mth 235 L4

Plan :

* Comparison
  Linear – Non-linear ODE

* Review : Linear ODE

* Non-linear ODE

* Properties of Solutions
  to non-linear ODE.

(2.4)

* Review: Linear ODE

Thrm (I): Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$, wit $t_2 > t_1$, and given constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, the IVP in the unknown $y : (t_1, t_2) \to \mathbb{R}$ given by

$$\boxed{y' = -a(t)\, y + b(t)}$$

$$\boxed{y(t_0) = y_0}$$

has the unique solution

(1)
$$\boxed{y(t) = \frac{1}{\mu(t)} \left[y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \right]}$$

with

$$\mu(t) = e^{A(t)}, \quad A(t) = \int_{t_0}^{t} a(s)\, ds.$$

Proof: base on the integrating factor method.

* [Thrm I assumes that the coefficients a, b are continuous in $(t_1, t_2) \subset \mathbb{R}$]

* Thrm I implies that the solutions to linear ODE satisfy:

(1) There is an explicit expression for the solution $y(t)$ (Eq (1)).

(2) For every initial condition $y_0 \in \mathbb{R}$ there is a unique solution.

(3) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution y is defined for all $t \in (t_1, t_2)$.

* None of these properties holds for solutions to Non-linear ODE

* Non-linear ODE

Def : [A differential eq.

$$y'(t) = f(t, y(t))$$

is called non-linear iff $f(t, u)$ is non-linear on its second argument.]

Examples

(1) $$y' = \frac{t^2}{y^3}$$

(2) $$y' = 2t\, y + \ln(y)$$

(3) See Sect. 2.2.

* <u>Recall the notation</u>

- Y, $Y(t)$ reserved for solutions of an ODE.

- $u \in \mathbb{R}$ denotes an independent variable.

<u>Example</u>

(1) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(t, u)$.

(2) $f(t, Y(t))$: evaluation of f at t and the solution Y of the ODE

$$Y'(t) = f(t, Y(t)).$$

Thm: (II) Fix a rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, and let $f : R \to \mathbb{R}$ be a function $(t, u) \longmapsto f(t, u)$.

If f, $\frac{\partial f}{\partial u}$ are continuous on R,

and $(t_0, y_0) \in R$,

then there exists $\hat{R} \subset R$ with $(t_0, y_0) \in \hat{R}$ such that the IVP

$$\boxed{y'(t) = f(t, y(t))}$$

$$\boxed{y(t_0) = y_0}$$

has a unique solution on $\hat{R}$.

<u>Proof</u>: Not given.

* **Comparison of Thrms I and II.**

(1) No explicit expression for the solution y in II.

(2) Non-uniqueness of solutions to IVP may happen at points (t, u) where $\frac{\partial f}{\partial u}(t,u)$ is not continuous.

(3) A change in the initial condition y_0 may change the domain where the solution $y(t)$ is defined.

* Example. (1) [Find all solutions of
$$y'(t) = \frac{t^2}{1 + y^4(t)}$$
]

(Separable non-linear ODE)

Sol.

$$(1 + y^4)\, y' = t^2$$

$$\int (1 + y^4)\, y' \, dt = \int t^2 \, dt + c$$

$$\int (1 + u^4) \, du = \int t^2 \, dt + c$$

$$u + \frac{u^5}{5} = \frac{t^3}{3} + c$$

$$\boxed{\frac{y^5(t)}{5} + y(t) = \frac{t^3}{3} + c}$$

implicit expression only.

* <u>Example</u> (2)

$$\left[\begin{array}{l} \text{Find all sols. IVP} \\ y'(t) = y^{1/3}(t) \\ y(0) = 0 \end{array}\right]$$

(separable non-linear ODE)

<u>Notice</u>:

$$\left[\begin{array}{ll} f(t,u) = u^{1/3} & \\ \dfrac{\partial f}{\partial u} = \dfrac{1}{3}\dfrac{1}{u^{2/3}} & \text{. Not continuous at } u = 0. \end{array}\right]$$

Solution of IVP exists but it is not unique

<u>First</u> <u>sol.</u> $\boxed{y_1(t) = 0}$, $t \in \mathbb{R}$

Second sol.

