mth 235 L8

Plan:
* Review 1 for Exam 1

* Change of Exam date
  Monday → Tuesday

* Homogeneous eqs. (2.2)

* Bernoulli eqs. (2.4)

* Announcements

- Exam date change : Monday $\rightarrow$ Tuesday
- The exam will be taken in recitations. Your TA will proctor the exam.
- Monday class : Review 2.
- Monday afternoon : Review in MLC.

* Exam 1

- 6 problems , 50 min.
- No multiple choice questions
- No notes, No books, No calculators.
- Problems similar to homeworks.

* Homogeneous Eqs. (2.2 Separable eqs.)

- Homogeneous eqs. are NOT separable.
- Homogeneous eqs can be transformed into separable eqs.

Def: [The differential eq.

$$y'(t) = f(t, y(t))$$

is called homogeneous iff

$$f(t, y) = F\left(\frac{y}{t}\right)$$

for some function $F : \mathbb{R} \to \mathbb{R}$.]

Example [Show that the eq. below is homogeneous

$$(t-y)y' - 2y + 3t + \frac{y^2}{t} = 0$$
]

Sol:

$$(t-y)\,y' = 2y - 3t - \frac{y^2}{t}$$

$$y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{(t-y)}$$

$$y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{(t-y)} \; \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)}$$

$$\boxed{y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{\left(1 - \left(\frac{y}{t}\right)\right)}}$$ Homogeneous

$$f(t,y) = \frac{2y - 3t - y^2/t}{t - y}, \qquad F(x) = \frac{2x - 3 - x^2}{1 - x}$$

$$\boxed{f(t,y) = F\left(\frac{y}{t}\right)}$$

<u>Example:</u> [Is the eq. below homogeneous ?

$$y' = \frac{t^2}{1 - y^3}$$

]

<u>Sol:</u>

$$y' = \frac{t^2}{(1 - y^3)} \qquad \frac{\left(\frac{1}{t^3}\right)}{\left(\frac{1}{t^3}\right)}$$

$$y' = \frac{\frac{1}{t}}{\frac{1}{t^3} - \left(\frac{y}{t}\right)^3}$$

The eq is Not homogeneous.

Thm: If the differential eq for y

$$\boxed{y'(t) = f(t, y(t))}$$

is homogeneous, then the differential eq for

$$\boxed{v(t) = \frac{y(t)}{t}}$$

is separable.

<u>Proof:</u> $y' = f(t, y)$ homogeneous $\Rightarrow$

$\Rightarrow$ $\boxed{y' = F\left(\frac{y}{t}\right)}$ for some $F: \mathbb{R} \to \mathbb{R}$.

Introduce: $\boxed{v(t) = \frac{y(t)}{t}}$

$$y = t\, v$$

$$\boxed{y' = v + t\, v'}$$

$$v + t\, v' = F(v)$$

$$\boxed{v' = \frac{(F(v) - v)}{t}}$$

separable eq.

Example: Find all solutions of

$$y' = \frac{t^2 + 3y^2}{2ty}$$

$$f(t,y) = \frac{t^2+3y^2}{2ty}$$

Sol.

The eq. is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \quad \frac{(1/t^2)}{(1/t^2)}$$

(choose highest power of y for the exponent of $\cancel{1/t}$).

$$\boxed{y' = \frac{1 + 3(y/t)^2}{2(y/t)}}.$$

$$\boxed{v = \frac{y}{t}} \Rightarrow y = t\,v \Rightarrow \boxed{y' = v + t\,v'}$$

$$v + t\,v' = \frac{1 + 3v^2}{2v}$$

$$F(v) = \frac{1+3v^2}{2v}$$

$$t\,v' = \frac{1 + 3v^2}{2v} - v$$

$$t\,v' = \frac{1 + 3v^2 - 2v^2}{2v}$$

$$t V' = \frac{1+V^2}{2V} \quad \Rightarrow \quad \boxed{V' = \frac{1}{t}\left(\frac{1+V^2}{2V}\right)}$$ separable.

