mth 235 L9

Plan:
* Review 2 for Exam 1
* Linear eqs.
* separable eqs
* Homogeneous eqs
* Applications
* Bernoulli eqs.
* Exact eqs.
* Exact eqs with integrating factors.

* Example: $\left[\begin{array}{l}\text{Find all solutions } y \text{ of} \\ (t^2+y^2)(t+y\,y')+2=0\end{array}\right]$

Sol:

$$(t^2+y^2)\,t + (t^2+y^2)\,y\;y' + 2 = 0$$

$$\boxed{(t^2y+y^3)\,y' + (t^3+t\,y^2+2) = 0}$$

Not linear, Not separable

Not homogeneous, Not Bernoulli.

Exact ?

$$N = t^2y+y^3 \quad \Rightarrow \quad \partial_t N = 2ty$$

$$M = t^3+t\,y^2+2 \quad \Rightarrow \quad \partial_y M = 2ty$$

$$\boxed{\partial_t N = \partial_y M} \qquad \boxed{\text{exact !}}$$

There exists ψ sol. of

$$\partial_y \psi = N \qquad (1)$$

$$\partial_t \psi = M \qquad (2)$$

(1) $\Rightarrow$ $\partial_y \psi = t^2 y + y^3$

$$\psi = t^2 \frac{y^2}{2} + \frac{y^4}{4} + g(t)$$

(2) $t y^2 + g'(t) = \partial_t \psi = M = t^3 + t y^2 + 2$

$$g'(t) = t^3 + 2$$

$$g(t) = \frac{t^4}{4} + 2t$$

$$\psi = t^2 \frac{y^2}{2} + \frac{y^4}{4} + \frac{t^4}{4} + 2t$$

$$\boxed{\frac{y^4(t)}{4} + \frac{t^2}{2} y^2(t) + 2t + \frac{t^4}{4} = c_0}$$

* **Example** [Find the explicit sol. y of

$$y' = \frac{t(t^2+e^t)}{4y^3}, \quad y(0) = -\sqrt{2}.$$]

Sol:

$$\boxed{4y^3 y' = t^3 + te^t}$$

separable eq.

$$\int 4y^3(t)\, y'(t)\, dt = \int (t^3 + te^t)\, dt + c$$

substitution: $u = y(t)$, $du = y'(t)\, dt$

$$\int 4u^3\, du = \int (t^3 + te^t)\, dt + c$$

$$\boxed{u^4 = \frac{t^4}{4} + \int te^t\, dt + c}$$

Integration by parts:

$$\int t e^t \, dt = t e^t - \int e^t \, dt$$

$$f = t \qquad g' = e^t$$

$$f' = 1 \qquad g = e^t$$

$$\boxed{\int t e^t = (t-1) e^t}$$

$$\boxed{y^4(t) = \frac{t^4}{4} + (t-1) e^t + c}$$

Initial condition : $y(0) = -\sqrt{2}$

$$(-\sqrt{2})^4 = 0 + (0-1)\, 1 + c$$

$$4 = -1 + c \Rightarrow \boxed{c = 5}$$

$$\boxed{y^4(t) = \frac{t^4}{4} + (t-1) e^t + 5}$$

implicit form.

The explicit solution is one of:

$$y_{\pm}(t) = \pm \left[\frac{t^4}{4} + (t-1)\, e^t + 5 \right]^{1/4}$$

From the initial condition

$$y(0) = -\sqrt{2} < 0$$

we conclude that the solution is

$$\boxed{y(t) = - \left[\frac{t^4}{4} + (t-1)\, e^t + 5 \right]^{1/4}}$$

* **Example:** Find all sol. y of

$$y' = \frac{3y^2 - t^2}{2ty}.$$

Sol:

The eq. is homogeneous, since the sum of the t and y exponents on every term gives the same result; 2 in this case.

$$y' = \frac{3y^2 - t^2}{2ty} \quad \frac{(1/t^2)}{(1/t^2)}$$

$$y' = \frac{3y^{2} - t^2}{2ty} \quad \frac{(1/t^{2})}{(1/t^2)}$$

$$\boxed{y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}}$$

$$\boxed{v = \frac{y}{t}} \quad \Rightarrow \quad y = t\,v$$

$$\boxed{y' = v + t\,v'}$$

$$v + t\,v' = \frac{3v^2 - 1}{2v}$$

$$t\,v' = \frac{3v^2 - 1}{2v} - v$$

$$t\,v' = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$t\,v' = \frac{v^2 - 1}{2v}$$

$$\boxed{\frac{2v}{v^2 - 1}\,v' = \frac{1}{t}}$$

separable eq.

$$\int \frac{2v\,v'}{v^2-1}\,dt = \int \frac{dt}{t} + c_0$$

Substitution: $u = v^2 - 1$, $du = 2v\,v'\,dt$

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0$$

$$\ln|u| = \ln|t| + c_0$$

$$e^{\ln|u|} = e^{\ln|t| + c_0} = e^{\ln|t|}\,e^{c_0}$$

$$|u| = |t|\,c_1 \quad , \quad c_1 = e^{c_0}$$

$$\boxed{|v^2 - 1| = c_1\,|t|} \quad , \quad v = \frac{y}{t}$$

$$\left|\frac{y^2}{t^2} - 1\right| = c_1\,|t|$$

$$\boxed{|y^2 - t^2| = c_1\,|t|^3}$$

implicit sols.

* Example: Tank with $V_0 = 100$ gal,
Q_0 : Not given ;
$t_0 = 0$ Fresh water poured in;
mixing mechanism.