$$y' = y^{1/3}$$

$$y^{-1/3}\, y' = 1$$

$$\int y^{-1/3}\, y'\, dt = \int dt + c$$

$$\int u^{-1/3}\, du = \int dt + c$$

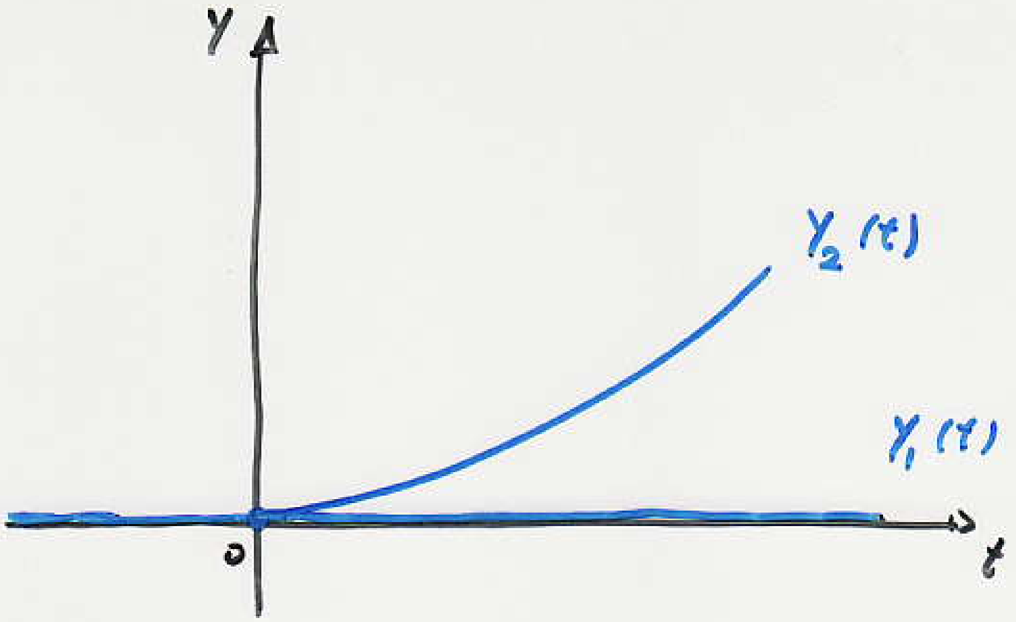
$$\frac{3}{2}\, u^{2/3} = t + c$$

$$\frac{3}{2}\, \left[y(t)\right]^{2/3} = t + c$$

$$\boxed{y(t) = \left[\frac{2}{3}\,(t+c)\right]^{3/2}}$$

$$0 = y(0) = \left[\frac{2}{3}\, c\right]^{3/2} \Rightarrow \boxed{c = 0}$$

$$\boxed{y_2(t) = \left[\frac{2}{3}\, t\right]^{3/2}}$$



The IVP has two solutions

* Example (3)

$$\left[\begin{array}{l} \text{Find } y \text{ sol. of IVP} \\ y'(t) = y^2(t) \\ y(0) = y_0 \end{array}\right]$$

(separable non-linear ODE)

Sol.

$$\frac{y'}{y^2} = 1$$

$$\int \frac{y'}{y^2}\,dt = \int dt + c$$

$$\int \frac{du}{u^2} = \int dt + c$$

$$-\frac{1}{u} = t + c$$

$$-\frac{1}{y(t)} = t + c$$

$$\boxed{y(t) = -\frac{1}{t+c}}$$

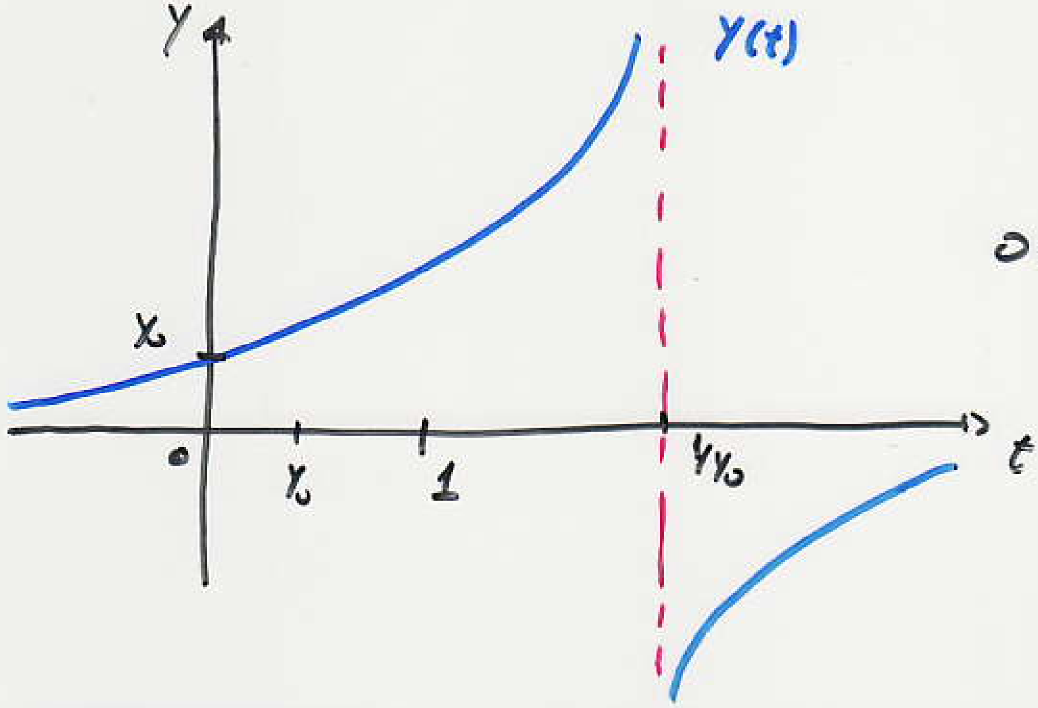
$$y_0 = y(0) = -\frac{1}{c} \Rightarrow \boxed{c = -\frac{1}{y_0}}$$

