mth 235 L16

Plan:

* Equations with singular points
* Euler Differential eq.

$$(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0$$

* Solutions near x_0
* The roots of the indicial polynomial.
* (Equations with regular-singular points)

(5.4)

* Equations with Singular points

Recall: [$x_0 \in \mathbb{R}$ is a singular point of

$$\boxed{\mathbb{P}(x)\, y'' + Q(x)\, y' + R(x)\, y = 0} \quad (1)$$

iff $\boxed{\mathbb{P}(x_0) = 0.}$]

- We are interested in finding sols. to (1) near a singular point x_0.
- The order of the eq. changes near a singular point.

as $x \to x_0$ $\left\{ \begin{array}{l} \text{- two l.i. solutions remain bounded} \\ \qquad \text{or} \\ \text{- only one sol. remain bounded} \\ \qquad \text{or} \\ \text{- none sol. remain bounded.} \end{array} \right.$

- It is known how to find sols. of eqs. near singular points in the case that the points are not so singular.

- Such points will be called regular singular points.

- Main example of an equation with regular singular points is the Euler differential eq.

* The Euler equation

Def: Given constants $x_0, \alpha, \beta \in \mathbb{R}$, the differential equation

$$\boxed{(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0} \quad (2)$$

is called the Euler equation.

Remarks:

- Eq. (2) has variable coefficients.
- Solutions of (2) are not of the form

$$y = e^{rx}$$

- $x_0 \in \mathbb{R}$ is a singular point of (2).
- We are interested in finding sols. to (2) arbitray close to x_0
- Particular case $x_0 = 0$ is

$$\boxed{x^2\, y'' + \alpha\, x\, y' + \beta\, y = 0} \quad (3)$$

$x_0 = 0$ is a singular point.

* <u>Main idea to find solutions to (3)</u>

- <u>Recall</u> the constant coeff. case:

$$\boxed{y'' + a_1 y' + a_0 y = 0} \qquad (4)$$

we looked for solutions $\boxed{y = e^{rx}}$, since the exponential can be canceled out of eq. (4)

$$\boxed{(r^2 + a_1 r + a_0)\,\cancel{e^{rx}} = 0,} \qquad (5)$$

So the number r of the characteristic eq.

$$\boxed{p(r) = r^2 + a_1 r + a_0 = 0}$$

- In the case of Euler eq

$$\boxed{x^2 y'' + \alpha x y' + \beta y = 0}$$

exponentials functions do not have the property in (4), since for $y = e^{rx}$

$$\underbrace{(x^2 r^2 + \alpha x r + \beta)}_{\text{depends on } x.}\,\cancel{e^{rx}} = 0$$

Idea: Look for solutions $\boxed{y(x) = x^r}$

$y'(x) = r\, x^{r-1} \Rightarrow \boxed{x\, y' = r\, x^r}$

$y''(x) = r(r-1)\, x^{r-2} \Rightarrow \boxed{x^2 y'' = r(r-1)\, x^r}$

Introduce y into $x^2 y'' + \alpha x y' + \beta y = 0$

$$\left(r(r-1) + \alpha r + \beta \right) x^r = 0$$

so r must be solution of

$$\boxed{q(r) = r(r-1) + \alpha r + \beta = 0}$$

Indicial eq. or Euler characteristic eq.

r must be a root of $q(r)$.

Thrm: Given constants $x_0, \alpha, \beta \in \mathbb{R}$ consider the Euler eq.

$$\boxed{(x-x_0)^2\, y'' + \alpha\,(x-x_0)\, y' + \beta\, y = 0} \quad (6)$$

Let r_1, r_2 be roots of the indicial polynomial

$$\boxed{q(r) = r(r-1) + \alpha r + \beta}\,.$$

(a) If $r_1 \neq r_2$, then the general sol. of (6) is

$$\boxed{y(x) = c_1\, |x-x_0|^{r_1} + c_2\, |x-x_0|^{r_2}}$$

with c_1, c_2 constants.

(b) If $r_1 = r_2$, then the general sol. of (6) is

$$\boxed{y(x) = c_1\, |x-x_0|^{r_1} + c_2\, \ln(|x-x_0|)\, |x-x_0|^{r_1}}$$

with c_1, c_2 constants.

<u>Example</u>: Find the general sol. of

<u>Sol</u>:

$$x^2 y'' + 4x y' + 2y = 0 \qquad (7)$$

we look for sols. $y = x^r$ $\quad x > 0$.

$$x^2 r(r-1) x^{r-2} + 4x r x^{r-1} + 2x^r = 0$$

$$(r(r-1) + 4r + 2) x^r = 0$$

r must be root of the indicial polynomial

$$q(r) = r(r-1) + 4r + 2 = 0$$

$$= r^2 + 3r + 2. = 0$$

$$r = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$\boxed{r_1 = -1} \qquad \boxed{r_2 = -2}$$

$$\boxed{\begin{array}{l} y_1(x) = x^{-1} \\ \hline y_2(x) = x^{-2} \end{array}}$$

fundamental solutions of (7) for $x > 0$.

