mth 235 L11

Plan:

* $y'' + a_1 y' + a_0 y = 0$
* Repeated roots of the characteristic polynomial
* Reduction of order method.

(3.4)

* **Review: Constant coefficient eqs.**

Given constants $a_1, a_0 \in \mathbb{R}$, consider

$$\boxed{y'' + a_1 y' + a_0 y = 0}$$

with characteristic polynomial having roots

$$\boxed{r = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}} \to r_1, \; r_2$$

(1) If $\boxed{a_1^2 - 4a_0 > 0}$, then: $\begin{cases} y_1(t) = e^{r_1 t} \\ y_2(t) = e^{r_2 t} \end{cases}$

(2) If $\boxed{a_1^2 - 4a_0 < 0}$, then $\begin{cases} y_1(t) = e^{\alpha t}\cos(\beta t) \\ y_2(t) = e^{\alpha t}\sin(\beta t) \end{cases}$

Where $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$

(3) If $\boxed{a_1^2 - 4a_0 = 0}$, then $\{ \; y_1(t) = e^{-\frac{a_1}{2} t}$

* Questions:

Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4} y = 0 \qquad (1)$$

have two linearly independent solutions?

or

Is every solution to eq. (1) proportional to

$$y_1(t) = e^{-\frac{a_1}{2}t} \quad ?$$

* <u>Plausibility argument.</u> (Not a proof)

(To form an intuition about what is happening.)

(Intuition is subjective.)

- Case (3) can be seen as the limit $\beta \to 0$ in case (2)

- Let us study the solutions of the differential eq. in case (2) as $\beta \to 0$ for fixed t.

$$\boxed{\underset{\beta}{Y_1}(t) = e^{-\frac{a_1}{2}t}\cos(\beta t)} \xrightarrow[\boxed{\beta \to 0}]{} \boxed{e^{-\frac{a_1}{2}t} = Y_1(t)}$$

(t : fixed)

Since :

$$\boxed{\cos(\beta t) \underset{\beta \to 0}{\longrightarrow} 1}$$

$$\boxed{\underset{\beta}{Y_2}(t) = e^{-\frac{a_1}{2}t}\sin(\beta t)} \xrightarrow[\boxed{\beta \to 0}]{} \boxed{\beta t\, e^{-\frac{a_1}{2}t}} \to \underline{0}$$

(t : fixed)

Since :

$$\boxed{\frac{\sin(\beta t)}{\beta t} \underset{\beta \to 0}{\longrightarrow} 1}$$

$$y_{1\beta}(t) = e^{-\frac{a_1}{2}t}\cos(\beta t) \xrightarrow[\beta \to 0]{} e^{-\frac{a_1}{2}t} = y_1(t)$$

$$y_{2\beta}(t) = e^{-\frac{a_1 t}{2}}\sin(\beta t) \xrightarrow[\beta \to 0]{} \beta t e^{-\frac{a_1}{2}t} = \beta t y_1(t) \to 0$$

Is $\boxed{y_2(t) = t\, y_1(t)}$ solution of (1) ?

Introducing y_2 into eq. (1), gives the answer:

$\boxed{\text{yes.}}$

Since y_2 is not proportional to y_1,

y_2, y_1 are a fundamental set for eq. (1)

Thrm: If the coefficients $a_1, a_0 \in \mathbb{R}$ in the eq.

$$\boxed{y'' + a_1 y' + a_0 y = 0} \qquad (1)$$

satisfy $\boxed{a_1^2 = 4a_0}$,
then

$$\boxed{y_1(t) = e^{r_1 t}}, \quad \boxed{y_2(t) = t\, e^{r_1 t}},$$

with $\boxed{r_1 = -\frac{a_1}{2}}$, are

fundamental solutions of eq. (1).

We give a proof based on a new idea:

Reduction of order method.

<u>Proof</u>: The characteristic eq. is

$$\boxed{r^2 + a_1 r + a_0 = 0}$$

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}, \quad \boxed{a_1^2 = 4a_0} \Rightarrow$$

$$\boxed{r_1 = r_2 = -\frac{a_1}{2}}.$$

Therefore r_1 satisfies both.

$$\boxed{r_1^2 + a_1 r_1 + a_0 = 0} \quad (2) \qquad \boxed{2r_1 + a_1 = 0} \quad (3)$$

It is clear that $y_1(t) = e^{r_1 t}$ is solution of Eq. (1)

A second solution y_2 not proportional to y_1 can be found as follows:

(D'Alembert ~1750)

