mth 235 L3

Plan: * Application: Physics
(Mathematical modeling)

* Main Example:
Salt in a water tank.

(2.3)

* Application: Physics

- Physics studies Natural processes.
- Mathematical description of this process

  (Differential eq.)

- Analysis of the mathematical description

  (Find solutions of differential eqs. and study such solutions.)

- The mathematical description predicts the behaviour of the natural process.
- Compare predictions with nature.

* Problem: Test the conservation of mass law.

* Particular Situation: Salt concentration in water.

* Main ideas of the test.

  - We assume that the mass of both salt and water is conserved.

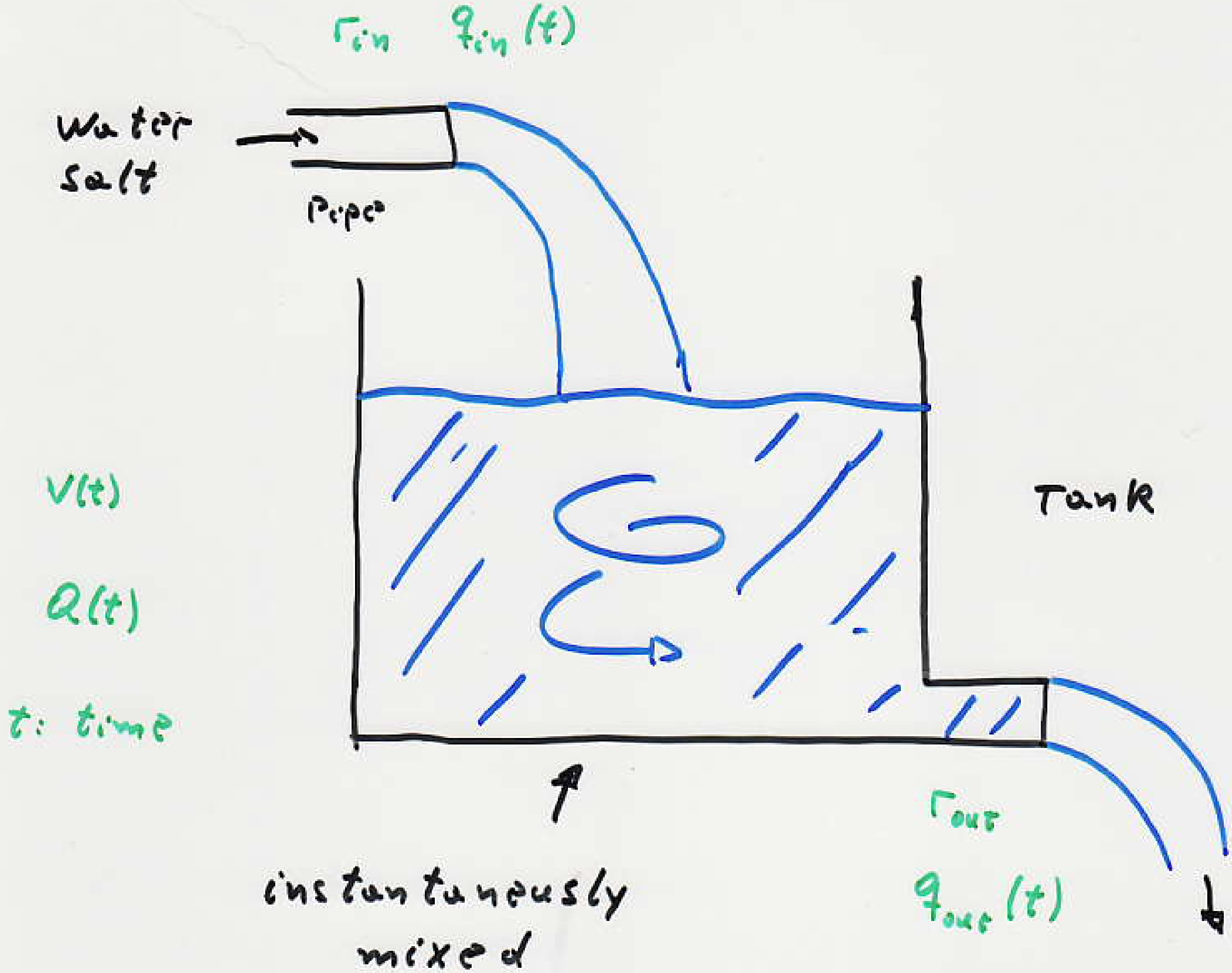
  - With that assumption we construct a mathematical model of certain situation

  - We study the predictions of this mathematical description

  - [If the predictions agree with the observation of the natural process,

    Then we conclude that the conservation of mass law holds for water and salt.]

