mth 235 L6

Plan: * Second order linear ODE.

* Homogeneous eqs.

* Superposition property

* Constant coefficients

* Characteristic equation

* Main Result. (3.1)

* Second order linear differential eqs.

Def: Given functions $a_1, a_0, b : \mathbb{R} \to \mathbb{R}$, the eq. in the unknown $y : \mathbb{R} \to \mathbb{R}$

(1) $$\boxed{y'' + a_1(t)\, y' + a_0(t)\, y = b(t)}$$

is called a second order linear ODE. with variable coefficients.

Def: Eq (1) is called homogeneous iff

$$b(t) = 0 \qquad t \in \mathbb{R}$$

Def: Eq. (1) is called of constant coefficients iff a_1, a_0, b are constants.

* <u>Examples</u> : constant coefficients, homogeneous second order, linear ODE.

(1) $y'' + 5y' + 6y = 0$

(2) $2y'' - 3y' + y = 0$

* <u>Examples</u> : Variable coefficients, non-homogeneous, second order, linear ODE.

(1) $y'' + 2t\, y' - \frac{\ln(t)}{t}\, y = e^{3t}$

(2) Newton's second law of motion for a point particle of mass m moving in one space dimension under a force f.

$$\boxed{m\, y''(t) = f(t)}$$

$$(ma = f)$$

* The superposition property

Any linear combination of solutions to linear, homogeneous ODE is also a solution.

Propos. If the functions $y_1, y_2 : \mathbb{R} \to \mathbb{R}$ are solutions of the homogeneous linear ODE

$$\boxed{y'' + a_1(t)\, y' + a_0(t)\, y = 0} \tag{2}$$

then the function

$$\boxed{c_1\, y_1(t) + c_2\, y_2(t)}$$

is also a solution for all $c_1, c_2 \in \mathbb{R}$.

<u>Proof</u>: Show that $(c_1 y_1 + c_2 y_2)$ is sol. of (2).

$$(c_1 y_1 + c_2 y_2)'' + a_1(t)\,(c_1 y_1 + c_2 y_2)' + a_0(t)\,(c_1 y_1 + c_2 y_2) =$$

$$= c_1 \left[y_1'' + a_1(t)\, y_1' + a_0(t)\, y_1 \right]$$

$$+ c_2 \left[y_2'' + a_1(t)\, y_2' + a_0(t)\, y_2 \right]$$

$$= 0. \qquad \blacksquare$$

* Constant coefficients homogeneous ODE

The main ideas to find solutions of second order, constant coeff., homogeneous, linear ODE can be found by trial and error.

We show these ideas in the following:

* **Example**: [Find all functions

$$y(t) = e^{rt}$$

with r constant, solutions of

$$y'' + 5\,y' + 6\,y = 0. \qquad (3)$$

]

Remark: [We use exponentials since the factors e^{rt} can be canceled out from the eq.]

Sol: $y(t) = e^{rt} \Rightarrow y'(t) = r\, e^{rt}$

$y''(t) = r^2 e^{rt}$

$$r^2 e^{rt} + 5\, r e^{rt} + 6\, e^{rt} = 0$$

$$(r^2 + 5r + 6)\, e^{rt} = 0$$

$\boxed{r^2 + 5r + 6 = 0}$ (characteristic eq.)

r must be a root of

$\boxed{P(r) = r^2 + 5r + 6}$ (characteristic polynomial)

$$r_\pm = \frac{-5 \pm \sqrt{25 - 24}}{2}$$

$$r_\pm = \frac{-5 \pm 1}{2} \Rightarrow \boxed{r_+ = -2}, \ \boxed{r_- = -3}.$$

The functions

$$\boxed{y_1(t) = e^{-2t}} \quad , \quad \boxed{y_2(t) = e^{-3t}}$$

are solution of Eq. (3).

The superposition property implies

(4) $$\boxed{y(t) = c_1\, e^{-2t} + c_2\, e^{-3t}} \quad , \quad c_1, c_2 \in \mathbb{R}$$

is also a solution.

* Remarks

(1) There are two free constants in (4)

(2) The ODE (3) is second order, so two integrations must be done to find the solution. This explain the origin of the two free constants in (4).

