mth 235 L14

Plan: * Review: Non-homogeneous eqs.

* Application: Vibrations and resonance.

(3.5)
(3.6)

* Review: Non-homogeneous eqs.

Problem: [Find the general sol. of

$$y'' + p(t)\, y' + q(t)\, y = g(t)$$]

Recall

$$\begin{bmatrix} \text{General Sol.} \\ \text{Non-homog.} \\ \text{eq.} \end{bmatrix} = \begin{bmatrix} \text{General sol.} \\ \text{Homogeneous} \\ \text{eq.} \end{bmatrix} + \begin{bmatrix} \text{Particular sol.} \\ \text{Non-Homog.} \\ \text{eq.} \end{bmatrix}$$

- undetermined coeff. (guess) { - constant coeff. and - g(t) in table.

- Variation of parameters { - variable coeff. or - g(t) not in table.

Example Verify that $y_1 = e^t$, $y_2 = t$ satisfy

$$\boxed{(1-t)\, y'' + t\, y' - y = 0},$$

and then find a particular sol. of

$$\boxed{(1-t)\, y'' + t\, y' - y = 2(t-1)^2 e^{-t}.}$$

Sol.

$$y_1 = e^t, \quad y_1' = e^t, \quad y_1'' = e^t$$

$$\big((1-t) + t - 1\big) e^t = (1 - t + t - 1) e^t \underline{= 0} \checkmark$$

$$y_2 = t, \quad y_2' = 1, \quad y_2'' = 0$$

$$(1-t)\, 0 + t - t \underline{= 0} \checkmark$$

- The eq. has variable coeff. $\therefore$

Variation of parameters.

$$y_1 = e^t \quad , \quad y_2 = t$$

$$W_{y_1,y_2} = \begin{vmatrix} e^t & e^t \\ t & 1 \end{vmatrix} = e^t - t\,e^t$$

$$\boxed{W_{y_1,y_2} = -(t-1)\,e^t}$$

$$g(t) \neq 2\,(t-1)^2\,e^{-t}$$

$$(1-t)\,y'' + t\,y' - y = 2\,(t-1)^2\,e^{-t}$$

$$y'' + \frac{t}{(1-t)}\,y' - \frac{1}{(1-t)} = \frac{2\,(t-1)^2}{-(t-1)}\,e^{-t}$$

$$\boxed{g(t) = -\,2\,(t-1)\,e^{-t}}$$

$$u_1' = -\frac{y_2 g}{W_{y_1 y_2}} = -\frac{t\left[-2(t-1)e^{-t}\right]}{-(t-1)e^{t}}$$

$$\boxed{u_1' = -2t\, e^{-2t}}$$

$$u_1 = -2\int t\, e^{-2t}\, dt \qquad f = t \quad g' = e^{-2t}$$

$$f' = 1 \quad g = -\frac{e^{-2t}}{2}$$

$$= -2\left[-\frac{t}{2}e^{-2t} - \int\left(-\frac{1}{2}\right)e^{-2t}\,dt\right]$$

$$= -2\left[-\frac{t\,e^{-2t}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2t}\right]$$

$$= t\,e^{-2t} + \frac{e^{-2t}}{2}$$

$$\boxed{u_1 = \left(t+\frac{1}{2}\right)e^{-2t}}$$

$$u_2' = \frac{y_1 g}{W_{y_1 y_2}} = \frac{e^t \left[-2(t-1) e^{-t} \right]}{-(t-1) e^t}$$

