mth 235 L17

Plan:

* Equations with regular-singular points.
* Examples

(5.4)

* Method to find a solution to eqs. with r.s.p.
* Example.

(5.5)

Recall: $x_0 \in \mathbb{R}$ is a singular point of

$$\mathbb{P}(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

iff holds $\mathbb{P}(x_0) = 0$.

* Equations with regular-singular points

Def: A singular point $x_0 \in \mathbb{R}$ of the eq.

$$\boxed{P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0} \qquad (1)$$

is called a regular-singular point iff hold

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \alpha \text{ is finite,}$$

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \beta \text{ is finite}$$

and both functions

$$\boxed{(x - x_0) \frac{Q(x)}{P(x)}}, \qquad \boxed{(x - x_0)^2 \frac{R(x)}{P(x)}}$$

admit convergent Taylor series at x_0.

Remark: [If x_0 is a rsp of (1) and

$$\mathbb{P}(x) \sim (x-x_0)^n \quad \text{near } x_0,$$

then

$$\left.\begin{aligned} Q(x) &\sim (x-x_0)^{n-1} \\ R(x) &\sim (x-x_0)^{n-2} \end{aligned}\right] \text{near } x_0.$$
]

Remark: [Equations with a rsp x_0 are equations close to an Euler eq. near x_0.]

Proof: x_0 rsp of

$$\boxed{\mathbb{P}(x)\, y'' + Q(x)\, y' + R(x)\, y = 0}$$

$$y'' + \frac{Q(x)}{\mathbb{P}(x)}\, y' + \frac{R(x)}{\mathbb{P}(x)}\, y = 0$$

$$(x-x_0)^2\, y'' + (x-x_0) \underbrace{\left[(x-x_0)\frac{Q(x)}{\mathbb{P}(x)}\right]}_{\sim \alpha} y' + \underbrace{\left[(x-x_0)^2 \frac{R(x)}{\mathbb{P}(x)}\right]}_{\sim \beta} y = 0$$

near x_0

Euler eq. $\boxed{(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0}$

* **Main Example**: [Show that x_0 is a r.s.p. of the Euler eq.

$$(x-x_0)^2 y'' + c_1 (x-x_0) y' + c_2 y = 0$$]

Sol:

$\mathbb{P}(x) = (x-x_0)^2 \quad \Rightarrow \quad x_0$ singular point.

$Q(x) = c_1 (x-x_0)$

$R(x) = c_2$

$$\boxed{(x-x_0)\frac{Q(x)}{\mathbb{P}(x)}} = (x-x_0)\frac{c_1(x-x_0)}{(x-x_0)^2} = \boxed{c_1 \xrightarrow[x \to x_0]{} c_1} \checkmark$$

$$\boxed{(x-x_0)^2\frac{R(x)}{\mathbb{P}(x)}} = (x-x_0)^2\frac{c_2}{(x-x_0)^2} = \boxed{c_2 \xrightarrow[x \to x_0]{} c_2} \checkmark$$

and $\quad (x-x_0)\dfrac{Q(x)}{\mathbb{P}(x)} = c_1 \quad , \quad (x-x_0)^2\dfrac{R(x)}{\mathbb{P}(x)} = c_2$

have Taylor series expansions at x_0.

$\boxed{x_0 \text{ is a r.s.p.}}$

<u>Example</u>: [Find the r.s.p. of the eq.

$$(1-x^2)\, y'' - 2x\, y' + \alpha(\alpha+1)\, y = 0.]$$

<u>Sol.</u>

$$\mathbb{P}(x) = (1-x^2) = (1+x)(1-x) \Rightarrow \boxed{\begin{matrix} x_1 = -1 \\ x_2 = 1 \end{matrix}}$$ singular points.

$Q(x) = -2x$

$R(x) = \alpha(\alpha+1)$

case $x_1 = -1$

$$(x+1)\frac{Q(x)}{\mathbb{P}(x)} = (x+1)\frac{(-2x)}{(1+x)(1-x)} = -\frac{2x}{(1-x)} \xrightarrow[x\to -1]{} 1$$

$$(x+1)^2\frac{R(x)}{\mathbb{P}(x)} = (x+1)^2\frac{\alpha(\alpha+1)}{(1+x)(1-x)} = \frac{(x+1)}{(1-x)}\alpha(\alpha+1) \xrightarrow[x\to -1]{} 0$$

functions $-\frac{2x}{1-x}$, $\frac{(x+1)}{(1-x)}\alpha(\alpha+1)$

have Taylor expansions at $x_1 = -1$.

$$\boxed{x_1 = -1 \quad \text{is a r.s.p.}}$$

<u>Exercise</u>: show $\boxed{x_2 = -1}$ is a r.s.p.

<u>Example</u>: [Find the r.s.p. of the eq.

$$(x+2)^2 (x-1)\, y'' + 3(x-1)\, y' + 2y = 0$$]

<u>Sol</u>:

$P(x) = (x+2)^2 (x-1) \quad \Rightarrow \quad \boxed{\begin{array}{l} x_1 = -2 \\ x_2 = 1 \end{array}}$ singular points.

