mth 235 L7

Plan: * Properties of solutions to second order linear ODE with variable coefficients.

* Existence, uniqueness of sols. to IVP

* l.d., l.i. functions and Wronskians

* General and fundamental solutions of ODE.

* Abel's Theorem on the Wronskian.

(3.2)

* The main result on IVP.

Existence and uniqueness of solutions to IVP for second order, linear ODE with variable coefficients.

Thm (I): If the functions $a_1, a_0, b : (t_1, t_2) \to \mathbb{R}$ are continuous and $t_0 \in (t_1, t_2)$, $y_0, y_1 \in \mathbb{R}$, then there exists a unique solution $y : (t_1, t_2) \to \mathbb{R}$ to the IVP

$$\boxed{y'' + a_1(t)\, y' + a_0(t)\, y = b(t)}$$

$$\boxed{y(t_0) = y_0}\ , \quad \boxed{y'(t_0) = y_1}$$

Proof: Not given.

Remarks:

(1) Thrm I is a generalization to variable coefficients of a result proven in the previous section for constant coefficients

(2) Solutions to second order ODE require two integrations to be obtained, so a unique solution to an IVP requires two initial conditions.

(3) No explicit formula for the solution (unlike first order linear ODE)

* Example : Find the longest interval $(t_1, t_2) \subset \mathbb{R}$ such that the IVP

$$(t-1)\, y'' - 3t\, y' + 4\, y = t\,(t-1)$$

$$y(-2) = 2 \quad , \quad y'(-2) = 1$$

has a unique sol.

Sol:

$$y'' - \frac{3t}{(t-1)}\, y' + \frac{4}{(t-1)}\, y = t$$

$$\left.\begin{array}{l} a_1(t) = -\dfrac{3t}{(t-1)} \\ \\ a_0(t) = \dfrac{4}{(t-1)} \end{array}\right| \Rightarrow \text{ continuous in } \quad \begin{array}{l} I_1 = (-\infty, 1) \\ I_2 = (1, \infty) \end{array}$$

Initial condition at $t_0 = -2 \in I_1$

Answer : $\boxed{I_1 = (-\infty, 1)}$

* The rest of the class is dedicated to show:

If y_1, y_2 are not proportional to each other and they are solutions to

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0 \,, \qquad (1)$$

then any other solution y to (1) is given by

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

for appropriate $c_1, c_2 \in \mathbb{R}$

* We need few definitions.

- Proportional functions (linearly dependent.)

- Wronskian

- State a precise, more general result.

* Linear dependence - independence of functions

Def: Two continuous functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are called linearly dependent (ld) iff there exists a non-zero constant $c \in \mathbb{R}$ such that for all $t \in (t_1, t_2)$ holds

$$y_1(t) = c \; y_2(t).$$

The functions y_1, y_2 are called linearly independent (li) iff they are not ld.

Remarks:

(1) $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ ld $\Leftrightarrow$ exist $c_1, c_2 \in \mathbb{R}$, not both zero s.t.
$c_1 y_1(t) + c_2 y_2(t) = 0$
for $t \in (t_1, t_2)$.

(2) $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ li $\Leftrightarrow$ The only $c_1, c_2 \in \mathbb{R}$ sol. of
$c_1 y_1(t) + c_2 y(t) = 0$
for $t \in (t_1, t_2)$ are
$c_1 = c_2 = 0$.

(3) Not defined in the textbook.

Example

(1) $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$
are l.d.

(2) Show that $y_1(t) = \sin(t)$
$y_2(t) = \sin(2t)$
are l.i.

Sol.

(2) Find $c_1, c_2 \in \mathbb{R}$ s.t.

$$c_1 \sin(t) + c_2 \sin(2t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

for $t = \frac{\pi}{2}$ $\quad \sin\left(\frac{\pi}{2}\right) = 1$, $\sin(\pi) = 0$

$$\boxed{c_1 = 0}$$

Then

$c_2 \sin(2t) = 0$,

for $t = \frac{\pi}{4}$ $\quad \sin\left(2\,\frac{\pi}{4}\right) = 1$

$$\boxed{c_2 = 0}$$

$$\boxed{\text{So } y_1, y_2 \text{ are l.i.}}$$

* The Wronskian and ld-li functions

[The Wronskian is a function that determines whether two functions are l.d or l.i.]