$$\boxed{u_2' = 2 e^{-t}}$$

$$u_2 = \int 2 e^{-t} \, dt \quad \Rightarrow \quad \boxed{u_2 = -2 e^{-t}}$$

$$y_p = u_1 y_1 + u_2 y_2$$

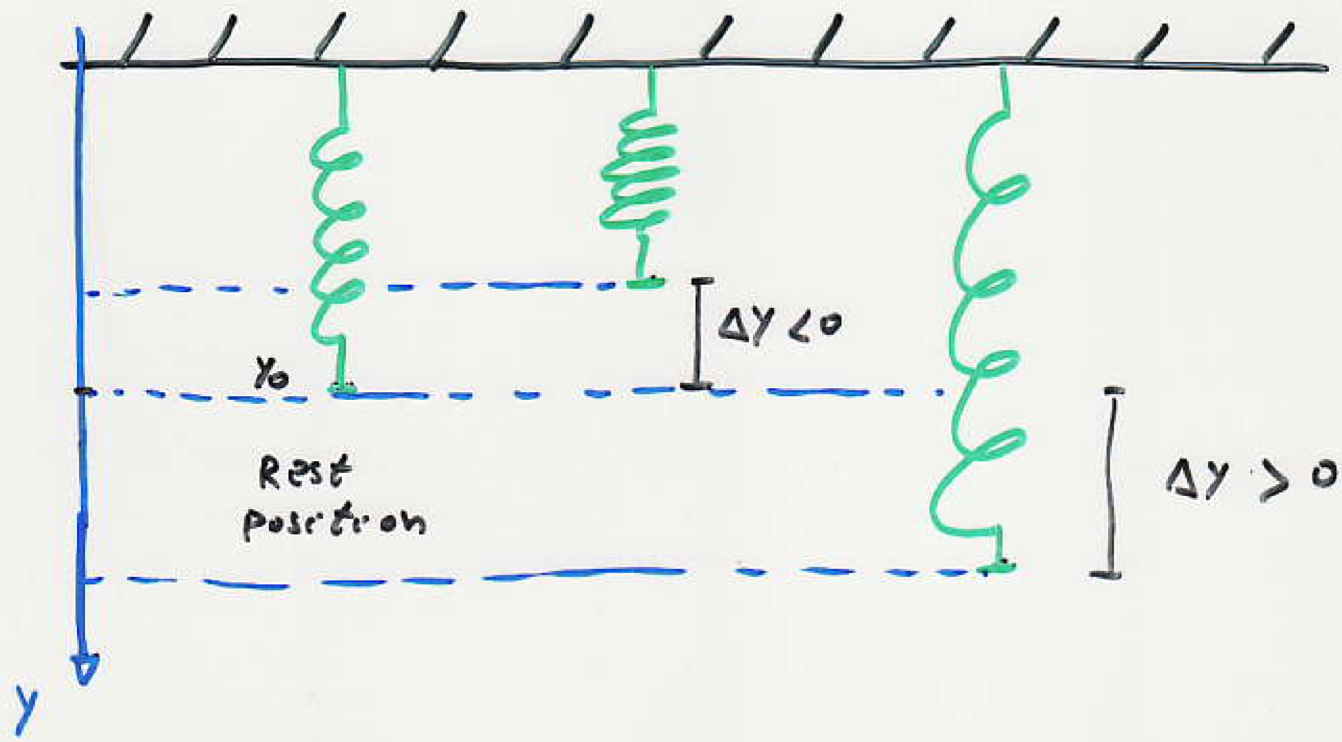
$$= \left(t + \tfrac{1}{2}\right) e^{-2t} \, e^t + \left(-2 e^{-t}\right) t$$

$$= \left[t + \tfrac{1}{2} - 2t \right] e^{-t}$$

$$\boxed{y_p = \left(-t + \tfrac{1}{2}\right) e^{-t}}$$

* Application : Vibration and Resonance.

