mth 235 L 12

Plan :

* $y'' + a_1 y' + a_0 y = b(t)$.
* Method of undetermined coefficients
* Examples.

(3.5)

Notation : Given functions $p, q$.

$$\boxed{L(y) = y'' + p(t)\, y' + q(t)\, y.}$$

So, the equation

$$\boxed{y'' + p(t)\, y' + q(t)\, y = f(t)}$$

can be written as

$$\boxed{L(y) = f}$$ non-homogeneous

$$\boxed{L(y) = 0}$$ homogeneous.

* Preliminar results on $L(y) = y'' + p(t)\, y' + q(t)\, y$

**Propos.** For all continuously differentiable functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ and all $c_1, c_2 \in \mathbb{R}$ holds

$$\boxed{L(c_1 y_1 + c_2 y_2) = c_1\, L(y_1) + c_2\, L(y_2).}$$

Proof:

$$\boxed{L(c_1 y_1 + c_2 y_2)} = (c_1 y_1 + c_2 y_2)'' + p(t)\,(c_1 y_1 + c_2 y_2)' + q(t)\,(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_1\, p(t)\, y_1' + c_1\, q(t)\, y_1 + c_2 y_2'' + c_2\, p(t)\, y_2' + c_2\, q(t)\, y_2$$

$$= \boxed{c_1\, L(y_1) + c_2\, L(y_2)}$$

■

**Propos.** Let $L(y) = y'' + p(t)\,y' + q(t)\,y$.

If $y_1, y_2$ are fundamental sols. of

$$\boxed{L(y) = 0\,.} \qquad (1)$$

and $y_p$ is a solution of

$$\boxed{L(y_p) = f\,,} \qquad (2)$$

then any other solution of (2) is given by

$$\boxed{y(t) = c_1\,y_1(t) + c_2\,y_2(t) + y_p(t)\,,} \qquad (3)$$

where $c_1, c_2 \in \mathbb{R}$.

**Notation:** The expression for $y$ in (3) is called the general solution of the non-homogeneous eq (2).

**Propos.** If $\boxed{f(t) = f_1(t) + \cdots + f_n(t)}$ , $n \geq 1$,

and there exist

$$\boxed{y_1(t), \ldots, y_n(t)}$$

such that

$$\boxed{L(y_i) = f_i} \quad , \quad i = 1, \ldots, n ,$$

then

$$\boxed{y_p(t) = y_1(t) + \cdots + y_n(t)}$$

is solution of

$$\boxed{L(y_p) = f.}$$

## * The method of undetermined coefficients

- [This is a method to find sols. to linear, non-homogeneous eqs.]

- [It consist in guessing the solution $y_p$ of the non-homogeneous eq.

$$L(y_p) = f$$

for particular source functions $f$.]

* **Summary of the method.**

- **Problem:** Given $\boxed{L(y) = y'' + a_1 y' + a_0 y,}$ with $a_1, a_2 \in \mathbb{R}$, find all solutions to

$$\boxed{L(y) = f.} \qquad (4)$$

(1) Find the general solution of the homogeneous eq.

$$\boxed{L(y_h) = 0.}$$

(2) If $\boxed{f(t) = f_1(t) + \cdots + f_n(t)}$, $n \geq 1$,

then look for $y_{p_i}$ solution of

$$\boxed{L(y_{p_i}) = f_i}.$$

Once $y_{p_i}$ are found, a particular solution of (4) is

$$\boxed{y_p = y_{p_1} + \cdots + y_{p_n}.}$$

(3) Given $f_1(t)$, guess $y_{p_1}(t)$ as follows:

| $f_1(t)$ (given) | $y_{p_1}(t)$ (guess) ($k$ not given). |
|---|---|
| $K e^{at}$ | $k e^{at}$ |
| $K t^m$, $m \geq 0$ | $k_m t^m + k_{m-1} t^{m-1} + \dots + k_0$ |
| $K \cos(at)$ or $K \sin(at)$ | $k_1 \cos(at) + k_2 \sin(at)$ |
| $K e^{at} \cos(bt)$ or $K e^{at} \sin(bt)$ | $e^{at} (k_1 \cos(bt) + k_2 \sin(bt))$ |
| $K t^m e^{at}$ | $e^{at} (k_m t^m + \dots + k_0)$. |

