$\frac{6}{(s^2+4)(s-2)^2} = \frac{3}{8}\mathcal{L}[\cos(2t)] - \frac{3}{8}\mathcal{L}[e^{2t}]$$

$$+ \frac{3}{4}\mathcal{L}[t e^{2t}]$$

$$\boxed{\frac{6}{(s^2+4)(s-2)^2} = \mathcal{L}\left[\frac{3}{8}\left(\cos(2t) - e^{2t} + 2t e^{2t}\right)\right]}$$

$$\mathcal{L}[y] = \mathcal{L}[(1-t)e^{2t}]$$

$$+ \mathcal{L}\left[\frac{3}{8}\left(\cos(2t) + (2t-1)e^{2t}\right)\right]$$

$$\boxed{y(t) = (1-t)e^{2t} + \frac{3}{8}(2t-1)e^{2t} + \frac{3}{8}\cos(2t)}$$