mth 235 L10

Plan:
* $y'' + a_1 y' + a_0 y = 0$
* characteristic polynomial with complex roots
* Two main sets of fundamental solutions
* Examples.

(3.3)

Room change: February 5 and 12

Lecture is at

Room 1281 Antony Hall

This Friday and next Friday.

[ Homework 3 due Friday 5.

Please write down yor section number ]

* **Review: Main result for complex roots**

**Thrm:** Given $a_1, a_0 \in \mathbb{R}$ consider the ODE

$$\boxed{y'' + a_1 y' + a_0 y = 0} \qquad (1)$$

If the characteristic polynomial

$$\boxed{p(r) = r^2 + a_1 r + a_0}$$

has complex roots

$$\boxed{r_1 = \alpha + i\beta}, \quad \boxed{r_2 = \alpha - i\beta},$$

with $\alpha, \beta \in \mathbb{R}$, then each set

$$\boxed{\tilde{y}_1(t) = e^{(\alpha + i\beta)t}}, \quad \boxed{\tilde{y}_2(t) = e^{(\alpha - i\beta)t}}$$

and

$$\boxed{y_1(t) = e^{\alpha t} \cos(\beta t)}, \quad \boxed{y_2(t) = e^{\alpha t} \sin(\beta t)}$$

is a fundamental set of Eq. (1).

* Recall: $\{y_1, y_2\}$ is a fundamental set of eq. (1)
iff $y_1, y_2$ are l.i.
iff $W_{y_1, y_2} \neq 0$.

* Recall: If $\{y_1, y_2\}$ is a fundamental set of eq. (1), then the functions

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

with $c_1, c_2$ arbitrary constants, is called the general solution of eq. (1).

Example: Find the general solution of

$$y'' - 2y' + 6y = 0 \tag{1}$$

Sol:

Looking for solutions $y(t) = e^{rt}$ we obtain the chacteristic eq.

$$r^2 - 2r + 6 = 0.$$

$$r = \frac{2 \pm \sqrt{4-24}}{2} = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm 2\sqrt{-5}}{2}$$

$$\boxed{r_1 = 1 + i\sqrt{5}} \qquad \boxed{r_2 = 1 - i\sqrt{5}}$$

Fundamental solutions:

$$\boxed{\tilde{y}_1(t) = e^{(1+i\sqrt{5})t}} \quad \boxed{\tilde{y}_2(t) = e^{(1-i\sqrt{5})t}}$$

The general sol. is

$$\boxed{y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}} \tag{2}$$

$$\boxed{\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}}$$

* Remarks

(1) The solutions in (2) include real-valued and complex-valued solutions.

(2) Real-valued solutions are difficult to take apart from complex-valued solutions in (2).

(3) Since eq. (1) has real coefficients, it is useful in applications to obtain the most general real-valued solution.

<u>Example</u>: [ Find the real-valued general solution of

$$y'' - 2y' + 6y = 0. \qquad (1)$$ ]

<u>Sol.</u>

Complex-valued fundamental solutions are:

$$\boxed{\tilde{y}_1(t) = e^{(1+i\sqrt{5})t}} \, , \quad \boxed{\tilde{y}_2(t) = e^{(1-i\sqrt{5})t}}$$

Any linear combination is also a solution of (1).
In particular

$$y_1(t) = \frac{1}{2}\left[\tilde{y}_1(t) + \tilde{y}_2(t)\right]$$

$$y_2(t) = \frac{1}{2i}\left[\tilde{y}_1(t) - \tilde{y}_2(t)\right]$$

Recalling that

$$e^{(1+\sqrt{5}i)t} = e^{t} \quad e^{i\sqrt{5}t}$$

$$e^{(1-\sqrt{5}i)t} = e^{t} \quad e^{-i\sqrt{5}t}$$

$$y_1(t) = \frac{1}{2}\left[ e^t e^{i\sqrt{5}t} + e^t e^{-i\sqrt{5}t} \right]$$

$$\boxed{y_1(t) = \frac{e^t}{2}\left( e^{i\sqrt{5}t} + e^{-i\sqrt{5}t} \right)}$$

$$\boxed{y_2(t) = \frac{e^t}{2i}\left( e^{i\sqrt{5}t} - e^{-i\sqrt{5}t} \right)}$$

<u>Claim</u>: these two functions are real-valued.

<u>Recall</u>: The Euler formula

$$\boxed{e^{i\sqrt{5}t} = \cos(\sqrt{5}t) + i\sin(\sqrt{5}t)}$$

and its complex-conjugate

$$\boxed{e^{-i\sqrt{5}t} = \cos(\sqrt{5}t) - i\sin(\sqrt{5}t)}$$

Then:

$$e^{i\sqrt{5}t} + e^{-i\sqrt{5}t} = 2\cos(\sqrt{5}t)$$

$$e^{i\sqrt{5}t} - e^{-i\sqrt{5}t} = 2i\sin(\sqrt{5}t)$$

Therefore.

$$\boxed{\begin{aligned} y_1(t) &= e^{t}\cos(\sqrt{5}t) \\ y_2(t) &= e^{t}\sin(\sqrt{5}t) \end{aligned}}$$

Real-valued solutions

They are not proportional to each other for all $t$, so they are a fundamental set for eq. (1).