(4) If any guess $y_{p_i}$ above satisfies the homogeneous eq.

$$\boxed{L(y_{p_i}) = 0},$$

then change the guess to

$$\boxed{t^s \; y_{p_i}}, \qquad \boxed{s \geq 1},$$

such that

$$\boxed{L(t^s y_{p_i}) \neq 0.}$$

(5) Impose the condition

$$\boxed{L(y_{p_i}) = f_i}$$

to find the constants $k_1, \dots, k_m$.

(6) The general solution of $L(y) = f$ is

$$\boxed{y(t) = y_h(t) + y_{p_1}(t) + \dots + y_{p_n}(t)}.$$

**Example** [ Find all solutions to

$$y'' - 3y' - 4y = 3e^{2t}$$ ]

**Sol:**

$\boxed{L(y) = y'' - 3y' - 4y}$ , $\boxed{f(t) = 3e^{2t}}$

- First, find sols. to $\boxed{L(y_h) = 0.}$

$$p(r) = r^2 - 3r - 4 = 0$$

$$r = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2}$$

$\boxed{r_1 = 4}$ $\boxed{r_2 = -1}$

$$\boxed{y_h(t) = c_1 e^{4t} + c_2 e^{-t}}$$

- Table says: Guess $\boxed{y_p(t) = k e^{2t}}$

- Since $\boxed{L(y_p) \neq 0}$, we do not modify the guess.

- Introduce $Y_p$ in : $\boxed{L(Y_p) = f}$

$$(k\,4 - 3\,k\,2 - 4k)\,e^{2t} = 3\,e^{2t}$$

$$4k - 6k - 4k = 3$$

$$-6k = 3$$

$$\boxed{k = -\frac{1}{2}}$$

$$\boxed{Y_p(t) = -\frac{1}{2}\,e^{2t}}$$

- The general solution of the non-homogeneous eq. is

$$\boxed{y(t) = c_1\,e^{4t} + c_2\,e^{-t} - \frac{1}{2}\,e^{2t}}$$

**Example** : [ Find the general sol. of

$$y'' - 3y' - 4y = 3e^{4t}$$ ]

**Sol:**

We know that the general sol. to the homogeneous eq. is

$$\boxed{y_h = c_1 e^{4t} + c_2 e^{-t}.}$$

Following the table, the guess $y_p$ is

$$\boxed{y_p = k e^{4t}.}$$

However, this guess satisfies:

$$\boxed{L(y_p) = 0}$$

So we modify the guess to:

$$\boxed{y_p(t) = k\, t\, e^{4t}}$$

$$y_p'(t) = k\,e^{4t} + 4k\,t\,e^{4t}$$

$$y_p''(t) = 4k\,e^{4t} + 4k\,e^{4t} + 16\,k\,t\,e^{4t}$$

- Introduce $y_p$ into $L(y_p) = \ell$.

$$\left[(8k + 16kt) - 3(k + 4kt) - 4kt\right]e^{4t} = 3\,e^{4t}$$

$$\left[(8 + 16t) - (3 + 12t) - 4t\right]k = 3$$

$$\left[5 + \underbrace{(16 - 12 - 4)}_{=0}\,t\right]k = 3$$

$$5k = 3 \implies \boxed{k = \frac{3}{5}}$$

So: $\boxed{y_p(t) = \frac{3}{5}\,t\,e^{4t}}$ and

$$\boxed{y(t) = c_1\,e^{4t} + c_2\,e^{-t} + \frac{3}{5}\,t\,e^{4t}}$$

**Example** [Find the general sol. of

$$y'' - 3y' - 4y = 2\sin(t)$$ ]

