mth 235 L18

Plan:
* Laplace Transform
* Review: Improper integrals
* Examples
* Properties of the Laplace transform. (6.1)
* Use in differential eqs. (6.2)

* The Laplace Transform

Def: The function $F: D_F \to \mathbb{R}$ is the **Laplace Transform** of a function $f: [0, \infty) \to \mathbb{R}$ iff holds

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t)\, dt.}$$

Remark: The domain D_F of F depends on the function f.

Notation: $F(s) = \mathcal{L}[f(t)]$.

* Review: Improper Integrals

$$\boxed{\int_{t_0}^\infty g(t)\, dt = \lim_{N \to \infty} \int_{t_0}^N g(t)\, dt}$$

Notation.

- Integral **converges** if limit exists
- Integral **diverges** if limit does not exist.

* Review of improper integrals

Example : [compute the improper integral

$$I = \int_0^\infty e^{-at}\,dt \quad , \quad a>0]$$

Sol:

$$\int_0^\infty e^{-at}\,dt = \lim_{N\to\infty} \int_0^N e^{-at}\,dt$$

$$= \lim_{N\to\infty} \left[\left(-\frac{1}{a}\right) e^{-at} \Big|_0^N \right]$$

$$= \lim_{N\to\infty} \left[-\frac{1}{a} \left(e^{-aN} - 1 \right) \right]$$

$(a>0)$

$$\boxed{\int_0^\infty e^{-at}\,dt = \frac{1}{a}}$$

* Examples of Laplace Transforms

(1) [Find $\mathcal{L}[1]$.]

Sol: $f(t) = 1$,

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, 1 \, dt$$

$$= \int_0^\infty e^{-st} \, dt$$

$$\boxed{\mathcal{L}[1] = \frac{1}{s}} \qquad \boxed{s > 0}$$

$$\boxed{F(s) = \frac{1}{s}} \qquad \boxed{D_F = (0, \infty)}$$

(2) [Find $\mathcal{L}[e^{at}]$.]

Sol: $f(t) = e^{at}$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt \quad , \quad s-a>0$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{(s-a)}} \qquad \boxed{s>a}$$

$$\boxed{F(s) = \frac{1}{s-a}} \qquad \boxed{D_F = (a, \infty)}$$

(3) [Find $\mathcal{L}[\sin(at)]$.]

Sol: $f(t) = \sin(at)$

$$\mathcal{L}[\sin(at)] = \int_0^\infty e^{-st} \sin(at)\, dt$$

(a) Integration by parts, twice.

(b) complex exponentials.

$$\sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat} \right)$$

$$\mathcal{L}[\sin(at)] = \int_0^\infty e^{-st} \frac{1}{2i}\left(e^{iat} - e^{-iat} \right) dt$$

$$= \frac{1}{2i} \int_0^\infty \left(e^{(-s+ia)t} - e^{-(s+ia)t} \right) dt$$

$$= \frac{1}{2i} \lim_{N \to \infty} \left[\int_0^N e^{(-s+ia)t}\, dt - \int_0^N e^{-(s+ia)t}\, dt \right]$$

$\mathcal{L}[\sin(at)] =$

$$= \frac{1}{2i} \lim_{N \to \infty} \left[\frac{1}{(-s+ia)} e^{(-s+ia)t} \Big|_0^N - \frac{(-1)}{(s+ia)} e^{-(s+ia)t} \Big|_0^N \right]$$

$$= \frac{1}{2i} \lim_{N \to \infty} \left[\frac{1}{(-s+ia)} \left(e^{-sN} e^{iaN} - 1 \right) + \frac{1}{(s+ia)} \left(e^{-sN} e^{-iaN} - 1 \right) \right]$$

$$= \frac{1}{2i} \left[-\frac{1}{(-s+ia)} - \frac{1}{(s+ia)} \right]$$

$$= \frac{1}{2i} \frac{-(s+ia) - (-s+ia)}{(-s^2 - a^2)}$$

$$= \frac{1}{2i} \frac{-2ia}{-(s^2+a^2)}$$

$$\boxed{\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}} \qquad \boxed{s > 0.}$$

* **More examples of Laplace Transforms**

- See table on page 317, textbook.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}$	$s > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

⋮

* Sufficient conditions for the existence of $\mathcal{L}[\ ]$.