The general sol. is

$$\boxed{y(t) = \left[c_1\cos(\sqrt{5}t) + c_2\sin(\sqrt{5}t)\right]e^{t}}$$

$y(t)$ is real-valued for $c_1, c_2 \in \mathbb{R}$.

is complex-valued for $c_1, c_2 \in \mathbb{C}$.

* **Remark** :

The proof of Thrm (case b) follows exactly the same ideas given in the example above replacing the roots

$$1 + \sqrt{5}\, i \longrightarrow \alpha + \beta i$$

$$1 - \sqrt{5}\, i \longrightarrow \alpha - \beta i$$

The real-valued fundamental sol. are

$$\boxed{y_1(t) = e^{\alpha t} \cos(\beta t)}$$

$$\boxed{y_2(t) = e^{\alpha t} \sin(\beta t)}$$

Example: $\left[ \text{Find real-valued fundamental solutions of} \quad y'' + 2y' + 6 = 0 \right]$ (3)

Sol:

$$P(r) = r^2 + 2r + 6 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 24}}{2} = \frac{-2 \pm \sqrt{-20}}{2} = \frac{-2 \pm 2\sqrt{-5}}{2}$$

$$\boxed{r_1 = -1 + i\sqrt{5}} \qquad \boxed{r_2 = -1 - i\sqrt{5}}$$

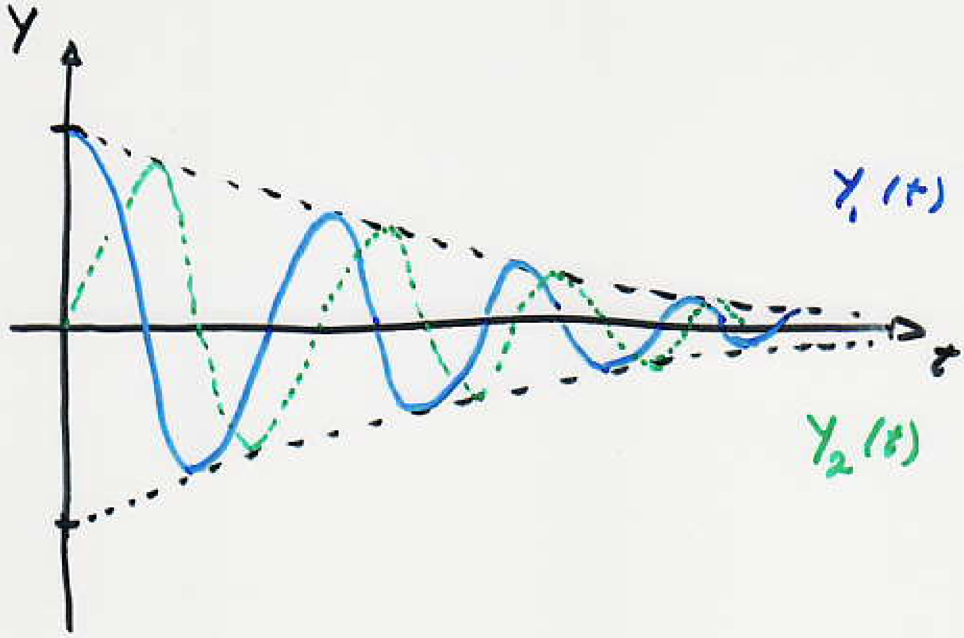
$$\alpha = -1, \quad \beta = \sqrt{5} \qquad (\alpha < 0)$$

Real-valued fundamental solutions are

$$\boxed{y_1(t) = e^{-t} \cos(\sqrt{5}\, t)}$$

$$\boxed{y_2(t) = e^{-t} \sin(\sqrt{5}\, t)}$$

# Remark




- Eqs. like (3) describe physical processes related to damped oscillations.

(pendulum with friction.)

**Example** $\left[\begin{array}{l}\text{Find the real-valued general sol. of} \\ \qquad y'' + 5y = 0. \qquad\qquad (4)\end{array}\right]$

**Sol:**

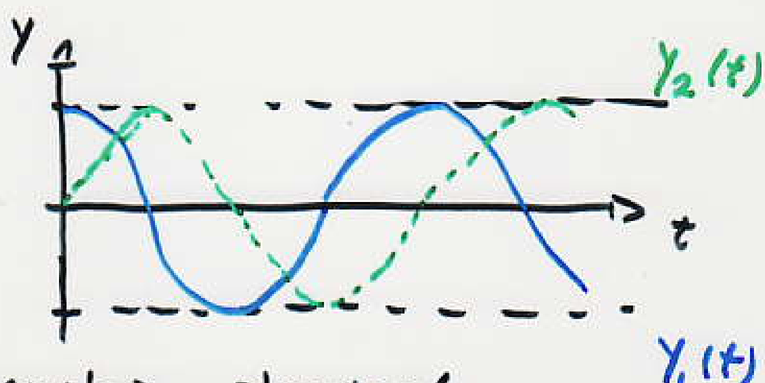
$P(r) = r^2 + 5 = 0 \quad \Rightarrow \quad r^2 = -5$

