mth 235 L15

Plan:

* $P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \quad (1)$

* Review Power Series

* Solutions of (1) using power series

* The case of regular points.

* Examples.

(5.2)

Read Section (5.1).

* Review of power series

Def : The power series of a function $y: \mathbb{R} \to \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is given by

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n}$$

Examples

(1) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

$x_0 = 0 \qquad |x| < 1$

(2) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$x_0 = 0 \quad , \quad x \in \mathbb{R}.$

(3) [Taylor series centered at $x_0 \in \mathbb{R}$,

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x-x_0)^n .$$]

(4) [Find the power series of $y(x) = \sin(x)$ centered at $x_0 = 0$.]

Sol:

$y(x) = \sin(x)$ $\qquad$ $y(0) = 0$

$y'(x) = \cos(x)$ $\qquad$ $y'(0) = 1$

$y''(x) = -\sin(x)$ $\qquad$ $y''(0) = 0$

$y'''(x) = -\cos(x)$ $\qquad$ $y'''(0) = -1$

$\vdots$ $\qquad$ $\vdots$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

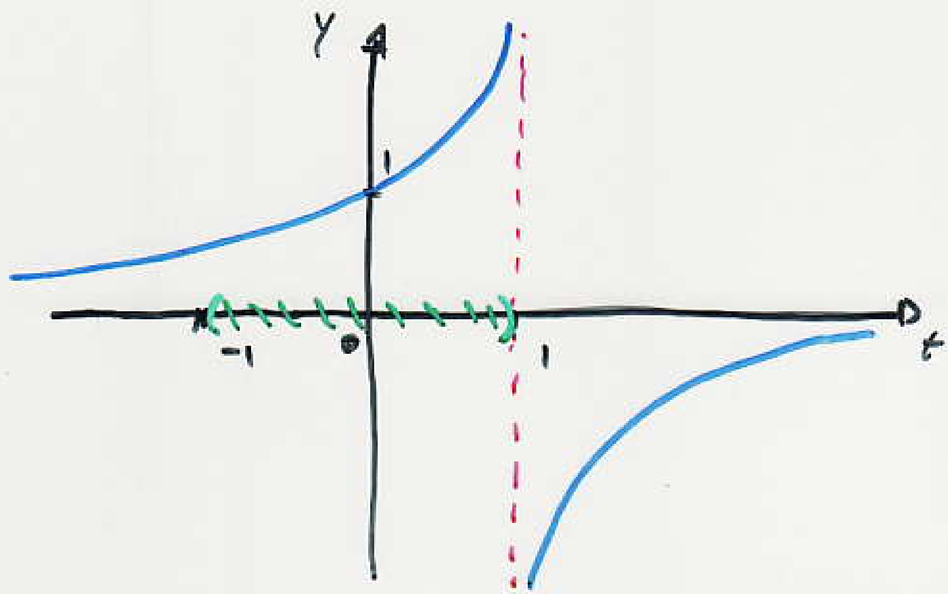
$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

(5) $$\boxed{\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

<u>Remark</u>: [The power series of a function y may not be defined on the whole domain of the function.]

<u>Example</u>

$y(x) = \frac{1}{1-x}$ defined on $D = \mathbb{R} - \{1\}$.



$y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$.

Def: The power series

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n}$$

converges absolutely iff

$$\boxed{\sum_{n=0}^{\infty} |a_n| \, |x-x_0|^n}$$

converges.

Example: The series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges but it does not converges absolutely, since

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

Def: The radius of convergence of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

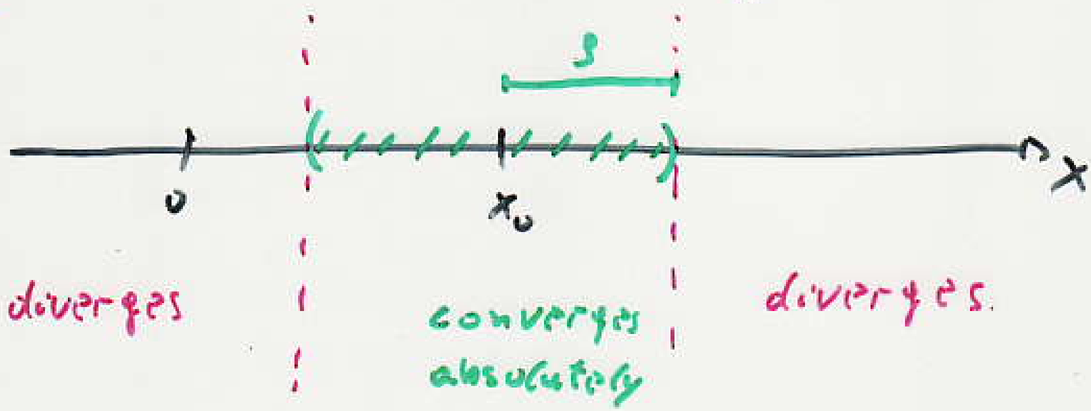
is the number $\rho \geq 0$ that satisfies **both**

(1) the series converges absolutely for

$$\boxed{|x - x_0| < \rho},$$

(2) the series diverges for

$$\boxed{|x - x_0| > \rho.}$$



Examples:

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has $\rho = 1$

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $\rho = \infty$.