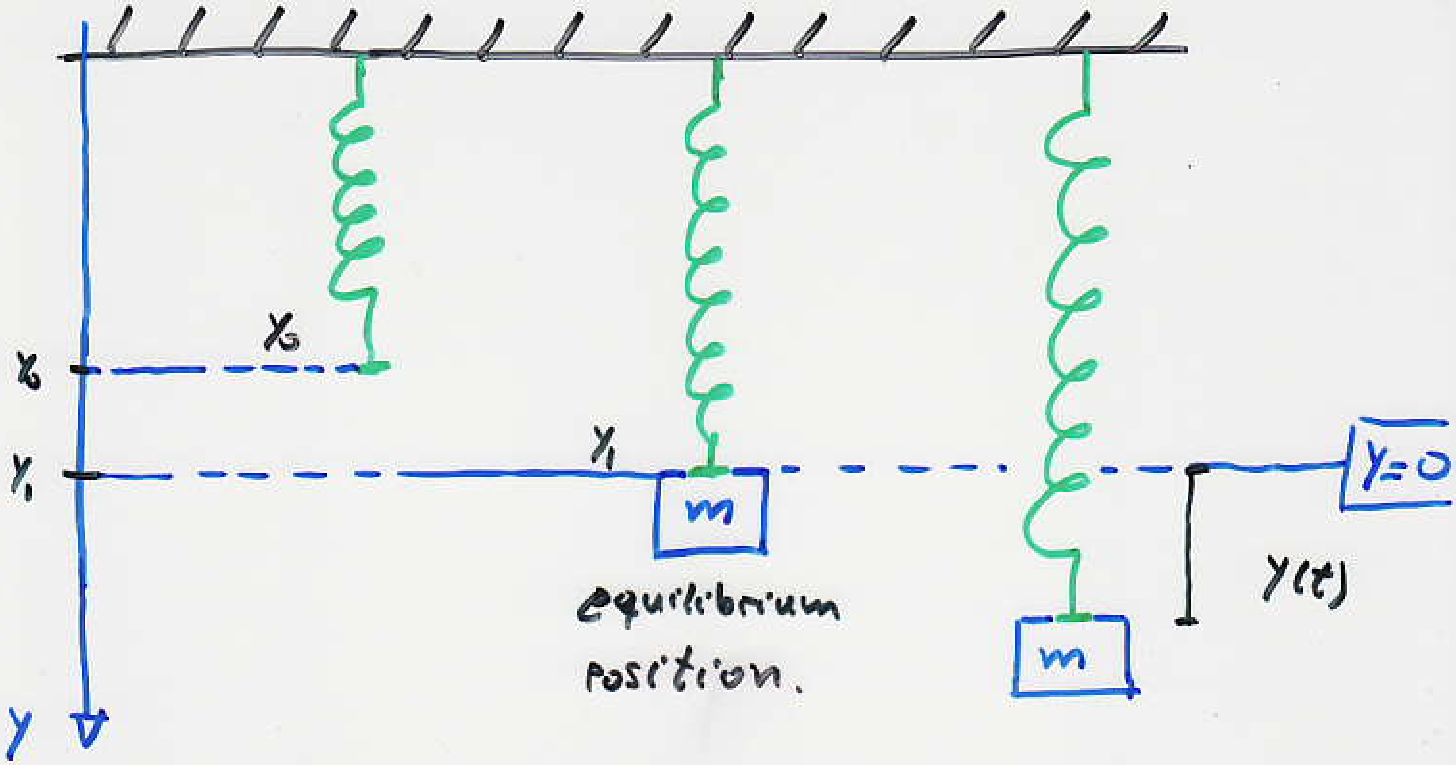
Problem : [Describe the motion of an object suspended from the end of a hanging spring.]



F_R : Restoring force from the spring.

$$\boxed{F_R = -k\,\Delta y} \qquad k > 0$$

k: spring constant.



At the equilibrium position:

$$\boxed{m g - k(y_1 - y_0) = 0}$$

$$\boxed{y_1 = \frac{m g}{k} + y_0}$$

The motion of the object is described by Newton's laws

$$\boxed{m \, y''(t) = -k \, y(t)}$$

$$y'' + \frac{k}{m} y = 0 \quad , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\boxed{y'' + \omega_0^2 \, y = 0} \quad \text{unforced vibration.}$$

$$\boxed{y'' + \omega_0^2 \, y = f(t)} \quad \text{forced vibration.}$$

<u>Notation</u>: ω_0 is called the natural frequency of the system.