$\boxed{r_1 = \sqrt{5}\, i} \qquad \boxed{r_2 = -\sqrt{5}\, i}$

$\alpha = 0 \quad , \quad \beta = \sqrt{5}.$

$$\boxed{\begin{array}{l} y_1(t) = \cos(\sqrt{5}\, t) \\ \hline y_2(t) = \sin(\sqrt{5}\, t) \end{array}}$$

$$\boxed{y(t) = c_1 \cos(\sqrt{5}\, t) + c_2 \sin(\sqrt{5}\, t)}$$
$\boxed{c_1, c_2 \in \mathbb{R}}$




**Remark**

Eqs like (4) describe physical processes without dissipation $(\alpha = 0)$.

* **Application: The RLC circuit.**

Consider an electric circuit with Resistance $R$, capacitor $C$, and inductance $L$, given by


$\begin{pmatrix} C \neq 0 \\ L \neq 0 \end{pmatrix}$

$I(t)$ : electric current.

The electric current flowing in such circuit satisfies:

$$\boxed{L\, I'(t) + R\, I(t) + \frac{1}{C} \int_{t_0}^{t} I(s)\, ds = 0}$$

Derivate both sides in eq. above

$$\boxed{L\, I'' + R\, I' + \frac{1}{C} I = 0}$$

$$\boxed{I'' + 2\left(\frac{R}{2L}\right) I' + \frac{1}{LC} I = 0}$$

$$\boxed{\alpha = \frac{R}{2L}} \qquad \boxed{\omega = \frac{1}{\sqrt{LC}}}$$

(usual definitions in physics.)

$$\boxed{I'' + 2\alpha I' + \omega^2 I = 0} \qquad (5)$$

<u>Example</u> [Find real-valued fundamental sols. to Eq. (5), in cases 1, 2 below.]

<u>Sol:</u>

$$p(r) = r^2 + 2\alpha r + \omega^2 = 0$$

$$r = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}}{2}$$

$$\boxed{r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}}$$

$$\alpha = \frac{R}{2L} \qquad \omega^2 = \frac{1}{LC}$$

Case 1: If $R = 0$, then $\alpha = 0$ and

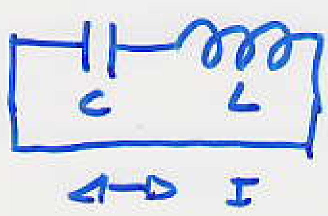
$$r_\pm = \pm \omega i.$$

Therefore:

$$\boxed{I_1(t) = \cos(\omega t)}$$

$$\boxed{I_2(t) = \sin(\omega t)}$$

current oscillates



case 2: If $\boxed{R < \sqrt{\frac{4L}{C}}}$, then

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC}$$

$$\iff \alpha^2 < \omega^2 .$$

Therefore.

$$\boxed{r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}}$$

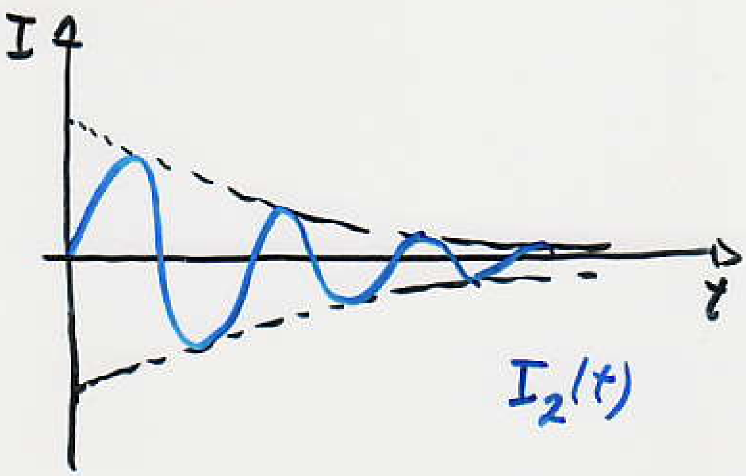
and:

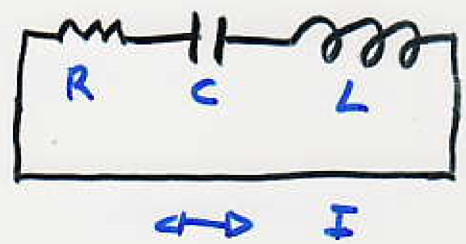
$$I_1(t) = e^{-\alpha t} \cos\left(\sqrt{\omega^2 - \alpha^2}\; t\right)$$

$$I_2(t) = e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2}\; t\right)$$

$\alpha = \frac{R}{2L}$

$\omega = \frac{1}{LC}$




The resistance damps the current oscillations.