* The Experimental Device.



r_{in} , r_{out} : constants

(time independents)

r_{in}, r_{out} : rates in and out of water entering and leaving the tank.

q_{in}, q_{out} : Salt concentration entering and leaving the tank.

$V(t)$: Water volume in tank.

$Q(t)$: Salt mass in tank.

* **Remark:** **Units**

$$[r_{in}] = [r_{out}] = \frac{\text{Volume}}{\text{Time}}.$$

$$[q_{in}] = [q_{out}] = \frac{\text{Mass}}{\text{Volume}}.$$

$$[V(t)] = \text{Volume}.$$

$$[Q(t)] = \text{Mass}.$$

* The mass conservation law provides the main eqs. of the mathematical description.

* **Main eqs.**

(1) $$\frac{dV(t)}{dt} = r_{in} - r_{out}$$

(2) $$\frac{dQ(t)}{dt} = r_{in}\, q_{in}(t) - r_{out}\, q_{out}(t)$$

Mass conservation

(3) $$q_{out}(t) = \frac{Q(t)}{V(t)}$$

instantaneously mixed hypothesis

(4) r_{in}, r_{out} : constants

* **Remark**: $$\left[\frac{dV}{dt}\right] = \frac{\text{Volume}}{\text{Time}} = [r]$$

$$[r\,q] = [r]\,[q] = \frac{\text{Volume}}{\text{Time}}\;\frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}$$

$$\left[\frac{dQ}{dt}\right] = \frac{\text{Mass}}{\text{Time}} = [r\,q]$$

* Analysis of the mathematical model

Eqs. (4), (1) imply:

$$\boxed{V(t) = (r_{in} - r_{out})\, t + V_0} \tag{5}$$

where : $\boxed{V(0) = V_0}$ initial volume of water in tank.

Eqs. (3), (2) imply:

$$\boxed{\frac{dQ(t)}{dt} = r_{in}\, q_{in}(t) - r_{out}\, \frac{Q(t)}{V(t)}} \tag{6}$$

Eqs. (6), (5) imply:

$$\boxed{\frac{dQ(t)}{dt} = r_{in}\, q_{in}(t) - \frac{r_{out}}{(r_{in} - r_{out})\, t + V_0}\, Q(t)}$$

Main eq. for $Q(t)$

* <u>Notation</u>

$$a(t) = \frac{r_{out}}{(r_{in} - r_{out})\, t + V_0}$$

given functions

$$b(t) = r_{in} \; q_{in}(t).$$

$$\boxed{Q'(t) = -\, a(t)\; Q(t) \; + b(t)}$$

Main Eq. for $Q(t)$

Linear ODE for $Q(t)$

solution: Integrating factor method.

$$Q(t) = \frac{1}{\mu(t)} \left[Q_0 + \int_0^t \mu(s)\, b(s)\, ds \right]$$

$$\mu(t) = e^{A(t)}, \quad A(t) = \int_0^t a(s)\, ds$$

$$Q(0) = Q_0$$

* Predictions for particular situations

case 1 : $\quad r_{in} = r_{out} = r$

(Example 1) $\quad q_{in}$: constant

Problem : $\left[\begin{array}{l}\text{If } r,\ q_{in},\ a_0,\ V_0 \text{ are given,} \\ \text{then find } Q(t).\end{array}\right]$

Sol :

$$Q'(t) = -a(t)\,Q(t) + b(t)$$

$$a(t) = \frac{r_{out}}{(r_{in} - r_{out})\,t + V_0} \Rightarrow \boxed{a(t) = \frac{r}{V_0} = a_0}$$

$$b(t) = r_{in}\, q_{in}(t) \Rightarrow \boxed{b(t) = r\, q_{in} = b_0}$$

IVP

$$\left[\begin{array}{l} Q'(t) = -a_0 \; Q(t) + b_0 \\ Q(0) = Q_0 \end{array}\right] \quad \begin{array}{r}(1)\\(2)\end{array}$$

integrating factor method.