$Q(x) = 3(x-1)$

$R(x) = 2$

case $x_1 = -2$.

$$\left[(x+2)\frac{Q(x)}{P(x)} = (x+2)\frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3}{(x+2)} \underset{x \to -2}{\longrightarrow} \pm\infty \right]$$

$$\boxed{x_1 = -2 \text{ is NOT a r.s.p.}}$$

case $x_2 = 1$

$$\left[(x-1)\frac{Q(x)}{P(x)} = (x-1)\frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3(x-1)}{(x+2)^2} \underset{x \to 1}{\longrightarrow} 0 \right] \checkmark$$

$$\left[(x-1)^2\frac{R(x)}{P(x)} = (x-1)^2\frac{2}{(x+2)^2(x-1)} = \frac{2(x-1)}{(x+2)^2} \underset{x \to 1}{\longrightarrow} 0 \right] \checkmark$$

Taylor series requirement ✓ $\boxed{x_2 = 1 \text{ is r.s.p.}}$

Example Find the r.s.p. of the eq.

$$x\,y'' + x\ln(|x|)\,y' + 3x\,y = 0$$

Sol:

$P(x) = x \quad \Rightarrow \quad \boxed{x_0 = 0}$ singular point

$Q(x) = x\ln(|x|)$

$R(x) = 3x$

$$x\,\frac{Q(x)}{P(x)} = \frac{x\,(x\ln(|x|))}{x} = x\ln(|x|) \underset{x\to 0}{\longrightarrow} 0 \;(\pm\infty)$$

$$x\ln(|x|) = \frac{\ln(|x|)}{\left(\frac{1}{x}\right)} \underset{x\to 0}{\longrightarrow} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = -x \underset{x\to 0}{\longrightarrow} 0$$

L'Hôpital's rule

$$x^2\,\frac{R(x)}{P(x)} = x^2\,\frac{3x}{x} = 3x^2 \underset{x\to 0}{\longrightarrow} 0$$

However, $x\ln(|x|)$ does not admit a Taylor series expansion at $x_0 = 0$.

$$\boxed{x_0 = 0 \text{ is NOT a r.s.p.}}$$

* Finding a solution near a r.s.p.

Recall: If x_0 is a r.s.p. of

$$\boxed{\mathbb{P}(x)\, y'' + Q(x)\, y' + R(x)\, y = 0} \quad (1)$$

then the eq. coeff. are close to the coeff. of the Euler eq.

$$\boxed{(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0} \quad (2)$$

where

$$\alpha = \lim_{x \to x_0} (x-x_0)\frac{Q}{\mathbb{P}}$$

$$\beta = \lim_{x \to x_0} (x-x_0)^2 \frac{R}{\mathbb{P}}.$$

Idea: If Eq. (1) is close to Eq. (2), then sols. to Eq. (1) may be close to sols. to Eq. (2)

Recall: At least one solution to an Euler eq. has the form

$$\boxed{y_E(x) = (x - x_0)^r}$$

* Method to find solutions near a r.s.p. x_0

(1) Look for solutions of the form

$$\boxed{y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n} \quad (3)$$

(2) Introduce (3) into (1) and find an equation for r and a recurrence relation for a_n.

(3) First, find the solutions for the number r.
Then, introduce the larger solution (any sol. if they are complex numbers) into the recurrence relation, and solve the recurrence relation to find the a_n.

Example: Find the first 4 terms of the power series expansion of a solution y near the r.s.p. $x_0 = 0$ of the eq.

$$x^2 y'' - x(x+3)\,y' + (x+3)\,y = 0$$

Recall: The sol. y near $x_0 = 0$ may be close to a solution of an Euler eq. with r.s.p. $x_0 = 0$. At least one sol. of the Euler eq. is

$$y_E = x^r$$

So we look for solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^{(n+r)}}$$

Find r and a_n.

Sol.

$$y(x) = \sum_{n=0}^{\infty} a_n \, x^{(n+r)}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) \, a_n \, x^{(n+r-1)}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) \, a_n \, x^{(n+r-2)}$$

$$\left[\begin{aligned} & x^2 \left[\sum_{n=0}^{\infty} (n+r)(n+r-1) \, a_n \, x^{(n+r-2)} \right] \\ & - x(x+3) \left[\sum_{n=0}^{\infty} (n+r) \, a_n \, x^{(n+r-1)} \right] \\ & + (x+3) \left[\sum_{n=0}^{\infty} a_n \, x^{(n+r)} \right] \qquad = 0. \end{aligned} \right]$$

$$\left[\begin{aligned} & \sum_{n=0}^{\infty} (n+r)(n+r-1) \, a_n \, x^{(n+r)} \\ & - (x+3) \sum_{n=0}^{\infty} (n+r) \, a_n \, x^{(n+r)} \\ & + (x+3) \sum_{n=0}^{\infty} a_n \, x^{(n+r)} \qquad = 0 \end{aligned} \right]$$

