mth 235 L 5

Plan:

* Exact ODE
* Definition, Examples
* Main Result.
* Generalizations: Integrating factor method.

(2.6)

* Exact ODE

Def: Given $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions

$$M, N : R \to \mathbb{R},$$

the ODE in the unknown $y : (t_1, t_2) \to \mathbb{R}$

$$\boxed{N(t, y(t))\, y'(t) + M(t, y(t)) = 0}$$

is called exact iff holds

$$\boxed{\partial_t N(t,u) = \partial_u M(t,u)}.$$

Notation:
$$\partial_t N = \frac{\partial}{\partial t} N$$

$$\partial_u M = \frac{\partial}{\partial u} M$$

<u>Example</u>: Show that the ODE below is exact:

$$2t\, y(t)\, y'(t) + 2t + y^2(t) = 0$$

<u>Sol</u>:

$$\underbrace{[\,2t\,y\,]}_{N(t,y)}\, y' + \underbrace{[\,2t + y^2\,]}_{M(t,y)} = 0$$

$$N(t,y) = 2t\,y \quad \Rightarrow \quad \partial_t N(t,y) = 2y$$

$$M(t,y) = 2t + y^2 \quad \Rightarrow \quad \partial_y M(t,y) = 2y$$

$$\boxed{\partial_t N = \partial_y M}$$

The ODE is exact.

<u>Remark</u>: The ODE is non-linear.

is not separable.

$\underline{\text{Example}}$: Is the ODE below exact ?

$$\sin(t)\, y' + t^2 e^y y' - y' = -y\cos(t) - 2t\, e^y.$$

$\underline{\text{Sol}}$:

$$\underbrace{\left[\sin(t) + t^2 e^y - 1\right]}_{N} y' + \underbrace{\left[y\cos(t) + 2t\, e^y\right]}_{M} = 0$$

$$N(t,y) = \sin(t) + t^2 e^y - 1$$

$$\underline{\partial_t N = \cos(t) + 2t\, e^y}$$

$$M(t,y) = y\cos(t) + 2t\, e^y$$

$$\underline{\partial_y M = \cos(t) + 2t\, e^y}$$

$$\boxed{\partial_t N = \partial_y M}$$

ODE is exact.

Example : [Is the ODE below exact ?

$$y' = -a(t)\, y + b(t).$$]

Sol:

$$\underbrace{y'}_{N} + \underbrace{[\, a(t)\, y - b(t) \,]}_{M} = 0$$

$$N(t,y) = 1 \quad \Rightarrow \quad \partial_t N = 0$$

$$M(t,y) = a(t)y - b(t) \quad \Rightarrow \quad \partial_y M = a(t)$$

$$\boxed{\partial_t N \neq \partial_y M} \qquad a(t) \neq 0$$

The ODE is not exact for $a(t) \neq 0$.

* Recall a result from Calculus.

Thrm: The continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy

$$\boxed{\partial_t N(t,u) = \partial_u M(t,u)}$$

iff there exists a twice continuously differentiable function

$$\psi : R \to \mathbb{R}$$

satisfying

$$\boxed{\begin{array}{c} \partial_u \psi(t,u) = N(t,u) \\ \hline \partial_t \psi(t,u) = M(t,u) \end{array}}.$$

Proof: $(\Leftarrow)$ Simpler:

$$\left.\begin{array}{l} \partial_t N = \partial_t \partial_u \psi \\ \partial_u M = \partial_u \partial_t \psi \end{array}\right] \Rightarrow \partial_t N = \partial_u M$$

$(\Rightarrow)$ Difficult: Poincaré, 1880.

Example:

Recall that the ODE below is exact,

$$2t\,y\,y' + 2t + y^2 = 0$$

because

$$\left.\begin{array}{lll} N = 2ty & \Rightarrow & \partial_t N = 2y \\ M = 2t + y^2 & \Rightarrow & \partial_y M = 2y \end{array}\right] \Rightarrow \underline{\partial_t N = \partial_y M}.$$

The function $\psi(t,y)$ in this case is:

$$\boxed{\psi(t,y) = t^2 + t\,y^2}$$

because:

$$\boxed{\begin{array}{ll} \partial_t \psi = 2t + y^2 & = M \\ \partial_y \psi = 2ty & = N \end{array}}$$

[The function ψ is crucial to find implicit solutions to the ODE.]