$$Q(t) = \frac{1}{\mu(t)} \left[Q_0 + \int_0^t \mu(s) \, b_0 \, ds \right]$$

$$\mu(t) = e^{a_0 t} ,$$

$$\int_0^t \mu(s) \, b_0 \, ds = \frac{b_0}{a_0} \left(e^{a_0 t} - 1 \right)$$

$$\boxed{Q(t) = \left(Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}}$$

$$\frac{b_0}{a_0} = (r\, q_{in}) \left(\frac{V_0}{r}\right) \Rightarrow \boxed{\frac{b_0}{a_0} = q_{in} V_0}$$

$$\boxed{Q(t) = \left(Q_0 - q_{in} V_0\right) e^{-\frac{r t}{V_0}} + q_{in} V_0}$$

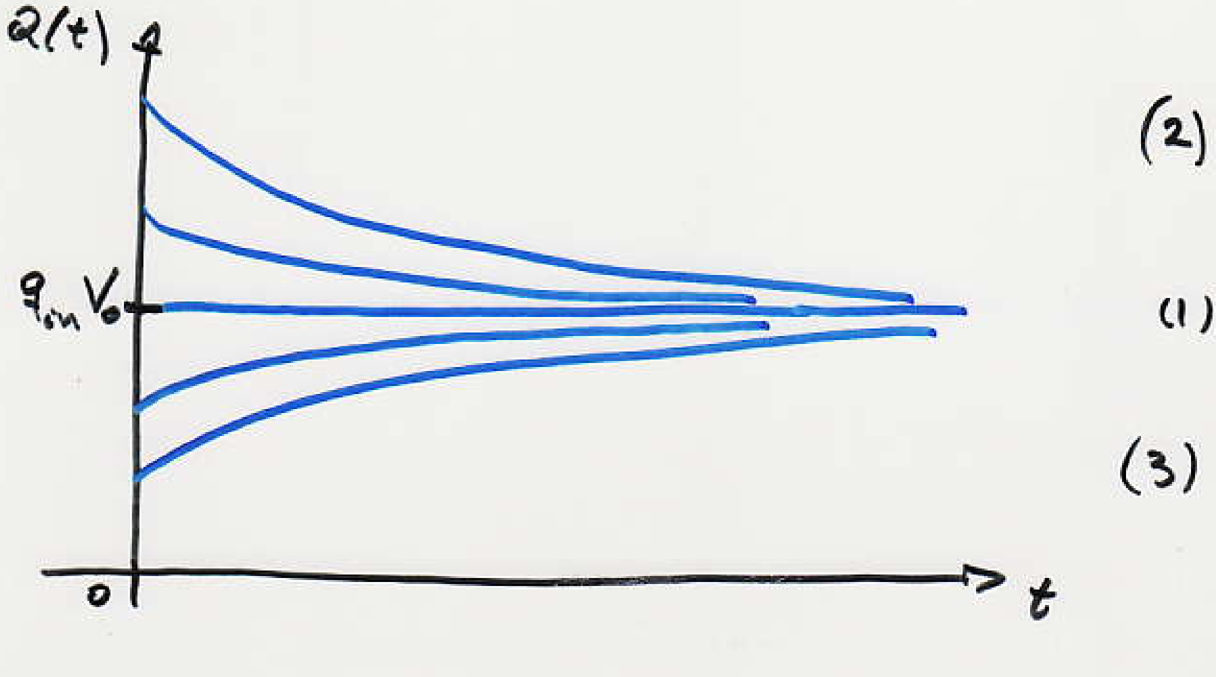
(predictions for $r_{in} = r_{out} = r$, q_{in} : const.)

<u>Particular cases</u>

(1) $\frac{Q_0}{V_0} = q_{in} \Rightarrow Q(t) = q_{in} V_0 = Q_0$

(2) $\frac{Q_0}{V_0} > q_{in}$

(3) $\frac{Q_0}{V_0} < q_{in}$



Case 2: V_0 given $(200\ \ell)$

(Probl. 1) $\frac{a_0}{V_0}$ given $\left(1\ \frac{gr}{\ell}\right)$

$q_{in} = 0$ (fresh water coming in.)

$r_{in} = r_{out} = r$ $\left(2\ \frac{\ell}{min}\right)$

Problem: [Find t_1 s.t. $\frac{Q(t)}{V(t)}$ is 1% the initial value.]