Thrm: If $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and there exist constants $K > 0$, $a > 0$ such that

$$\boxed{|f(t)| \leq K e^{at}}$$

(f is bounded by an exponential)

then $\boxed{F(s) = \mathcal{L}[f(t)]}$ exists for $\boxed{s > a.}$

* Properties of $\mathcal{L}[\ ]$.

(1) Linear property.

Thrm: [If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are defined and a, b are constants, then

$$\boxed{\mathcal{L}[af+bg] = a\,\mathcal{L}[f] + b\,\mathcal{L}[g].}$$
]

Idea of the proof: [Integration is a linear operation on functions.]

$$\int [a\, f(t) + b\, g(t)]\, dt = a \int f(t)\, dt + b \int g(t)\, dt.$$

(2) The $\mathcal{L}[f']$.

Thm: [If $f : [0, \infty) \to \mathbb{R}$ satisfies that $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are defined, then

$$\boxed{\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)}.$$
]

Proof: (integration by parts.)

$$\boxed{\mathcal{L}[f']} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \lim_{N \to \infty} \int_0^N e^{-st} f'(t)\, dt$$

$du = f'\, dt \qquad v = e^{-st}$

$u = f \qquad dv = -se^{-st}\,dt$

$$= \lim_{N \to \infty} \left[f(t) e^{-st} \Big|_0^N - \int_0^N (-s)\, e^{-st} f(t)\, dt \right]$$

$$= \lim_{N \to \infty} \left[\underbrace{f(N) e^{-sN}}_{\to 0} - f(0) + s \int_0^N e^{-st} f(t)\, dt \right]$$

$$= \boxed{s\,\mathcal{L}[f] - f(0)}.$$

■

(3) The $\mathcal{L}[f'']$.

Thrm: If $f : [0,\infty) \to \mathbb{R}$ satisfies that $\mathcal{L}[f]$, $\mathcal{L}[f']$ and $\mathcal{L}[f'']$ are defined, then

$$\boxed{\mathcal{L}[f''] = s^2\, \mathcal{L}[f] - s f(0) - f'(0).}$$

Proof:

$$\boxed{\mathcal{L}[f'']} = \mathcal{L}[(f')']$$

$$= s\, \mathcal{L}[(f')] - f'(0)$$

$$= s\left[s\, \mathcal{L}[f] - f(0)\right] - f'(0)$$

$$= \boxed{s^2\, \mathcal{L}[f] - s f(0) - f'(0).}$$

$\blacksquare$

* The $\mathcal{L}[\ ]$ and differential eqs.

- Laplace transforms can be used to find solutions of differential eqs. with constant coefficients.

- Idea of the method.

$$\mathcal{L}\left[\begin{array}{c}\text{differential eq.}\\ \text{for } y(t)\end{array}\right] \xrightarrow[\text{(1)}]{} \begin{array}{c}\text{Algebraic eq.}\\ \text{for } \mathcal{L}[y].\end{array} \rightarrow$$

$\xrightarrow[\text{(2)}]{}$ Solve the algebraic eq. for $\mathcal{L}[y]$ $\xrightarrow[\text{(3)}]{}$ Transform back to obtain $y(t)$ (using a Table.)

* Example : Use Laplace transform to find the sol. to the IVP

$$y' + 2y = 0 \quad , \quad y(0) = 3.$$

Sol:

$$\mathcal{L}[\,y' + 2y\,] = \mathcal{L}[0] = 0$$

① $$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0$$

$$\boxed{s\,\mathcal{L}[y] - y(0) + 2\,\mathcal{L}[y] = 0}$$ algebraic eq. for $\mathcal{L}[y]$.

$$s\,\mathcal{L}[y] - 3 + 2\,\mathcal{L}[y]$$ ← Initial condition.

②

$$(s+2)\,\mathcal{L}[y] = 3$$

$$\boxed{\mathcal{L}[y] = \frac{3}{s+2}}$$ Solved the algebraic eq. for $\mathcal{L}[y]$.

Table: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ $s > a$

for $a = -2$: $\boxed{\mathcal{L}[e^{-2t}] = \frac{1}{s+2}.}$ $\boxed{s > -2.}$

Recall: $\boxed{\mathcal{L}[3e^{-2t}]} = 3\,\mathcal{L}[e^{-2t}]$

$$= \boxed{\frac{3}{s+2}}$$

we conclude.

$$\mathcal{L}[y(t)] = \frac{3}{s+2} = \mathcal{L}[3e^{-2t}]$$

$$\boxed{y(t) = 3e^{-2t}}$$