Thm: (Ratio test) Given the power series

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n} \qquad (1)$$

introduce the number

$$\boxed{L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} .$$

Then, for all $x \in \mathbb{R}$ such that

(a) $|x-x_0| L < 1$ the series (1) converges;

(b) $|x-x_0| L > 1$ the series (1) diverges;

(c) $|x-x_0| L = 1$ inconclusive.

For $L \neq 0$ the radius of convergence of (1) is

$$\boxed{\rho = \frac{1}{L}}$$

Remark : on sumation indices.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + \cdots$$

$$= \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

$$= \sum_{j=2}^{\infty} a_{j-2} (x-x_0)^{j-2}$$

$$= \sum_{m=-3}^{\infty} a_{m+3} (x-x_0)^{m+3}$$

$$y'(x) = \sum_{n=0}^{\infty} n\, a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + \cdots$$

$$= \sum_{n=1}^{\infty} n\, a_n (x-x_0)^{n-1}$$

$$\boxed{\begin{array}{c} n-1 = m \\ n = m+1 \end{array}}$$

$$= \sum_{m=0}^{\infty} (m+1)\, a_{m+1} (x-x_0)^m$$

* Regular points equations (5.2)

We look for solutions $y(x)$ of

$$\boxed{\mathbb{P}(x)\, y'' + Q(x)\, y' + R(x)\, y = 0} \qquad (1)$$

around $x_0 \in \mathbb{R}$ where $\mathbb{P}(x_0) \neq 0$, using a power series representation of the solution centered at x_0, that is

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n .}$$

Def : The point $x_0 \in \mathbb{R}$ is called a regular point of eq. (1) iff

$$\boxed{\mathbb{P}(x_0) \neq 0 .}$$

The point x_0 is called a singular point of eq (1) iff

$$\boxed{\mathbb{P}(x_0) = 0 .}$$

Remark : - Eq (1) is second order near a regular point.

- The order of Eq. (1) does NOT change near a regular point.

* Summary of the Power Series Method.

(1) If x_0 is a regular point, propose

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^n . \qquad (2)$$

(2) Introduce (2) into eq. (1) and find a recurrence relation among the coefficients a_n

(3) Solve the recurrence relation in terms of free coefficients.

(4) If possible, add up the resulting series for $y(x)$.

<u>Example</u>: Find the general solution of

$$\boxed{y' + c\,y = 0}$$

using a power series around $x_0 = 0$.

<u>Sol</u>:

(We know the solution: $y(x) = a_0\, e^{-cx}$.)

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n\, x^n}\ , \qquad \boxed{x_0 = 0}$$

$$y'(x) = \sum_{n=0}^{\infty} n\, a_n\, x^{n-1} = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$$

$$n-1 = m \quad , \quad n = m+1$$

$$y'(x) = \sum_{m=0}^{\infty} (m+1)\, a_{m+1}\, x^m$$

$$\boxed{y'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n}$$

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n + \sum_{n=0}^{\infty} c\, a_n\, x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)\, a_{n+1} + c\, a_n \right] x^n = 0$$

$$\boxed{(n+1)\, a_{n+1} + c\, a_n = 0} \quad \boxed{n \geq 0.}$$

Recurrence Relation

$$\boxed{a_{n+1} = \frac{-c}{(n+1)}\, a_n}$$

$n=0$ $\qquad \boxed{a_1 = -c\, a_0}$

$n=1$ $\qquad a_2 = -\frac{c}{2}\, a_1 \quad \Rightarrow \quad \boxed{a_2 = \frac{c^2}{2}\, a_0}$

$n=2$ $\qquad a_3 = -\frac{c}{3}\, a_2 \quad \Rightarrow \quad \boxed{a_3 = -\frac{c^3}{(3)(2)}\, a_0}$

$\vdots$

n $\qquad \boxed{a_n = \frac{(-c)^n}{n!}\, a_0}$

Solved the recurrence relation.

$$\boxed{y(x) = \left[\sum_{n=0}^{\infty} \frac{(-c)^n}{n!}\, x^n \right] a_0}$$

power series

$$\boxed{y(x) = a_0\, e^{-cx}}$$

We add up the power series.