Def. The **Wronskian** of continuously differentiable functions $Y_1, Y_2 : (t_1, t_2) \to \mathbb{R}$ is the function $W_{Y_1 Y_2} : (t_1, t_2) \to \mathbb{R}$

$$\boxed{W_{Y_1 Y_2}(t) = Y_1(t)\, Y_2'(t) - Y_2(t)\, Y_1'(t)}$$

Remark: - If $A(t) = \begin{bmatrix} Y_1(t) & Y_1'(t) \\ Y_2(t) & Y_2'(t) \end{bmatrix}$

then $\boxed{W_{Y_1 Y_2}(t) = \det(A(t))}$

- $\boxed{W_{Y_1 Y_2} = \begin{vmatrix} Y_1 & Y_1' \\ Y_2 & Y_2' \end{vmatrix}}$ $(= \det(A))$

Example : Find $W_{y_1 y_2}$ for

(1) $y_1 = \sin(t)$, $y_2 = \sin(2t)$ li

(2) $y_1 = \sin(t)$, $y_2 = 2\sin(t)$ ld

Sol.

(1) $$W_{y_1 y_2} = \begin{vmatrix} \sin(t) & \cos(t) \\ \sin(2t) & 2\cos(2t) \end{vmatrix}$$

$$= 2\cos(2t)\sin(t) - \sin(2t)\cos(t)$$

$$= 2\left[\cos^2(t) - \sin^2(t)\right]\sin(t)$$

$$- 2\sin(t)\cos(t)\cos(t)$$

$$\boxed{W_{y_1 y_2} = -2\sin^3(t)}$$ Not identically zero

(2) $$W_{y_1 y_2} = \begin{vmatrix} \sin(t) & \cos(t) \\ 2\sin(t) & 2\cos(t) \end{vmatrix}$$

$$= 2\sin(t)\cos(t) - 2\sin(t)\cos(t)$$

$$\boxed{W_{y_1 y_2} = 0}$$ identically zero.

Thrm: [The continuously differentiable functions $y_1, y_2 : (t_1, t_2) \to R$ are ld iff

$$W_{y_1 y_2}(t) = 0 \qquad \text{for all } t \in (t_1, t_2)$$
]

* [The wronskian determines whether two functions are ld or li.]

* Importance of the wronskian

(1) Sometimes is not simple to decide whether two functions are proportional.

(2) The Wronskian is useful to study properties of solutions to ODE without having the explicit form of these solutions.

see Abel's Thrm later on

Example of (1) : [Show that y_1, y_2 are l.d,

$$y_1 = \cos(2t) - 2\cos^2(t)$$

$$y_2 = \cos(2t) + 2\sin^2(t) \;]$$

Sol:

$$W_{y_1,y_2} = [\cos(2t) - 2\cos^2(t)]\,\underbrace{[-2\sin(2t) + 4\sin(t)\cos(t)]}_{=0}$$

$$- [\cos(2t) + 2\sin^2(t)]\,\underbrace{[-2\sin(2t) + 4\cos(t)\sin(t)]}_{=0}$$

$$\boxed{W_{y_1,y_2} = 0}$$

* On l.d - l.i solutions to linear ODE

Thm: (II) Let y_1, y_2 be continuously differentiable solutions of

$$\boxed{y'' + a_1(t)\, y' + a_0(t)\, y = 0} \quad (1)$$

Where a_1, a_0 are continuous.

Then, every solution y to eq. (1) can be decomposed as

$$\boxed{y(t) = c_1\, y_1(t) + c_2\, y_2(t)}$$

for appropriate $c_1, c_2 \in \mathbb{R}$

iff y_1, y_2 are l.i, that is,

iff $\boxed{W_{y_1 y_2} \neq 0}$.

Proof: In lecture notes, and textbook.

* Thrm I justifies the following:

Def: Two solutions y_1, y_2 to the homogeneous eq.

$$\boxed{y'' + a_1(t)\, y' + a_0(t)\, y = 0} \qquad (1)$$

are called a fundamental set of (1) iff y_1, y_2 are l.i.,

iff $\boxed{W_{y_1 y_2} \neq 0}$

Def: The functions

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

for $c_1, c_2 \in \mathbb{R}$ arbitrary, is called the general solution of (1).