$$\boxed{-(x+3) \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{(n+r)} =}$$

$$= - \underbrace{\sum_{n=0}^{\infty} (n+r)\, a_n\, x^{(n+r+1)}}_{m = n+1} - \sum_{n=0}^{\infty} 3(n+r)\, a_n\, x^{n+r}$$

$$= - \sum_{m=1}^{\infty} (m+r-1)\, a_{m-1}\, x^{(m+r)} - \sum_{n=0}^{\infty} 3(n+r)\, a_n\, x^{(n+r)}$$

$$\boxed{= \sum_{n=1}^{\infty} -(n+r-1)\, a_{n-1}\, x^{(n+r)} + \sum_{n=0}^{\infty} -3(n+r)\, a_n\, x^{(n+r)}}$$

$$\boxed{(x+3)\sum_{n=0}^{\infty} a_n x^{(n+r)} =}$$

$$= \underbrace{\sum_{n=0}^{\infty} a_n x^{(n+r+1)}}_{m = n+1} + \sum_{n=0}^{\infty} 3 a_n x^{(n+r)}$$

$$= \sum_{m=1}^{\infty} a_{m-1} x^{(m+r)} + \sum_{n=0}^{\infty} 3 a_n x^{(n+r)}$$

$$\boxed{= \sum_{n=1}^{\infty} a_{n-1} x^{(n+r)} + \sum_{n=0}^{\infty} 3 a_n x^{(n+r)}}$$

$$\left[\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{(n+r)} + \sum_{n=1}^{\infty} -(n+r-1)\, a_{n-1}\, x^{(n+r)} + \sum_{n=0}^{\infty} -3(n+r)\, a_n\, x^{(n+r)} + \sum_{n=1}^{\infty} a_{n-1}\, x^{(n+r)} + \sum_{n=0}^{\infty} 3\, a_n\, x^{(n+r)} = 0\right]$$

$$\left[a_0 x^r \left(r(r-1) - 3r + 3 \right) + \sum_{n=1}^{\infty} \Big[(n+r)(n+r-1)\, a_n - (n+r-1)\, a_{n-1} - 3(n+r)\, a_n + a_{n-1} + 3\, a_n \Big] x^{(n+r)} = 0\right]$$

$$\boxed{r(r-1) - 3r + 3 = 0}$$

(indicial equation)

$$\boxed{\begin{aligned} &(n+r)(n+r-1)\,a_n - (n+r-1)\,a_{n-1} \\ &- 3(n+r)\,a_n + a_{n-1} + 3\,a_n = 0 \end{aligned}}$$

(recurrence relation)

$$\left[\begin{aligned} &[(n+r)(n+r-1) - 3(n+r) + 3]\,a_n \\ &- [(n+r-1) - 1]\,a_{n-1} = 0 \end{aligned} \right]$$

$$\left[\begin{aligned} &[(n+r)(n+r-1) - 3(n+r-1)]\,a_n \\ &- [(n+r-2)]\,a_{n-1} = 0 \end{aligned} \right]$$

$$\boxed{(n+r-1)(n+r-3)\,a_n - (n+r-2)\,a_{n-1} = 0}$$

$$\boxed{r(r-1) - 3r + 3 = 0}$$

indicial eq. $r^2 - 4r + 3 = 0$

$$r = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \Rightarrow$$

$\boxed{r_1 = 3}$ $\boxed{r_2 = 1.}$

$r_1 - r_2 = 2$
$\Downarrow$
integer

Select the bigger root : $\boxed{r_1 = 3}$

Introduce $r = 3$ into the recurrence relation :

$$\boxed{(n+2)\, n\, a_n - (n+1)\, a_{n-1} = 0}$$

$$a_n = \frac{(n+1)}{(n+2)\, n.}\, a_{n-1}$$

$n=1$ $\quad a_1 = \frac{2}{(3)(1)} a_0 \Rightarrow \boxed{a_1 = \frac{2}{3} a_0}$

$n=2$ $\quad a_2 = \frac{3}{(4)(2)} a_1 = \frac{3}{4(2)} \frac{2}{3} a_0 \Rightarrow \boxed{a_2 = \frac{1}{4} a_0}$

$n=3$ $\quad a_3 = \frac{4}{(5)(3)} a_2 = \frac{4}{(5)(3)} \frac{1}{4} a_0 \Rightarrow \boxed{a_3 = \frac{1}{15} a_0}$

Since $$y(x) = \sum_{n=0}^{\infty} a_n \, x^{(n+r)}$$

$$y(x) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \dots$$

$r = 3.$

$$y(x) = a_0 x^3 + \frac{2}{3} a_0 x^4 + \frac{1}{4} a_0 x^5 + \frac{1}{15} a_0 x^6 + \dots$$

$$\boxed{y(x) = a_0 x^3 \left[1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \dots \right]}$$