$$y(t) = -\frac{1}{\left(t - \frac{1}{y_0}\right)}$$

$$\boxed{y(t) = \frac{1}{\left(\frac{1}{y_0} - t\right)}}$$

$y \to \pm\infty$

$t \to \frac{1}{y_0}$



$0 < y_0 < 1$

$\boxed{\text{Domain: } \left(-\infty, \frac{1}{y_0}\right) \text{ (Physics)}}$

$\boxed{\text{Domain } \left(-\infty, \frac{1}{y_0}\right) \cup \left(\frac{1}{y_0}, +\infty\right)}$

* **Example** : [consider the non-linear ODE

$$y'(t) = \frac{1}{(t-1)(t+1)\,(y(t)-2)\,(y(t)+3)}.$$

Find the regions on $\mathbb{R}^2$ where Thrm II does not hold.]

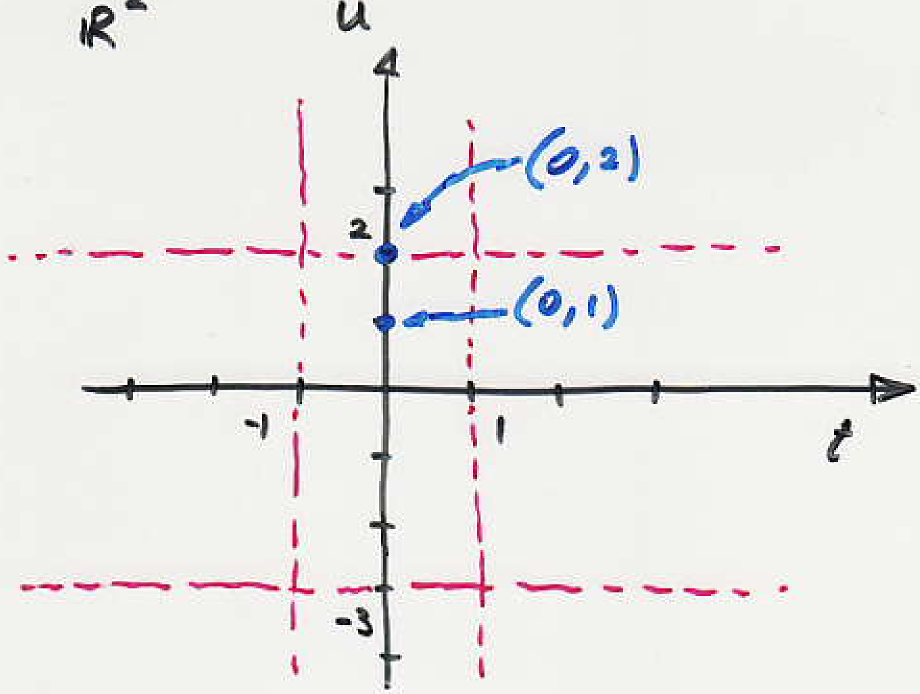
Sol.

$$f(t,u) = \frac{1}{(t-1)(t+1)\,(u-2)\,(u+3)}$$

not defined on the lines

$$\boxed{t=1, \quad t=-1, \quad u=2, \quad u=-3}.$$

So, f is not continuous on these lines.



$Y(0) = 2$

$Y(0) = 1$

(a) example : For the initial condition

$$Y(0) = 2$$

the hypotheses of Thrm II are Not satisfied.

(b) example : For the initial condition

$$Y(0) = 1$$

Thrm II holds on the rectangle

$$R = (-1, 1) \times (-3, 2).$$