mth 235 L 13

Plan:

* $y'' + p(t)\,y' + q(t)\,y = g(t)$
* Method of variation of parameters
* Examples.

(3.6)

* Variation of parameters

- This is a general method to find solutions to non-homogeneous eqs.

$$y'' + p(t)\, y' + q(t)\, y = g(t).$$

(Non-homogeneous, variable coefficients, function g continuous but otherwise arbitrary.)

- The method can be applied to more general equations than the undetermined coefficients method.

Thrm: Let $p, q, g : (t_1, t_2) \to \mathbb{R}$ be continuous functions and let y_1, y_2 be fundamental solutions of the homogeneous eq.

$$\boxed{y'' + p(t)\, y' + q(t)\, y = 0.}$$

Introdue the functions

$$\boxed{u_1(t) = -\int \frac{y_2\, g}{W_{y_1 y_2}}\, dt}$$

$$\boxed{u_2(t) = \int \frac{y_1\, g}{W_{y_1 y_2}}\, dt\,.}$$

Then, the function

$$\boxed{y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)}$$

is solution of the non-homogeneous eq.

$$\boxed{y'' + p(t)\, y' + q(t)\, y = g(t)\,.} \quad (1)$$

Furthermore, the general sol. of (1) is

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + y_p(t)\,,$$

with $c_1, c_2 \in \mathbb{R}$.

<u>Example.</u> [Find the general sol. of

$$y'' - 5y' + 6y = 2e^t.$$]

<u>Sol:</u>

<u>Notice</u>: - Undetermined coefficients method could be used.

- Now we use variation of parameters.

First: we need y_1, y_2 fundamental sols. of

$$\boxed{y'' - 5y' + 6y = 0.}$$

$$P(r) = r^2 - 5r + 6 = 0$$

$$r = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

$$\boxed{r_1 = 3}, \quad \boxed{r_2 = 2}$$

$$\boxed{y_1(t) = e^{3t}} \qquad \boxed{y_2(t) = e^{2t}}$$

Second: We need the Wronskian:

$$W_{y_1 y_2} = \begin{vmatrix} e^{3t} & 3e^{3t} \\ e^{2t} & 2e^{2t} \end{vmatrix}$$

$$= 2e^{5t} - 3e^{5t}$$

$$\boxed{W_{y_1 y_2} = -e^{5t}}$$

Recall the source function $\boxed{g(t) = 2e^{t}}$

Third: compute functions u_1, u_2.

Notice that:

$$u_1' = -\frac{y_2\, g}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1\, g}{W_{y_1 y_2}}$$

$$u_1' = -\frac{y_2\, g}{W_{y_1, y_2}} = -\frac{e^{2t} \quad 2e^{t}}{(-e^{5t})}$$

$$u_1' = 2\, e^{(3t-5t)} \Rightarrow \boxed{u_1' = 2\, e^{-2t}}$$

$$u_2' = \frac{y_1\, g}{W_{y_1, y_2}} = \frac{e^{3t} \quad 2e^{t}}{(-e^{5t})}$$

$$u_2' = -2\, e^{(4t-5t)} \Rightarrow \boxed{u_2' = -2\, e^{-t}}$$

Therefore

$$u_1 = \int 2\, e^{-2t}\, dt \Rightarrow \boxed{u_1 = -\, e^{-2t}}$$

$$u_2 = \int -2\, e^{-t}\, dt \Rightarrow \boxed{u_2 = 2\, e^{-t}}$$

Fourth: compute y_p.

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

$$= (-e^{-2t})\, e^{3t} + (2e^{-t})\, e^{2t}$$

$$= -e^{t} + 2e^{t}$$

$$\boxed{y_p(t) = e^{t}}$$

(compare with guessing method.)

The fundamental solution is:

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{2t} + e^{t}}$$

Proof of Thrm:

y_1, y_2 satisfy the eqs.

$$y_1'' + p(t)\, y_1' + q(t)\, y_1 = 0$$

$$y_2'' + p(t)\, y_2' + q(t)\, y_2 = 0$$

Introduce the function

$$\boxed{y_p = u_1\, y_1 + u_2\, y_2}$$

where u_1, u_2 are functions to be computed later on.

compute

$$\boxed{y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'}$$

$$\boxed{\begin{aligned} y_p'' = {} & u_1'' y_1 + 2 u_1' y_1' + u_1 y_1'' \\ & + u_2'' y_2 + 2 u_2' y_2' + u_2 y_2'' \end{aligned}}$$

y_p, y_p', y_p'' into the eq. $\boxed{y_p'' + p\,y_p' + q\,y_p = g}$

and then find the eqs. for u_1, u_2.

$$
\begin{aligned}
g = \big(& u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' \\
& + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' \big) \\
& + p\,(u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2') \\
& + q\,(u_1 y_1 + u_2 y_2)
\end{aligned}
$$

$$
\begin{aligned}
g = \; & u_1 (y_1'' + p\,y_1' + q\,y_1) \\
& + u_2 (y_2'' + p\,y_2' + q\,y_2) \\
& + u_1'' y_1 + u_2'' y_2 + 2 (u_1' y_1' + u_2' y_2') \\
& + p\,(u_1' y_1 + u_2' y_2)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
g = \; & u_1'' y_1 + u_2'' y_2 + 2 (u_1' y_1' + u_2' y_2') \\
& + p\,(u_1' y_1 + u_2' y_2) \qquad (2)
\end{aligned}
}
$$

<u>Key</u> <u>idea</u>: [Look f $u_1(t)$, $u_2(t)$ sol. of

$$\boxed{u_1' y_1 + u_2' y_2 = 0} \qquad (3) \;]$$

- So, the third term in (2) vanishes for such u_1, u_2.