The Thrm says that

$$\boxed{\begin{array}{l} y_1 = |x|^{-1} \\ \hline y_2 = |x|^{-2} \end{array}}$$

fundamental sols. for $x \neq 0$.

The general sol. is

$$\boxed{y(x) = c_1 |x|^{-1} + c_2 |x|^{-2}} \qquad c_1, c_2 \in \mathbb{R}$$

Example: $\left[\begin{array}{l} \text{Find the general sol. of} \\ x^2 y'' - 3x y' + 4y = 0 \end{array} \right]$

Sol:

we look for $y = x^r$, $x > 0$, so

$$\boxed{(r(r-1) - 3r + 4) x^r = 0}$$

r must be root of the indicial polynomial

$$q(r) = r(r-1) - 3r + 4 = 0$$

$$r - 4r + 4 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 16}}{2} \Rightarrow \boxed{r_1 = r_2 = 2}$$

repeated roots.

$$\boxed{y_1 = x^2}, \quad \boxed{y_2 = x^2 \ln(|x|).} \qquad x \neq 0$$

(fundamental solutions)

The general sol. is

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln(|x|)} \qquad x \neq 0.$$

<u>Proof</u> <u>of</u> <u>Thrm</u>: (existence) (case $x_0 = 0$)

Propose $\boxed{y = x^r}$ $x > 0$.

Then r must be sol. of.

$$\boxed{q(r) = r(r-1) + \alpha r + \beta = 0}$$

$$\boxed{r_{\substack{1\\2}} = -\frac{(\alpha-1)}{2} \pm \frac{1}{2}\sqrt{(\alpha-1)^2 - 4\beta}}$$

(1) $(\alpha-1)^2 - 4\beta > 0 \Rightarrow$ $\boxed{y_1 = x^{r_1}}$ $\boxed{y_2 = x^{r_2}}$ fundamental sols. $x > 0$.

This case includes:

$\boxed{r_1, r_2 \in \mathbb{R}}$ and $\begin{bmatrix} r_1 = \lambda + i\mu \\ r_2 = \lambda - i\mu \end{bmatrix} \in \mathbb{C}$

(2) $(\alpha - 1) - 4\beta = 0 \quad \Rightarrow \quad r_1 = r_2 = -\frac{(\alpha-1)}{2}$.

$$\boxed{Y_1 = x^{r_1}} \qquad x > 0.$$

- Find a solution Y_2 l.i. to Y_1.

- <u>Reduction</u> <u>of</u> <u>order</u> <u>method</u>.

we look for

$$\boxed{Y_2 = V(x) \; x^{r_1}}$$

$$\boxed{Y_2' = V' \, x^{r_1} + V \, r_1 \, x^{r_1 - 1}}$$

$$\boxed{\begin{aligned} Y_2' = {} & V'' \, x^{r_1} + 2 r_1 V' x^{r_1 - 1} \\ & + V \, r_1 (r_1 - 1) \, x^{r_1 - 2} \end{aligned}}$$

Introduce Y_2 into

$$x^2 Y_2'' + \alpha X Y_2' + \beta Y_2 = 0$$

$$0 = v'' x^{r_1+2} + 2r_1 v' \overline{x^{r_1+1}} + v\, r_1(r_1-1)\, \overline{x^{r_1}}$$

$$+ \alpha\left(v' \overline{x^{r_1+1}} + v\, r_1 \overline{x^{r_1}}\right)$$

$$+ \beta\, v\, \overline{x^{r_1}}$$

$$0 = v'' x^{r_1+2} + (2r_1+\alpha)\, v' x^{r_1+1}$$

$$+ \left(r_1(r_1-1) + \alpha r_1 + \beta\right) v\, x^{r_1}$$

Recall:

$$\boxed{r_1(r_1-1) + \alpha r_1 + \beta = 0} \qquad r_1 : \text{root of } \not{q}.$$

$$r_1 = -\frac{(\alpha_1-1)}{2} \implies \boxed{2r_1 + \alpha = 1}$$

r_1 the only root of $\not{q}$

$$0 = v'' x^{r_1+2} + v' x^{r_1+1}$$

$$\boxed{v'' x + v' = 0}$$

$$u = v' \quad \Rightarrow \quad x\,u' + u = 0$$

$$x\,u' = -u$$

$$\frac{u'}{u} = -\frac{1}{x}$$

$$\ln u = -\ln x = \ln\left(\frac{1}{x}\right)$$

$$\boxed{u = \frac{1}{x}} \qquad u = v'$$

$$v' = \frac{1}{x} \quad \Rightarrow \quad \boxed{v(x) = \ln(x)} \qquad x > 0$$

$$\boxed{y_1 = x^{r_1}} \; , \quad \boxed{y_2 = x^{r_1} \ln(x)} \qquad x > 0.$$

For $x<0$, introduce the change of variable:

$$\hat{x} = -x \qquad \Rightarrow \hat{x} > 0.$$

$$\boxed{x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0}$$

$\boxed{x = -\hat{x}}$, $\quad \frac{dy}{dx} = \frac{dy}{d\hat{x}} \frac{d\hat{x}}{dx} = -\frac{dy}{d\hat{x}}$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-\frac{dy}{d\hat{x}}\right)$$

$$= (-1)^2 \frac{d^2 y}{d\hat{x}^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{dy}{d\hat{x}}} \qquad \boxed{\frac{d^2 y}{dx^2} = \frac{d^2 y}{d\hat{x}^2}}$$

$$\boxed{\hat{x}^2 \frac{d^2 y}{d\hat{x}^2} + \alpha \hat{x} \frac{dy}{d\hat{x}} + \beta y = 0} \quad \hat{x} > 0$$

$$\boxed{y_1 = \hat{x}^{r_1}}, \quad \boxed{y_2 = \hat{x}^{r_2}} \qquad \boxed{\hat{x} = -x}$$