Example : Find the general sol. of
$$y'' + y = 0$$
using power series centered at $x_0 = 0$

Sol:

(we know the solution is
$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$
)

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\boxed{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0}$$

$$\sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}$$

$m = n-2$

$n = m+2$

$$= \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2}\, x^{m}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + a_n \right] x^n = 0$$

$$\boxed{(n+2)(n+1)\, a_{n+2} + a_n = 0} \qquad n \geq 0.$$

Recurrence Relation.

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)}} \qquad n \geq 0$$

$$\boxed{a_2 = -\frac{a_0}{2}} \qquad \boxed{a_3 = -\frac{a_1}{(3)(2)}}$$

$$a_4 = -\frac{a_2}{(4)(3)} \qquad a_5 = -\frac{a_3}{(5)(4)}$$

$$= \frac{a_0}{4!} \qquad = \frac{a_1}{5!}$$

$$\boxed{a_{2k} = \frac{(-1)^k}{(2k)!} a_0} \qquad \boxed{a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1}$$

We solved the recurrence relation.

a_0, a_1 arbitrary, not fixed by the recurrence relation.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \underbrace{\sum_{k=0}^{\infty} a_{2k} x^{2k}}_{\text{even powers}} + \underbrace{\sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}}_{\text{odd powers}}$$

$$\boxed{\begin{aligned} y(x) = &\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right] a_0 \\ &+ \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\right] a_1 \end{aligned}}$$

$$\boxed{y(x) = a_0 \cos(x) + a_1 \sin(x).}$$

<u>Recall:</u>

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

(Taylor expansions.)

<u>Example</u>: Find the first 4 terms of the power series sol. of

$$\boxed{y'' - x\, y = 0}. \qquad (1)$$

centered at $\boxed{x_0 = 2.}$

<u>Sol.</u>

(1) is a variable coefficients equation.

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n\, (x-2)^n}$$

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n\, (x-2)^{n-1}$$

$$\boxed{y''(x) = \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, (x-2)^{n-2}}$$

$$x\, y(x) = x \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$= \sum_{n=0}^{\infty} a_n\, x\, (x-2)^n$$

$$= \sum_{n=0}^{\infty} a_n \left[(x-2) + 2 \right] (x-2)^n$$

$$x\,y(x) = \sum_{n=0}^{\infty} a_n (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n .$$

$$0 = y'' - x\,y$$

$$0 = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^{n+1}$$

$$- \sum_{n=0}^{\infty} 2\, a_n\, (x-2)^n$$

$(n-2 = m)$ $(n+1 = m)$ $(n = m)$

$$0 = \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} (x-2)^m - \sum_{m=1}^{\infty} a_{m-1} (x-2)^m$$

$$- \sum_{m=0}^{\infty} 2\, a_m\, (x-2)^m$$

$(m = n)$

$$\boxed{\begin{aligned} 0 = {} & 2\,a_2 - 2\,a_0 \\ & + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1} - 2a_n \right] (x-2)^n \end{aligned}}$$

$$\boxed{\begin{aligned} & a_2 - a_0 = 0 \\ & (n+2)(n+1)\, a_{n+2} - 2a_n - a_{n-1} = 0 \end{aligned}} \qquad n \geq 1$$

Recurrence Relation.

$$\boxed{a_2 = a_0} \qquad a_{n+2} = \frac{2a_n + a_{n-1}}{(n+2)(n+1)}$$

$$\boxed{a_3 = \frac{2a_1 + a_0}{(3)(2)}}$$

$$a_4 = \frac{2a_2 + a_1}{(4)(3)} \Rightarrow \boxed{a_4 = \frac{2a_0 + a_1}{(4)(3)}}$$

$$y(x) = a_0 + a_1 (x-2) + a_2 (x-2)^2 + a_3 (x-2)^3 + \cdots$$

$$y(x) = a_0 + a_1 (x-2)^2 + a_0 (x-2)^2 + \frac{(2a_1 + a_0)}{6} (x-2)^3$$

$$+ \cdots$$

$$\boxed{\begin{aligned} y(x) = \; & a_0 \left[1 + (x-2)^2 + \tfrac{1}{6}(x-2)^3 + \cdots \right] \\ & + a_1 \left[(x-2)^2 + \tfrac{1}{3}(x-2)^3 + \cdots \right] \end{aligned}}$$