Express : $\boxed{y_2(t) = v(t) \quad y_1(t)}$

and find the equation that function v satisfies from the condition

$$\boxed{y_2'' + a_1 y_2' + a_0 y_2 = 0.}$$

$$y_2(t) = v(t) \quad e^{r_1 t}$$

$$y_2'(t) = v'(t)\, e^{r_1 t} + r_1 \, v(t) \, e^{r_1 t}$$

$$y_2''(t) = v''(t)\, e^{r_1 t} + 2 r_1 \, v'(t)\, e^{r_1 t} + r_1^2 \, v(t) e^{r_1 t}$$

$$\left[\left(v'' + 2 r_1 v' + r_1^2 v \right) + a_1 \left(v' + r_1 v \right) + a_0 v \right] e^{r_1 t} = 0$$

$$\left[v'' + 2 r_1 v' + r_1^2 v + a_1 v' + a_1 r_1 v + a_0 v \right] e^{r_1 t} = 0$$

$$v'' + \underbrace{(2r_1 + a_1)}_{=0} v' + \underbrace{(r_1^2 + a_1 r_1 + a_0)}_{=0} v = 0$$

Eq. for v: $\boxed{v'' = 0}$

Solution: $\boxed{v(t) = c_1 + c_2 t}$

Therefore: $\boxed{y_2(t) = (c_1 + c_2 t)\, e^{r_1 t}}$

If $c_2 \neq 0$, then y_2, y_1 l.i.

Simplest choice: $\boxed{c_1 = 0, \quad c_2 = 1}$

then:

$$\boxed{y_1(t) = e^{r_1 t}}, \quad \boxed{y_2(t) = t\, e^{r_1 t}}$$

are fundamental solutions of eq. (1).

The general sol. is

$$\boxed{y(t) = (c_1 + c_2 t)\, e^{r_1 t}}$$

∎

* **Example** : [Find the general sol. of

$$9y'' + 6y' + y = 0.$$]

Sol:

The characteristic eq. is:

$$9r^2 + 6r + 1 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 36}}{(2)(9)} \Rightarrow \boxed{r_1 = r_2 = -\frac{1}{3}}$$

$$\boxed{y(t) = (c_1 + c_2 t)\, e^{-t/3}}$$

* **Example:** Find the sol. to the IVP:

$$9y'' + 6y' + y = 0 \qquad (4)$$

$$y(0) = 1 \quad , \quad y'(0) = \frac{5}{3} .$$

Sol:

The general sol. of (4) is

$$\boxed{y(t) = (c_1 + c_2 t)\, e^{-t/3}}$$

$$\boxed{y'(t) = (c_1 + c_2 t)\left(\frac{-1}{3}\right) e^{-t/3} + c_2\, e^{-t/3}}$$

$$1 = y(0) = c_1 \quad \Rightarrow \quad \boxed{c_1 = 1}$$

$$\frac{5}{3} = y'(0) = -\frac{c_1}{3} + c_2 \qquad c_2 = \frac{5}{3} + \frac{1}{3}$$

$$\boxed{c_2 = 2}$$

$$\boxed{y(t) = (1 + 2t)\, e^{-t/3}}$$

* **The reduction of order method:**

The same idea used in the proof of Thrm above can be used in variable coefficients eqs.

Thrm: Given continuous functions $p, q : (t_1, t_2) \to \mathbb{R}$, let $y_1 : (t_1, t_2) \to \mathbb{R}$ be solution of

$$\boxed{y'' + p(t)\, y' + q(t)\, y = 0.} \qquad (5)$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ is solution of

$$\boxed{y_1(t)\, v'' + \left[2\, y_1'(t) + p(t)\, y_1(t)\right] v' = 0,} \qquad (6)$$

then the functions

$$\boxed{y_1(t)}\ , \quad \boxed{y_2(t) = v(t)\, y_1(t)}$$

are fundamental solutions of (5).

<u>Remark</u>: The reason for the name "Reduction of order method" is that v does not appear in (6).

$y_1' (v')' + (2y_1' + p)(v') = 0$ is first order in (v').

* Example: Find a fundamental set of solutions to

$$t^2 y'' + 2t\, y' - 2y = 0 \qquad (6)$$

knowing that $y_1(t) = t$ is a solution.

Sol:

We express $y_2(t) = v(t)\, y_1(t)$.