Summary: $\boxed{r_1 \neq r_2}$

$x > 0$: $y_1 = x^{r_1}$, $y_2 = x^{r_2}$

$x < 0$: $y_1 = (-x)^{r_1}$, $y_2 = (-x)^{r_2}$

$\boxed{r_1 = r_2}$

$x > 0$: $y_1 = x^{r_1}$ $y_2 = x^{r_1} \ln(x)$

$x < 0$: $y_1 = (-x)^{r_1}$ $y_2 = (-x)^{r_1} \ln(-x)$

Therefore:

$$\boxed{y_1 = |x|^{r_1} \quad , \quad y_2 = |x|^{r_2}} \quad , \quad \boxed{r_1 \neq r_2} \quad , \quad \boxed{x \neq 0.}$$

$$\boxed{y_1 = |x|^{r_1} \quad , \quad y_2 = |x|^{r_1} \ln(|x|)} \quad , \quad \boxed{r_1 = r_2} \quad , \quad \boxed{x \neq 0.}$$

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Remark: The case of complex roots

The Euler eq.

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad (6)$$

in the case that

$$f(r) = r(r-1) + \alpha r + \beta$$

has roots

$$r = -\frac{(\alpha-1)}{2} \pm \frac{1}{2}\sqrt{(\alpha-1)^2 - 4\beta}$$

with $\boxed{(\alpha-1)^2 - 4\beta < 0}$

Denote

$$\boxed{\mu = \sqrt{4\beta - (\alpha-1)^2}}$$

$$\boxed{\lambda = -\frac{(\alpha-1)}{2}}$$

The roots have the form

$$\boxed{r_1 = \lambda + i\mu} \qquad \boxed{r_2 = \lambda - i\mu}$$

- complex-valued fundamental sols. of (6)

$$\boxed{\tilde{y}_1 = |x|^{\lambda + i\mu}} \quad , \quad \boxed{\tilde{y}_2 = |x|^{\lambda - i\mu}}$$

Proposition: Real valued fundamental sols. of (6) in the case that q has complex roots are:

$$\boxed{y_1 = |x|^{\lambda} \cos\left[\mu \ln(|x|)\right]}$$

$$\boxed{y_2 = |x|^{\lambda} \sin\left[\mu \ln(|x|)\right]}$$

<u>Proof:</u> Given $\tilde{Y}_1$, $\tilde{Y}_2$, introduce:

$$\boxed{Y_1 = \frac{1}{2}\left(\tilde{Y}_1 + \tilde{Y}_2\right)}$$

$$\boxed{Y_2 = \frac{1}{2i}\left(\tilde{Y}_1 - \tilde{Y}_2\right)}$$

$$\tilde{Y}_1 = |x|^{\lambda + i\mu}$$

$$= |x|^{\lambda}\, |x|^{i\mu}$$

$$= |x|^{\lambda}\, e^{\ln\left(|x|^{i\mu}\right)}$$

$$= |x|^{\lambda}\, e^{i\mu \ln|x|} \qquad (\text{Euler eq.})$$

$$\boxed{\tilde{Y}_1 = |x|^{\lambda}\left[\cos(\mu \ln|x|) + i \sin(\mu \ln|x|)\right]}$$

$$\boxed{\tilde{Y}_2 = |x|^{\lambda}\left[\cos(\mu \ln|x|) - i \sin(\mu \ln|x|)\right]}$$

$$\boxed{Y_1 = |x|^{\lambda} \cos(\mu \ln|x|)}$$

$$\boxed{Y_2 = |x|^{\lambda} \sin(\mu \ln|x|)}$$

∎

Example: [Find real-valued fundamental sols. of

$$x^2 y'' - 3x y' + 13 y = 0$$]

Sol:

$$f(r) = r(r-1) - 3r + 13 = 0$$

$$r^2 - 4r + 13 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$$

$$\boxed{r_1 = 2 + 3i} \qquad \boxed{r_2 = 2 - 3i}$$

$$\boxed{\tilde{Y}_1 = |x|^{2+3i}} \qquad \boxed{\hat{Y}_2 = |x|^{2-3i}}$$

complex-Val. sols.

$$\boxed{\begin{array}{l} Y_1 = x^2 \cos(3 \ln|x|) \\ \hline Y_2 = x^2 \sin(3 \ln|x|) \end{array}}$$

Real-valued fundamental sols.