(3) An IVP for eq (3) has a unique solution if the IVP contains two initial conditions.

Def: Given the ODE with constant coeff.

$$\boxed{y'' + a_1 y' + a_0 y = 0}, \qquad (5)$$

the characteristic polynomial and the characteristic equation associated with eq. (5) are respectively given by

$$\boxed{p(r) = r^2 + a_1 r + a_0}, \qquad \boxed{p(r) = 0}$$

The ideas given in the previous example can be summarized as follows:

Propos. If r_1, r_2 are the roots of the characteristic polynomial associated with eq. (5), then

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}} \qquad (6)$$

is solution of (5) for all $c_1, c_2 \in \mathbb{R}$.

* **Remark** :

[Later on we will see that the solutions given in (6) are indeed all solutions of eq. (5).]

This property motivates the following :

Def : [The function in (6) is called the general solution of eq. (5).]

* <u>Example</u>: Find the sol. to the IVP

$$\left[\begin{array}{l} y'' + 5y' + 6y = 0 \\ y(0) = 1 , \quad y'(0) = -1 \end{array}\right] \begin{array}{r} (7) \\ (8) \end{array}$$

<u>Sol</u>:

First find the general sol. of (7).

We have already found that:

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-3t}.} \quad \text{(General sol.)}$$

therefore:

$$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

The initial condition (8) implies

$$\left| \begin{array}{l} 1 = y(0) = c_1 + c_2 \\ -1 = y'(0) = -2c_1 - 3c_2 \end{array} \right| \Rightarrow \boxed{c_2 = 1 - c_1}$$

$$-1 = -2c_1 - 3(1-c_1) \Rightarrow -1 = -2c_1 - 3 + 3c_1$$

$$\boxed{c_1 = 2} \quad \boxed{c_2 = -1} \quad \Rightarrow \quad \boxed{y(t) = 2e^{-2t} - e^{-3t}}$$

* **Example:** $\left[\begin{array}{l}\text{Find the general sol. of} \\ 2y'' - 3y' + y = 0.\end{array}\right]$

Sol.

we look for solutions of the form:

$$y(t) = e^{rt} \Rightarrow y'(t) = r e^{rt}, \quad y''(t) = r^2 e^{rt}.$$

$$(2r^2 - 3r + 1) e^{rt} = 0$$

$$\boxed{p(r) = 2r^2 - 3r + 1} \qquad \boxed{p(r) = 0}$$

$$r = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$$

$$\boxed{r_1 = 1}, \quad \boxed{r_2 = \frac{1}{2}}$$

The general sol. is

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{t/2}} \qquad c_1, c_2 \in \mathbb{R}.$$

* The main result.

Thm: Given $a_1, a_0 \in \mathbb{R}$, consider the ODE

$$\boxed{y'' + a_1\, y' + a_0\, y = 0\,.} \qquad (9)$$

Let r_1, r_2 be roots of the characteristic polynomial

$$\boxed{p(r) = r^2 + a_1\, r + a_0}$$

Let $c_1, c_2 \in \mathbb{R}$ be arbitrary.

Then, every solution of (9) belongs to only one of the following cases:

(a) If $r_1 \neq r_2$ and $r_1, r_2 \in \mathbb{R}$, then

$$\boxed{y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.}$$

(b) If $r_1 \neq r_2$ and

$\boxed{r_1 = \alpha_0 + i\beta_0}$, $\boxed{r_2 = \alpha_0 - i\beta_0}$.

then

$$\boxed{y(t) = \left[c_1 \cos(\beta_0 t) + c_2 \sin(\beta_0 t)\right] e^{\alpha_0 t}}.$$

(c) If $r_1 = r_2 \in \mathbb{R}$, then

$$\boxed{y(t) = (c_1 + c_2 t)\, e^{r_1 t}}.$$

Furthermore, the IVP for (9) with initial conditions

$$y(t_0) = y_0 \quad , \quad y'(t_0) = y_1$$

has a unique sol. for any $y_0, y_1 \in \mathbb{R}$.

<u>Remark</u>: case (a) $\rightarrow$ (3.1)

case (b) $\rightarrow$ (3.3)

case (c) $\rightarrow$ (3.4)