* **Example**: Show that $y_1 = \sqrt{t}$, $y_2 = \frac{1}{t}$ form a fundamental set for

$$2t^2\, y'' + 3t\, y' - y = 0 \qquad (1)$$

Sol:

First, show that y_1, y_2 are solutions to (1)

$$y_1 = t^{1/2}, \qquad y_1' = \frac{1}{2} t^{-1/2}, \qquad y_1'' = -\frac{1}{4} t^{-3/2}$$

$$2t^2 \left(-\frac{1}{4}\right) t^{-3/2} + 3t\, \frac{1}{2} t^{-1/2} - t^{1/2} =$$

$$= -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0$$

y_1 is sol.

$$y_2 = t^{-1}, \qquad y_2' = -t^{-2}, \qquad y_2'' = 2t^{-3}$$

$$2t^2\, 2t^{-3} + 3t\left(-t^{-2}\right) - t^{-1} =$$

$$= 4t^{-1} - 3t^{-1} - t^{-1} = 0$$

y_2 is sol.

Second, show that y_1, y_2 are l.i

$$W_{y_1,y_2} = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & \frac{1}{2} t^{-1/2} \\ t^{-1} & -t^{-2} \end{vmatrix}$$

$$= -t^{-2} t^{1/2} - \frac{1}{2} t^{-1/2} t^{-1}$$

$$= -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$\boxed{W_{y_1,y_2} = -\frac{3}{2} t^{-3/2}}$$

y_1, y_2 are l.i.

* Abel's Thrm. (~ 1820)

Thrm: (II) Let $p, q : (t_1, t_2) \to \mathbb{R}$ be continuous functions and y_1, y_2 be two solutions of

$$\boxed{y'' + p(t)\, y' + q(t)\, y = 0.}$$

Then, $W_{y_1 y_2}$ is a continuously differentiable function satisfying the first order ODE

$$\boxed{W'_{y_1 y_2} + p(t)\, W_{y_1 y_2} = 0,}$$

and so, it is given by

$$\boxed{W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, e^{-\int_{t_0}^{t} p(s)\,ds}}$$

with $t_0 \in (t_1, t_2)$

Proof: in Lecture notes and textbook.

* One can know the Wronskian of two solutions without having these solutions

corollary : $W_{y_1,y_2}(t_1) = 0 \quad t_1 \in (t_1, t_2)$

iff

$W_{y_1,y_2}(t) = 0 \quad \text{for all} \quad t \in (t_1, t_2).$

Example : Find the Wronskian of two solutions of

$$t^2 y'' - t(t+2)\, y' + (t+2)\, y = 0$$

$t \neq 0$

Sol:

We do not need to find sols y_1, y_2.

$$y'' - \frac{(t+2)}{t}\, y' + \frac{(t+2)}{t^2}\, y = 0$$

$$\boxed{P(t) = \left(1 + \frac{2}{t}\right)} \quad , \quad q(t) = \frac{2+t}{t^2} .$$

So W satisfies the eq

$$\boxed{W' - \left(1 + \frac{2}{t}\right) W = 0}$$

first order linear ODE.

$$W' = \left(1 + \frac{2}{t}\right) W$$

$$\int_{t_0}^{t} \frac{W'(s)}{W(s)}\, ds = \int_{t_0}^{t} \left(1 + \frac{2}{s}\right) ds$$

$$\ln W(t) - \ln W(t_0) = (t - t_0) + 2\left[\ln t - \ln t_0\right]$$

$$\ln\left(\frac{W(t)}{W(t_0)}\right) = t - t_0 + 2\ln\left(\frac{t}{t_0}\right)$$

$$= t - t_0 + \ln\left(\frac{t^2}{(t_0)^2}\right)$$

$$\frac{W(t)}{W(t_0)} = e^{t-t_0} \quad e^{\ln(t^2/t_0^2)}$$

$$= e^{t-t_0} \quad \frac{t^2}{t_0^2}$$

$$\boxed{W(t) = W(t_0) \quad \frac{t^2}{t_0^2} \quad e^{(t-t_0)}}$$

$$\boxed{W(t) = c_0 \quad t^2 e^t}$$