The eq. for v is obtained from the condition that y_2 be solution of (6),

$$t^2 y_2'' + 2t\, y_2' - 2y_2 = 0.$$

We need to compute:

$$y_2 = v\, t$$

$$y_2' = v'\, t + v$$

$$y_2'' = v''\, t + 2v'$$

$$t^2 (v'' t + 2v') + 2t\,(v' t + v) - 2vt = 0$$

$$t^3 v'' + (2t^2 + 2t^2) v' + \underbrace{(2t - 2t)}_{=0} v = 0$$

$$t^3 v'' + 4t^2 v' = 0$$

$$\boxed{v'' + \frac{4}{t} v' = 0}$$

$$u = v'$$

$$\boxed{u' + \frac{4}{t} u = 0}$$

first order, linear. integrating factor method.

$$a(t) = \frac{4}{t}, \quad A(t) = 4 \ln(t) = \ln(t^4)$$

$$\mu(t) = e^{A(t)} \Rightarrow \boxed{\mu(t) = t^4}$$

$$t^4 u' + 4t^3 u = 0$$

$$(t^4 u)' = 0$$

$$\boxed{u = \frac{c_0}{t^4}}$$

$$v' = \frac{c_0}{t^4} = c_0 t^{-4}$$

$$\boxed{v(t) = -\frac{c_0}{3} t^{-3} + c_1}$$

$$y_2(t) = \left(-\frac{c_0}{3} t^{-3} + c_1\right) y_1(t)$$

We can choose : $\boxed{c_1 = 0}$

$\boxed{c_0 = -3.}$

$$\left.\begin{array}{l} y_2(t) = t^{-3} y_1(t) \\ \\ y_1(t) = t \end{array}\right] \Rightarrow \boxed{\begin{array}{l} y_2(t) = \dfrac{1}{t^2} \\ \hline y_1(t) = t \end{array}}$$

fundamental solutions of (1).

<u>Proof of Thrm</u> :

$$\left[\begin{aligned} y_2 &= v\ y_1 \\ y_2' &= v'\ y_1 + v\ y_1' \\ y_2'' &= v''\ y_1 + 2v'\ y_1' + v\ y_1'' \end{aligned}\right]$$

$$y_2'' + p(t)\ y_2' + q(t)\ y_2 = 0$$

$$v''\ y_1 + 2v'\ y_1' + v\ y_1'' +$$
$$p(t)\ [\ v'\ y_1 + v\ y_1'\ ] +$$
$$q(t)\ [\ v\ y_1\ ] = 0$$

$$v''\ y_1 + [\ 2y_1' + p(t)\ y_1\ ]\ v' +$$
$$+ \underbrace{[\ y_1'' + p(t)\ y_1' + q(t)\ y_1\ ]}_{=0}\ v = 0$$

$$\boxed{y_1\ v'' + [\ 2y_1' + p\ y_1\ ]\ v' = 0}$$

first order, linear for $u = v'$

We now show that $y_2 = v\,y_1$ and y_1 are fundamental sols.

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & y_1' \\ v y_1 & v' y_1 + v y_1' \end{vmatrix}$$

$$= y_1 \left(v' y_1 + v y_1' \right) - v\, y_1\, y_1'$$

$$\boxed{W_{y_1 y_2} = v' (y_1)^2}$$

we need to find v'.

$$\boxed{u = v'}$$

$$\boxed{y_1\, u' + \left(2 y_1' + p\, y_1 \right) u = 0}$$

$$\frac{u'}{u} = -\frac{1}{y_1} \left(2 y_1' + p\, y_1 \right)$$

$$\boxed{\frac{u'}{u} = -2\,\frac{y_1'}{y_1} - p}$$

$$\ln(u) = -2\ln(y_1) - \int p(t)\,dt$$

$$\mathbb{P}(t) = \int p(t)\,dt$$

$$\ln(u) = \ln\left((y_1)^{-2}\right) - \mathbb{P}(t)$$

$$u = e^{\ln(y_1^{-2}) - \mathbb{P}(t)} = e^{\ln(y_1^{-2})}\, e^{-\mathbb{P}(t)}$$

$$\boxed{u(t) = \frac{1}{y_1^2}\, e^{-\mathbb{P}(t)}}$$

$$\boxed{v'(t) = \frac{1}{[y_1(t)]^2}\, e^{-\mathbb{P}(t)}}$$

$$W_{y_1 y_2} = v'(y_1)^2 \implies \boxed{W_{y_1 y_2} = e^{-\mathbb{P}(t)}}$$

$$W_{y_1 y_2} \neq 0$$

y_1, y_2 are fundamental solutions of (6).