Find r_i, r_o such that:
(1) Water volumen is constant;
(2) the time to reduce the initial salt Q_0 to $e^{-5} Q_0$ is 25 min.

Sol:

(1) Volumen of water constant

$$\boxed{r_i = r_o.}$$

$$\frac{d}{dt} V(t) = r_i - r_o = 0 \quad \Leftrightarrow \quad \boxed{V(t) = V_0}$$

(2) First find $Q(t)$.

$$\frac{d}{dt} Q(t) = r_i \, q_i - r_o \, q_o(t)$$

mixing mechanism

$$= r_i \, q_i - r_o \frac{Q(t)}{V(t)}$$

$$Q' = r_i q_i - \frac{r_o}{V_o} Q(t)$$

$$r_i = r_o = r$$

$$Q' = r q_i - \frac{r}{V_o} Q(t)$$

Fresh water is poured in:

$$\boxed{q_i = 0}$$

$$\boxed{Q' = -\frac{r}{V_o} Q(t)}$$ differential eq. for Q.

$$\boxed{Q(t) = Q_o e^{-\frac{r}{V_o} t}}$$

condition for r: $\boxed{e^{-5} Q_o = Q(25) = Q_o e^{-\frac{r}{V_o} 25}}$

$$e^{-5} = e^{-\frac{r}{V_o} 25} \iff -5 = -\frac{r}{V_o} 25$$

$$r = \frac{V_o}{5} \iff \boxed{r = 20 \ \frac{gal}{min}}$$

* Example : $\left[\begin{array}{l} \text{Find all sols. } y \text{ to} \\ y' = \dfrac{2}{t} y - \dfrac{\sin(t)}{t} y^2 \\ y(2\pi) = 2\pi \quad , \quad t > 0. \end{array} \right]$

Sol:

Bernoulli eq. with $n = 2$.

$$\boxed{\frac{y'}{y^2} = \frac{2}{t}\frac{1}{y} - \frac{\sin(t)}{t}}$$

$$\boxed{v = \frac{1}{y}} \quad \Rightarrow \quad \boxed{v' = (-1)\frac{y'}{y^2}}$$

$$-v' = \frac{2}{t} v - \frac{\sin(t)}{t}$$

$$\boxed{v' + \frac{2}{t} v = \frac{\sin(t)}{t}}$$ linear eq.

Integrating factor method.

$a(t) = \frac{2}{t}$, $b(t) = \frac{\sin(t)}{t}$.

$\mu(t) = e^{A(t)}$, $A(t) = \int \frac{2}{t}\, dt$

$A(t) = 2\ln(t)$ $(t>0)$

$$\boxed{A(t) = \ln(t^2)}$$

$\mu(t) = e^{\ln(t^2)}$

$$\boxed{\mu(t) = t^2}$$

$$t^2 V' + t^2 \frac{2}{t} V = t^2 \frac{\sin(t)}{t}$$

$$\boxed{t^2 V' + 2t V = t \sin(t)}$$

total derivative

$$(t^2 V)' = t \sin(t)$$

$$\int (t^2 V)' \, dt = \int t \sin(t) \, dt + c$$

$$\boxed{t^2 V(t) = \int t \sin(t) \, dt + c}$$

Integration by parts:

$$\int t \sin(t) \, dt = -t \cos(t) - \int \left(-\cos(t)\right) dt$$

$$f = t \qquad g' = \sin(t)$$

$$f' = 1 \qquad g = -\cos(t)$$

$$\boxed{\int t \sin(t) \, dt = -t \cos(t) + \sin(t)}$$

$$\boxed{t^2 V(t) = -t \cos(t) + \sin(t) + c}$$ Sol. to linear eq.

$$V = \frac{1}{y}$$

$$\frac{t^2}{y(t)} = -t\cos(t) + \sin(t) + c$$

$$\boxed{y(t) = \frac{t^2}{\sin(t) - t\cos(t) + c}}$$

Initial condition: $y(2\pi) = 2\pi$

$$2\pi = y(2\pi) = \frac{4\pi^2}{0 - 2\pi\,(1) + c}$$

$$1 = \frac{2\pi}{c - 2\pi} \iff c - 2\pi = 2\pi$$

$$\boxed{c = 4\pi}$$

$$\boxed{y(t) = \frac{t^2}{\sin(t) - t\cos(t) + 4\pi}}$$

* <u>Example</u>: [Find if the eq. below can be converted into an exact eq.

$$y' + \frac{y}{t} - \sin(t) = 0 .$$

If yes, find the integrating factor.]

<u>Sol</u>:

$$N = 1 \qquad \Rightarrow \qquad \partial_t N = 0$$

$$M = \frac{y}{t} - \sin(t) \qquad \Rightarrow \qquad \partial_y M = \frac{1}{t} .$$

$$\boxed{\partial_t N \neq \partial_y M.}$$

$$\frac{1}{N} \left(\partial_y M - \partial_t N \right) = \frac{1}{1} \left(\frac{1}{t} - 0 \right)$$

$$\boxed{\frac{1}{N} \left(\partial_y M - \partial_t N \right) = \frac{1}{t}}$$

independent of y.

$$\frac{\mu'}{\mu} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu) = \ln(t)$$

$$\boxed{\mu(t) = t}$$ integrating factor

Verify: $\boxed{t\,y' + y - t\sin(t) = 0}$

$$\tilde{N} = t \quad \Rightarrow \quad \partial_t \tilde{N} = 1$$

$$\tilde{M} = y - t\sin(t) \quad \Rightarrow \quad \partial_y \tilde{M} = 1$$

$$\boxed{\partial_t \tilde{N} = \partial_y \tilde{M}}$$