<u>Example</u> [Given w_0, w_1, with $\boxed{w_0 \neq w_1}$,

find y sol. to IVP

$$\boxed{y'' + w_0^2\, y = \cos(w_1 t)}$$

$\boxed{y(0) = 0}$, $\boxed{y'(0) = 0}$]

<u>sol.</u>

constant coeff. eq. with source function

$$\boxed{g(t) = \cos(w_1 t).}$$

Undetermined coeff. method. (guess.)

First. Find fundamental sols. of the homogeneous eq.

$$p(r) = r^2 + w_0^2 = 0$$

$\boxed{r_1 = w_0\, i}$ $\boxed{r_2 = -w_0\, i}$

$$\boxed{\begin{array}{l} y_1(t) = \cos(w_0 t) \\ \hline y_2(t) = \sin(w_0 t) \end{array}}$$

fundamental sols.

(real valued)

Second : proposed particular sol. of the non-homogeneous eq.

$$\boxed{y_p = k_1 \cos(\omega_1 t) + k_2 \sin(\omega_1 t)}$$

Since $\omega_1 \neq \omega_0$, y_p is **not** sol. of homogeneous eq.

Third : Find k_1 , k_2 .

$$\boxed{y_p' = -k_1 \omega_1 \sin(\omega_1 t) + k_2 \omega_1 \cos(\omega_1 t)}$$

$$\boxed{y_p'' = -k_1 \omega_1^2 \cos(\omega_1 t) - k_2 \omega_1^2 \sin(\omega_1 t)}$$

$$\left[-k_1 \omega_1^2 \cos(\omega_1 t) - k_2 \omega_1^2 \sin(\omega_1 t)\right]$$

$$+ \omega_0^2 \left[k_1 \cos(\omega_1 t) + k_2 \sin(\omega_1 t)\right] = \cos(\omega_1 t)$$

$$k_1 (\omega_0^2 - \omega_1^2) \cos(\omega_1 t)$$

$$+ k_2 (\omega_0^2 - \omega_1^2) \sin(\omega_1 t) = \cos(\omega_1 t)$$

$$t=0 \implies k_1 (\omega_0^2 - \omega_1^2) = 1$$

$$t = \frac{\pi}{2}\frac{1}{\omega_1} \implies k_2 (\omega_0^2 - \omega_1^2) = 0$$

$$\omega_0 \neq \omega_1 \implies \boxed{k_2 = 0}$$

$$\boxed{k_1 = \frac{1}{(\omega_0^2 - \omega_1^2)}}$$

$$\boxed{y_p(t) = \frac{1}{(\omega_0^2 - \omega_1^2)} \cos(\omega_1 t)}$$

General sol. of non-homog. eq.

$$\boxed{y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{1}{(\omega_0^2 - \omega_1^2)} \cos(\omega_1 t)}$$

We are solving an IVP.

$$\boxed{y'(t) = -c_1 w_0 \sin(w_0 t) + c_2 w_0 \cos(w_0 t) - \frac{w_1}{(w_0^2 - w_1^2)} \sin(w_1 t)}$$