- Any solution of (3) must satisfy

$$0 = (u_1' y_1 + u_2' y_2)'$$

that is

$$\boxed{0 = u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2'} \qquad (4)$$

Eqs. (2) and (4) imply

$$\boxed{u_1' y_1' + u_2' y_2' = g} \qquad (5)$$

[If u_1, u_2 solve (3) and (5), then y_p is solution of

$$\boxed{y_p'' + p\, y_p' + q\, y_p = g}$$
]

Find $u_1(t)$, $u_2(t)$ solution of

$$\boxed{\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= g. \end{aligned}}$$

(3)

(5)

Algebraic equations for u_1', u_2'.

$$\boxed{u_2' = -\frac{y_1}{y_2} u_1'}$$

introduce it in (5).

$$y_1' u_1' + y_2' \left(-\frac{y_1}{y_2}\right) u_1' = g$$

$$\frac{\left(y_1' y_2 - y_1 y_2'\right)}{y_2} u_1' = g$$

$$-W_{y_1 y_2} \frac{u_1'}{y_2} = g$$

$$\boxed{u_1' = -\frac{y_2\, g}{W_{y_1,y_2}}} \quad , \qquad u_2' = -\frac{y_1}{y_2}\, u_1'$$

$$\boxed{u_2' = \frac{y_1\, g}{W_{y_1,y_2}}}$$

Therefore:

$$\boxed{u_1 = \int -\frac{y_2\, g}{W_{y_1 y_2}}\, dt} \quad , \quad \boxed{u_2 = \int \frac{y_1\, g}{W_{y_1 y_2}}\, dt.}$$

and $\qquad y_p = u_1\, y_1 + u_2\, y_2$

satisfies

$$y_p'' + p\, y_p' + q\, y_p = g. \qquad \blacksquare$$

<u>Example</u>

Find a particular solution of

$$\boxed{t^2 y'' - 2y = 3t^2 - 1}$$

knowing that

$\boxed{y_1 = t^2}$, $\boxed{y_2 = \frac{1}{t}}$

are fundamental sols. of

$$\boxed{t^2 y'' - 2y = 0.}$$

<u>Sol:</u>

Find $W_{y_1 y_2} = \begin{vmatrix} t^2 & 2t \\ \frac{1}{t} & -\frac{1}{t^2} \end{vmatrix}$

$$= t^2\left(-\frac{1}{t^2}\right) - \frac{1}{t}(2t)$$

$$= -1 - 2$$

$$\boxed{W_{y_1 y_2}(t) = -3}$$

Notice: $g(t) \neq 3t^2 - 1$

$$t^2 y'' - 2y = 3t^2 - 1$$

$$\boxed{y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}}$$

$$\boxed{g(t) = 3 - \frac{1}{t^2}}$$

Find u_1, u_2 knowing:

$y_1 = t^2$ $\quad y_2 = \frac{1}{t}$, $\quad W_{y_1 y_2} = -3$ $\quad$, $g = 3 - \frac{1}{t^2}$

$$u_1' = -\frac{y_2 \, g}{W_{y_1 y_2}} = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{(-3)}$$

$$\boxed{u_1' = t^{-1} - \frac{t^{-3}}{3}}$$

$$\boxed{u_1 = \ln(t) + \frac{1}{6}\; t^{-2}}$$

$$u_2' = \frac{y_1\, g}{W_{y_1, y_2}} = t^2 \left(3 - \frac{1}{t^2}\right) \frac{1}{(-3)}$$

$$\boxed{u_2' = -t^2 + \frac{1}{3}}$$

$$\boxed{u_2 = -\frac{t^3}{3} + \frac{t}{3}}$$

compute the particular solution

$$y_p = u_1\, y_1 + u_2\, y_2$$

$$= \left(\ln(t) + \frac{t^{-2}}{6}\right) t^2 + \frac{1}{3}(-t^3 + t)\, \frac{1}{t}$$

$$= \ln(t)\, t^2 + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3}$$

$$\boxed{y_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, t^2}$$

Notice: $\quad y_1 = t^2 \quad$, so

$$\boxed{y_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, y_1(t)}$$

Denoting: $\quad L(y) = y'' - \frac{2}{t^2}\, y \quad$,

recall: $\quad 0 = L(y_1)$

and

$$\begin{aligned} g &= L(y_p) \\ &= L\left(t^2 \ln(t) + \tfrac{1}{2} - \tfrac{1}{3}\, y_1\right) \\ &= L\left(t^2 \ln(t) + \tfrac{1}{2}\right) - \tfrac{1}{3}\, L(y_1) \end{aligned}$$

$$g = L\left(t^2 \ln(t) + \tfrac{1}{2}\right)$$

so: $$\boxed{\tilde{y}_p = t^2 \ln(t) + \frac{1}{2}}$$

is another particular sol. of

$$L(\tilde{y}_p) = g.$$