* The main result

Thrm: The solution $y : (t_1, t_2) \to \mathbb{R}$ to the exact ODE

$$\boxed{N(t, y(t))\, y'(t) + M(t, y(t)) = 0}$$

can be given implicitly by

$$\boxed{\Psi(t, y(t)) = c}$$

where $\Psi : R \to \mathbb{R}$ is a function satisfying

$$\boxed{\begin{array}{c} \partial_u \Psi(t, u) = N(t, u) \\ \hline \partial_t \Psi(t, u) = M(t, u) \end{array}}$$

and c is a constant.

Proof: $N \; y' + M = 0$

$\partial_t N = \partial_y M \Rightarrow$ exists ψ such that

$$\partial_y \psi = N$$

$$\partial_t \psi = M$$

Therefore:

$$(\partial_y \psi) \; y' + (\partial_t \psi) = 0$$

$$\frac{d}{dt}\left[\psi(t, y(t)) \right] = 0$$

$$\boxed{\psi(t, y(t)) = c}$$

∎

Example: Find all sols. to

$$2tY\ Y' + 2t + Y^2 = 0$$

Sol:

Recall that the eq. is exact:

$N = 2tY \Rightarrow \partial_t N = 2Y$

$M = 2t + Y^2 \Rightarrow \partial_y M = 2Y$.

Then, there exists ψ satisfying

$$\boxed{\begin{array}{l} N = \partial_y \psi \quad (1) \\ \hline M = \partial_t \psi \quad (2) \end{array}}$$

We now find ψ

$(1) \Rightarrow \quad \partial_y \psi = 2tY$

$$\int (\partial_y \psi)\, dY = \int 2t\, Y\, dY + g(t)$$

$$\psi = 2t \int Y\, dY + g(t)$$

$$\psi(t,Y) = 2t\, \frac{Y^2}{2} + g(t)$$

$$\boxed{\psi = t\,y^2 + g(t)}$$

(2) $$\partial_t \psi = M$$

$$y^2 + g'(t) = 2t + y^2$$

$$g'(t) = 2t$$

$$\boxed{g(t) = t^2}$$

Therefore : $$\boxed{\psi(t,y) = t\,y^2 + t^2}$$

The solutions $y(t)$ of the ODE satisfy

$$\boxed{t\,y^2(t) + t^2 = c}\ , \quad c \in \mathbb{R}.$$

implicit expression.

$$\boxed{\psi(t, y(t)) = c}$$

<u>Example</u>: Find all solutions of

$$(\sin(t) + t^2 e^y - 1)\, y' + y\cos(t) + 2t e^y = 0$$

<u>Sol</u>:

Recall that the eq. is exact:

$N = \sin(t) + t^2 e^y - 1 \quad \Rightarrow$

$$\Rightarrow \quad \boxed{\partial_t N = \cos(t) + 2t e^y}$$

$M = y\cos(t) + 2t e^y \quad \Rightarrow$

$$\Rightarrow \quad \boxed{\partial_y M = \cos(t) + 2t e^y}$$

Therefore, there exists ψ satisfying

$$\boxed{\begin{array}{l} N = \partial_y \psi \quad (1) \\ M = \partial_t \psi \quad (2) \end{array}}$$

We now find ψ

(1) $\Rightarrow$ $\partial_y \Psi = N$

$$\partial_y \Psi = \sin(t) + t^2 e^y - 1$$

$$\boxed{\Psi(t,y) = y\sin(t) + t^2 e^y - y + g(t)}$$

(2) $\Rightarrow$ $\partial_t \Psi = M$

$$y\cos(t) + 2te^y + g'(t) = y\cos(t) + 2te^y$$

$$g'(t) = 0$$

$$g(t) = c_0$$

$$\boxed{\Psi(t,y) = y\sin(t) + t^2 e^y - y + c_0}$$

Therefore, the sols. $y(t)$ satisfy

$$\boxed{y(t)\sin(t) + t^2 e^{y(t)} - y(t) = c_1}$$

implicit expression.