Problem for a separable eq.

$$\frac{2V}{(1+V^2)} V' = \frac{1}{t}$$

$$\int \frac{2V\, V'}{1+V^2}\, dt = \int \frac{dt}{t} + c_0$$

Substitution: $\quad u = 1 + V^2(t) \quad \Rightarrow du = 2V\, V'\, dt$

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0$$

$$\ln(u) = \ln(t) + c_0$$

$$\left[u = e^{\ln(t) + c_0} = e^{c_0}\, e^{\ln(t)} = e^{c_0}\, t \right]$$

denote : $\quad c_1 = e^{c_0}$

$$\boxed{u = c_1 t}$$

Substitute back $u = 1 + V^2(t)$

$$\boxed{1 + V^2(t) = c_1 t}$$

Substitute back $V = \frac{y}{t}$,

$$1 + \left(\frac{y(t)}{t}\right)^2 = c_1 t$$

$$\boxed{t^2 + y^2(t) = c_1 t^3}$$

implicit solutions.

$$\boxed{y(t) = \pm\sqrt{c_1 t^3 - t^2}}$$

explicit solutions.

* The Bernoulli eq.

The Bernoulli eq. is a non-linear eq. that can be transformed into a linear eq.

Def: Given functions p, q and an integer n, the eq. on y,

$$y' + p(t)\, y = q(t)\, y^n$$

is called the Bernoulli eq.

Thrm: A Bernoulli eq with $n \neq 0$, $n \neq 1$, in the unknown y can be transformed into a linear eq in the unknown

$$\boxed{v(t) = \frac{1}{y^{n-1}(t)}}$$

Proof : $y' + p(t)\,y = q(t)\,y^n$ $\quad n \neq 0$, $n \neq 1$

$$\frac{y'}{y^n} + \frac{p(t)}{y^{n-1}} = q(t)$$

$$V = \frac{1}{y^{n-1}} = y^{1-n}$$

$$V' = (1-n)\,y^{1-n-1}\,y'$$

$$V' = (1-n)\,\frac{y'}{y^n} \Rightarrow \boxed{\frac{y'}{y^n} = \frac{1}{(1-n)}\,V'}$$

$$\frac{1}{(1-n)}\,V' + p(t)\,V = q(t)$$

$$\boxed{V' + (1-n)\,p(t)\,V = (1-n)\,q(t)}$$

linear eq. for V. ∎

<u>Example</u>: Find all solutions to

$$y' = a_0 y - b_0 y^3$$

where a_0, b_0 are constants.

<u>Sol</u>.

The eq. is a Bernoulli eq. with constant coefficients.

$$\boxed{\frac{y'}{y^3} = \frac{a_0}{y^2} - b_0}$$

$$\boxed{v = \frac{1}{y^2}} = y^{-2}$$

$$v' = (-2)\, y^{-3}\, y' = -2\,\frac{y'}{y^3} \Rightarrow$$

$$\Rightarrow \boxed{\frac{y'}{y^3} = -\frac{1}{2} v'}$$

$$-\frac{1}{2} v' = a_0 v - b_0$$

$$\boxed{V' = -2a_0 V + 2b_0}$$ linear eq. for V.

$$V' + 2a_0 V = 2b_0$$ integrating factor method.

$$\mu(t) = e^{2a_0 t}$$

$$e^{2a_0 t} V' + (2a_0) e^{2a_0 t} V = 2b_0 e^{2a_0 t}$$

$$(e^{2a_0 t} V)' = 2b_0 e^{2a_0 t}$$

$$\int (e^{2a_0 t} V)' \, dt = \int 2b_0 e^{2a_0 t} \, dt + c$$

$$e^{2a_0 t} V(t) = 2b_0 \frac{1}{(2a_0)} e^{2a_0 t} + c$$

$$\boxed{V(t) = \frac{b_0}{a_0} + c\, e^{-2a_0 t}}$$

solution of the linear eq for V.

substitute back $\quad v(t) = \dfrac{1}{y^2(t)}$.

$$\boxed{\frac{1}{y^2(t)} = \frac{b_0}{a_0} + c\, e^{-2a_0 t}}$$

implicit form.

$$\frac{1}{y^2} = \frac{b_0}{a_0} + \frac{c}{e^{2a_0 t}}$$

$$= \frac{b_0\, e^{2a_0 t} + c a_0}{a_0\, e^{2a_0 t}}$$

$$y^2(t) = \frac{a_0\, e^{2a_0 t}}{b_0\, e^{2a_0 t} + a_0 c}$$

$$\boxed{y(t) = \pm \sqrt{\frac{a_0\, e^{2a_0 t}}{b_0\, e^{2a_0 t} + a_0 c}}}$$

explicit form.