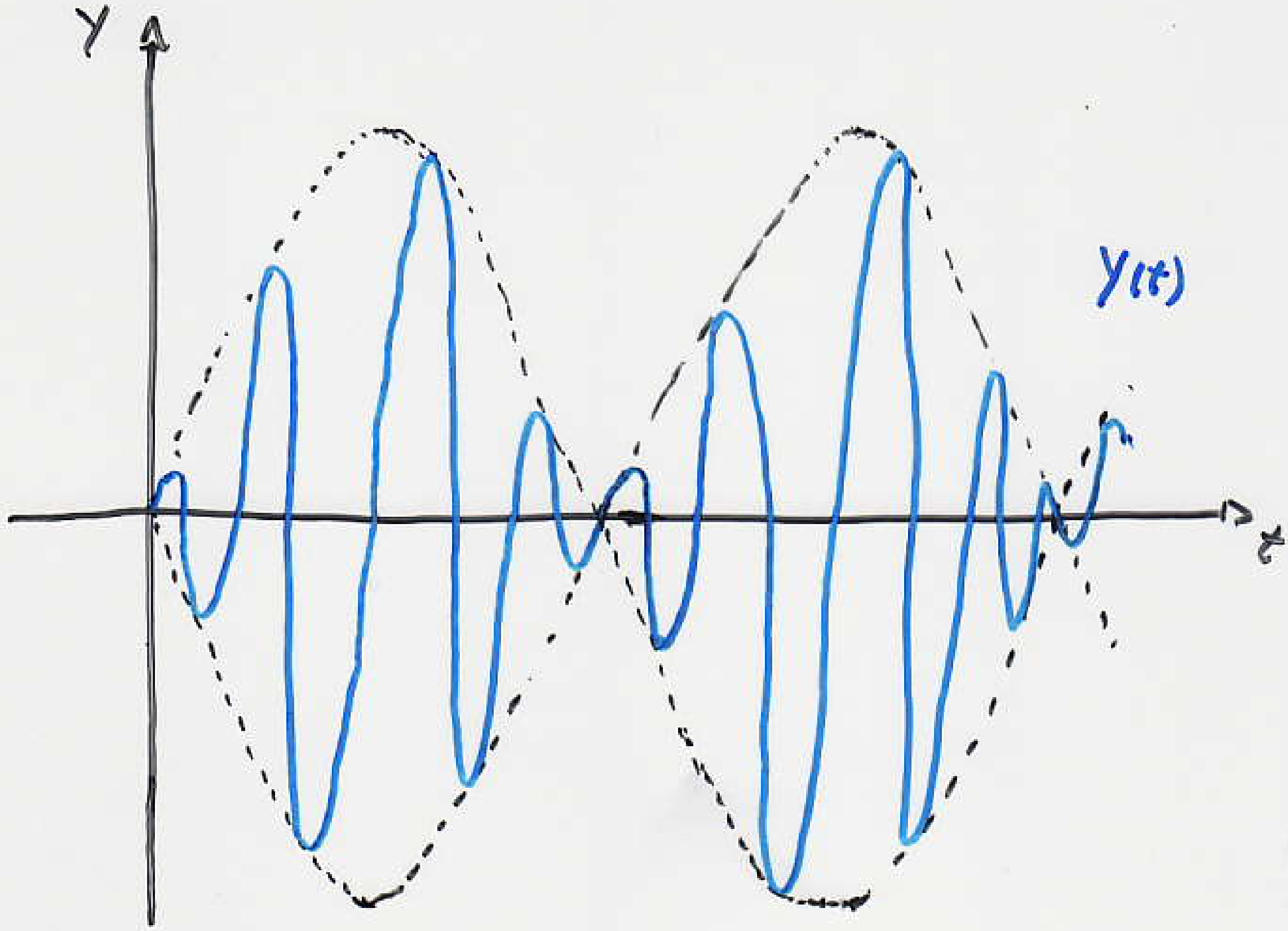
Initial conditions.

$$0 = y(0) = c_1 + \frac{1}{(w_0^2 - w_1^2)}$$

$$0 = y'(0) = c_2 w_0$$

Therefore: $\boxed{c_1 = -\frac{1}{(w_0^2 - w_1^2)}}$ $\boxed{c_2 = 0}$.

$$\boxed{y(t) = \frac{1}{(w_0^2 - w_1^2)} \left[\cos(w_1 t) - \cos(w_0 t) \right]}$$


Y
y(t)
t

<u>Example</u>: Given ω_0, ω_1, with $\boxed{\omega_0 = \omega_1}$, find y sol. of IVP

$$\boxed{y'' + \omega_0^2\, y = \cos(\omega_1 t)}$$

$$\boxed{y(0) = 0} \qquad \boxed{y'(0) = 0.}$$

<u>Sol</u>.