* Generalization: Integrating factor method.

There exist Non-exact ODE that can be converted into exact ODE by multiplying the equation by an appropriate function, called integrating factor.

Thrm: If the ODE

$$\left[N(t,y)\, y'(t) + M(t,y) = 0 \right]$$

is not exact, that is,

$$\left[\partial_t N \neq \partial_y M \right],$$

and if the function

$$\boxed{\frac{1}{N(t,y)} \left[\partial_y M(t,y) - \partial_t N(t,y) \right]}$$

does NOT depend on y,

Then the equation

$$\boxed{\mu(t)\, N(t,y)\, y' + \mu(t)\, M(t,y) = 0}$$

is exact, where $\mu(t)$ is solution of

$$\boxed{\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t,y)} \left[\partial_y M(t,y) - \partial_t N(t,y) \right].}$$

<u>Example</u>: Find all sols. y to the eq.

$$(3ty+y^2) + (t^2+ty)\,y' = 0$$

<u>Sol</u>:
Verify if the eq. is exact

$$\underbrace{(t^2+ty)}_{N}\,y' + \underbrace{(3ty+y^2)}_{M} = 0$$

$$N = t^2 + ty \quad \Rightarrow \quad \partial_t N = 2t + y$$

$$M = 3ty + y^2 \quad \Rightarrow \quad \partial_y M = 3t + 2y$$

$$\boxed{\partial_t N \neq \partial_y M}$$

The eq. is NOT exact.

However:

$$\frac{1}{N}\left(\partial_y M - \partial_t N\right) =$$

$$= \frac{1}{(t^2 + tY)}\left[(3t+2Y) - (2t+Y)\right]$$

$$= \frac{1}{(t^2+tY)}\,(t+Y)$$

$$= \frac{1}{t\,(t+Y)}\,(t+Y)$$

$$= \frac{1}{t} \Rightarrow \boxed{\frac{1}{N}\left(\partial_y M - \partial_t N\right) = \frac{1}{t}}$$

(independent of Y)

So, there exists $\mu(t)$ sol. of

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{N}\left(\partial_y M - \partial_t N\right) = \frac{1}{t}$$

$$\boxed{\frac{\mu'}{\mu} = \frac{1}{t}}$$

We find an integrating factor μ.

$$\int \frac{\mu'}{\mu}\, dt = \int \frac{1}{t}\, dt$$

$$\ln \mu(t) = \ln(t)$$

$$\boxed{\mu(t) = t}$$

Multiply the original ODE by $\mu = t$.

$$\underbrace{(t^3 + t^2 y)}_{\tilde{N}}\, y' + \underbrace{(3t^2 y + t y^2)}_{\tilde{M}} = 0$$

Verify that the new ODE is exact:

$$\tilde{N} = t^3 + t^2 y \quad \Rightarrow \quad \partial_t \hat{N} = 3t^2 + 2ty$$

$$\bar{M} = 3t^2 y + t y^2 \quad \Rightarrow \quad \partial_y \bar{M} = 3t^2 + 2ty$$

$$\boxed{\partial_t \hat{N} = \partial_y \tilde{M}}$$

Since $\boxed{\tilde{N}(t,y)\, y' + \tilde{M}(t,y) = 0}$

is exact, we find the solution in the usual way.

Find ψ sol. of

$$\boxed{\begin{array}{c} \partial_y \psi = \tilde{N} \\ \hline \partial_t \psi = \tilde{M} \end{array}} \quad \begin{array}{c} (1) \\ (2) \end{array}$$

$(1) \Rightarrow \quad \partial_y \psi = t^3 + t^2 y$

$$\boxed{\psi = t^3 y + \frac{t^2}{2} y^2 + g(t)}$$

$(2) \Rightarrow$

$$3t^2 y + t y^2 + g'(t) = 3t^2 y + t y^2$$

$$g'(t) = 0$$

$$\boxed{g(t) = c_0}$$

$$\boxed{t^3 y(t) + \frac{t^2}{2} y^2(t) = c_1}$$