**Sol:**

- The sols. to the homogeneous eq. are

$$\boxed{y_h(t) = c_1 e^{4t} + c_2 e^{-t}}$$

- The table says that a guess for $y_p$ is

$$\boxed{y_p(t) = k_1 \cos(t) + k_2 \sin(t).}$$

- Check: $L(y_p) \neq 0$. ✓

- compute:

$$y_p'(t) = -k_1 \sin(t) + k_2 \cos(t)$$

$$y_p''(t) = -k_1 \cos(t) - k_2 \sin(t).$$

- Introduce $y_p$ into $L(y_p) = f$.

$$\left(-k_1 \cos(t) - k_2 \sin(t)\right)$$
$$-3\left(-k_1 \sin(t) + k_2 \cos(t)\right)$$
$$-4\left(k_1 \cos(t) + k_2 \sin(t)\right) = 2\sin(t)$$

$$\left(-k_1 - 3k_2 - 4k_1\right)\cos(t)$$
$$+\left(-k_2 + 3k_1 - 4k_2\right)\sin(t) = 2\sin(t)$$

$$\left(-5k_1 - 3k_2\right)\cos(t) + \left(3k_1 - 5k_2\right)\sin(t) = 2\sin(t)$$

$\boxed{t=0}$ $\qquad$ $\boxed{-5k_1 - 3k_2 = 0}$

$\boxed{t=\frac{\pi}{2}}$ $\qquad$ $\boxed{3k_1 - 5k_2 = 2}$

$$k_1 = -\frac{3}{5}\, k_2 .$$

$$\left( 3\left(-\frac{3}{5}\right) - 5 \right) k_2 = 2$$

$$\left( -\frac{9}{5} - 5 \right) k_2 = 2$$

$$-\frac{(9+25)}{5}\, k_2 = 2$$

$$k_2 = -\frac{10}{34} \quad \Rightarrow \quad \boxed{k_2 = -\frac{5}{17}}$$

$$\boxed{k_1 = \frac{3}{17}}$$

So:

$$\boxed{y_p(t) = \frac{1}{17}\left[ 3\cos(t) - 5\sin(t) \right]}$$

and

$$\boxed{\begin{aligned} y(t) = {} & c_1 e^{4t} + c_2 e^{-t} \\ & + \frac{1}{17}\left[ 3\cos(t) - 5\sin(t) \right] \end{aligned}}$$

<u>Example</u> : [ Find the general sol. of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t)$$ ]

<u>Sol</u>:

We know that the sol. $y$ is given by

$$\boxed{y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t)}$$

where.

$$\boxed{y_h(t) = c_1 e^{4t} + c_2 e^{2t}}$$

and $y_{p_1}$ is sol. of $\boxed{L(y_{p_1}) = 3e^{2t}}$

and $y_{p_2}$ is sol. of $\boxed{L(y_{p_2}) = 2\sin(t).}$

We have found that:

$$\boxed{y_{p_1}(t) = -\frac{1}{2} e^{2t}}$$

$$\boxed{y_{p_2}(t) = \frac{1}{17}\left(3\cos(t) - 5\sin(t)\right)}$$

So: $$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t} + \frac{1}{17}\left[3\cos(t) - 5\sin(t)\right]}$$

Examples:

For : $y'' - 3y' - 4y = 3e^{2t}\sin(t)$

guess: $\boxed{y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.}$

For : $y'' - 3y' - 4y = 3t^2 e^{2t}$

guess : $\boxed{y_p(t) = \left(k_0 + k_1 t + k_2 t^2\right) e^{2t}}$

For: $y'' - 3y' - 4y = 3t\sin(t)$

guess: $\boxed{y_p(t) = (1 + k_1 t)\left(k_2 \sin(t) + k_3 \cos(t)\right)}$