The eq. is: $\quad y'' + \omega_0^2\, y = \cos(\omega_0 t)$

Fundamental sols. of homogeneous eq.

$$\boxed{\begin{array}{l} y_1 = \cos(\omega_0 t) \\ \hline y_2 = \sin(\omega_0 t) \end{array}}$$

proposed $\boxed{\tilde{y}_p = k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)}$

is the wrong guess, since $\tilde{y}_p$ is solution of the homogeneous eq.

Modify the guess:

$$\boxed{y_p = t\left[k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)\right]}$$

we need to compute y_p', y_p''.

$$\boxed{\begin{aligned} y_p' &= \left[k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)\right] \\ &\quad + t\left[-k_1 \omega_0 \sin(\omega_0 t) + k_2 \omega_0 \cos(\omega_0 t)\right] \end{aligned}}$$

$$\begin{aligned} y_p'' &= \left[-k_1 \omega_0 \sin(\omega_0 t) + k_2 \omega_0 \cos(\omega_0 t)\right] \\ &\quad + \left[-k_1 \omega_0 \sin(\omega_0 t) + k_2 \omega_0 \cos(\omega_0 t)\right] \\ &\quad + t\left[-k_1 \omega_0^2 \cos(\omega_0 t) - k_2 \omega_0^2 \sin(\omega_0 t)\right] \end{aligned}$$

$$\boxed{\begin{aligned} y_p'' &= \left[-2 k_1 \omega_0 \sin(\omega_0 t) + 2 k_2 \omega_0 \cos(\omega_0 t)\right] \\ &\quad - \omega_0^2 t\left[k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)\right] \end{aligned}}$$

Introduce y_p into $y_p'' + \omega_0^2 y_p = \cos(\omega_0 t)$

$$\left[-2 k_1 \omega_0 \sin(\omega_0 t) + 2 k_2 \omega_0 \cos(\omega_0 t)\right]$$

$$- \omega_0^2 t \left[k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)\right]$$

$$+ \omega_0^2 t \left[k_1 \cos(\omega_0 t) + k_1 \sin(\omega_0 t)\right]$$

$$= \cos(\omega_0 t)$$

$$\boxed{-2 k_1 \omega_0 \sin(\omega_0 t) + 2 k_2 \omega_0 \cos(\omega_0 t) = \cos(\omega_0 t)}$$

$t = 0 \Rightarrow 2 k_2 \omega_0 = 1 \Rightarrow \boxed{k_2 = \frac{1}{2\omega_0}}$

$t = \frac{\pi}{2} \frac{1}{\omega_0} \Rightarrow -2 k_1 \omega_0 = 0 \Rightarrow \boxed{k_1 = 0}$

$$\boxed{y_p = \frac{t}{2\omega_0} \sin(\omega_0 t)}$$

The general sol. is

$$\boxed{y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{t}{2\omega_0} \sin(\omega_0 t).}$$

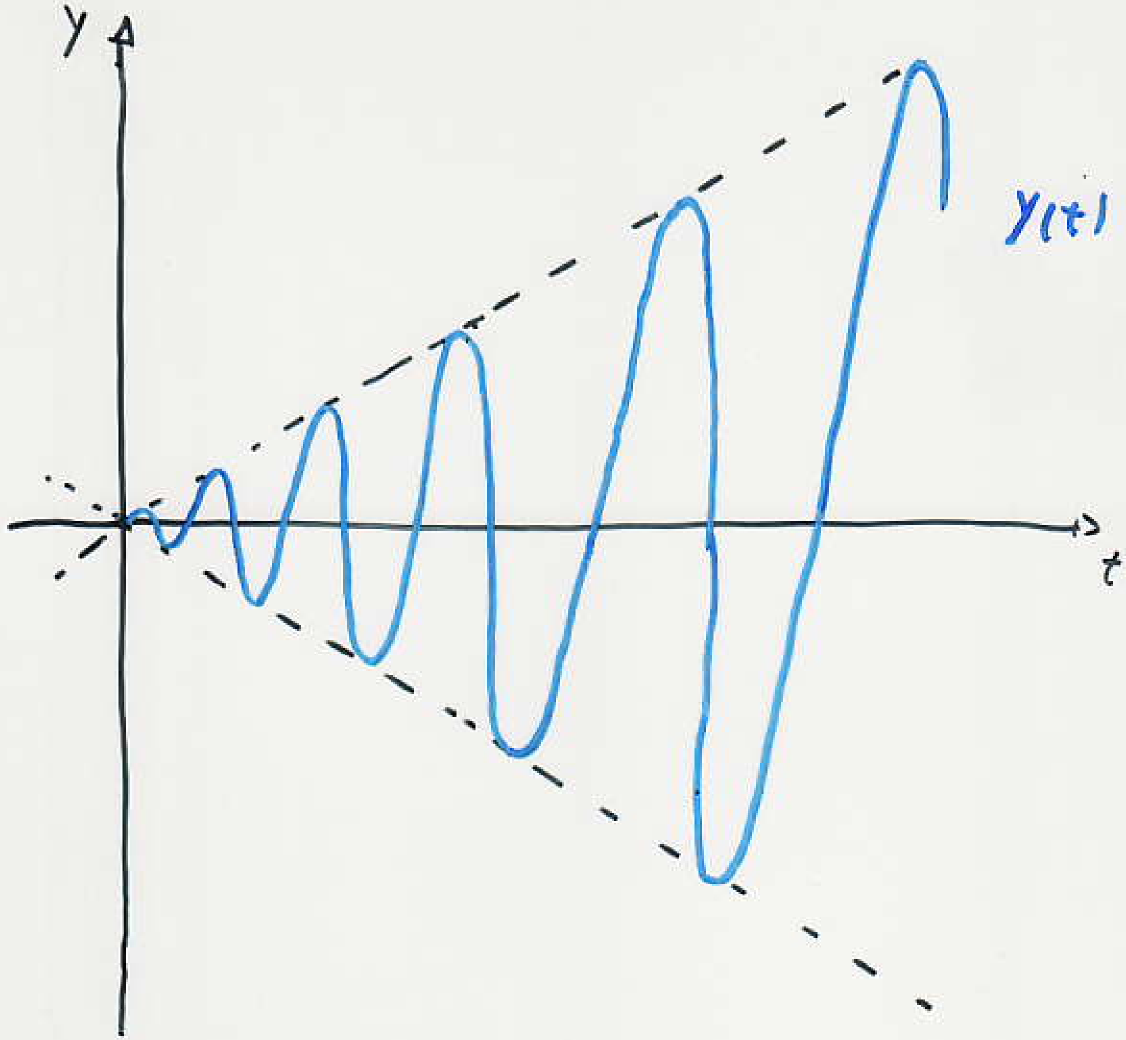
$$y'(t) = -c_1 \omega_0 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega_0 t) + \frac{1}{2\omega_0} \sin(\omega_0 t) + \frac{t}{2} \cos(\omega_0 t).$$

The initial conditions are:

$$0 = y(0) = c_1 \quad \Rightarrow \quad \boxed{c_1 = 0}$$

$$0 = y'(0) = c_2 \omega_0 \quad \Rightarrow \quad \boxed{c_2 = 0}$$

$$\boxed{y(t) = \frac{t}{2\omega_0} \sin(\omega_0 t)}$$



Resonance: $\omega_0 = \omega_1$

The oscillation amplitude grows in time until the physical system breaks down.