Sol.:

$$Q'(t) = -a(t)\,Q(t) + b(t).$$

$$b(t) = r_{in}\,q_{in} = 0 \quad \Rightarrow \quad \boxed{b(t) = 0}$$

$$a(t) = \frac{r_{out}}{(r_{in} - r_{out})\,t + V_0} \quad \Rightarrow \quad \boxed{a(t) = \frac{r}{V_0} = a_0}$$

IVP

$$\left[\begin{array}{l} Q'(t) = -a_0 \; Q(t) \\ Q(0) = Q_0 \end{array}\right]$$

$$Q(t) = \left(Q_0 - \frac{b_0}{a_0}\right) e^{-a_0 t} + \frac{b_0}{a_0} \implies$$

$$Q(t) = Q_0 \; e^{-a_0 t}$$

$$\boxed{Q(t) = Q_0 \; e^{-\frac{r t}{V_0}}} \qquad (1)$$

Find t_1 : $\quad \boxed{\dfrac{Q(t_1)}{V(t_1)} = \dfrac{Q_0}{V_0} \; \dfrac{1}{100}} \qquad (2)$

Recall $\boxed{V(t) = V_0}$ $\quad (r_{in} = r_{out})$

$$\frac{Q(t_1)}{V_0} = \frac{Q_0}{V_0} \; \frac{1}{100}$$

$$\boxed{Q(t_1) = \frac{Q_0}{100}}$$ condition for t_1

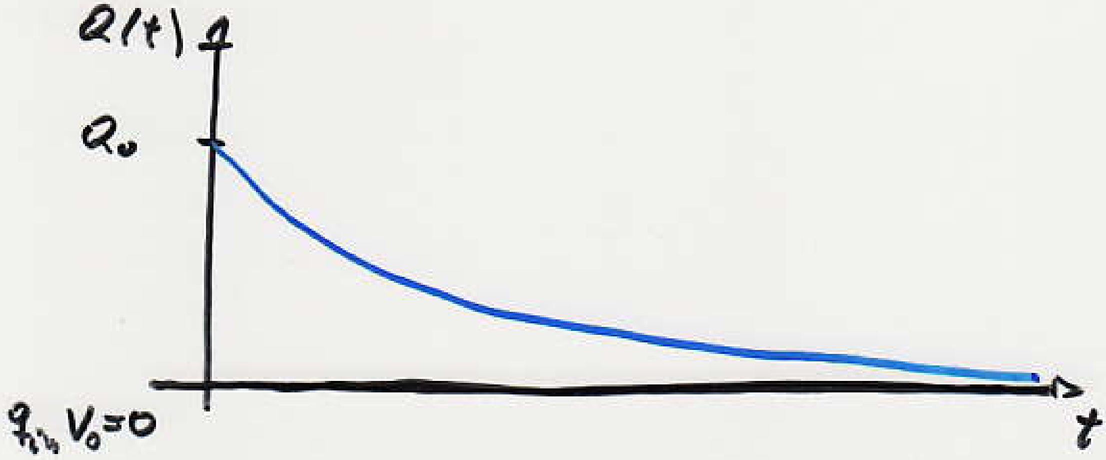
$$\frac{Q_0}{100} = Q(t_1) = Q_0 \, e^{-\frac{\Gamma}{V_0} t_1}$$

$$\frac{1}{100} = e^{-\frac{\Gamma}{V_0} t_1}$$

$$\ln\left(\frac{1}{100}\right) = -\frac{\Gamma}{V_0} t_1$$

$$\ln(100) = \frac{\Gamma}{V_0} t_1$$

$$\boxed{t_1 = \frac{V_0}{\Gamma} \ln(100)}$$



$Q_0 = 0$ (initially fresh water)

Case 3: V_0 given (10^6 gal)

(Example 3). $r_{in} = r_{out} = r$ given $\left(5 \times 10^6 \frac{gal}{Year}\right)$

$$q_{in}(t) = 2 + \sin(2t) \quad \left(\frac{gr}{gal}\right)$$

Problem: [Find $Q(t)$.]

Sol. $$Q'(t) = -a(t)\, Q(t) + b(t)$$

$$a(t) = \frac{r_{out}}{(r_{in} - r_{out})\,t + V_0} \Rightarrow \boxed{a(t) = \frac{r}{V_0} = a_0}$$

$$b(t) = r_{in}\, q_{in}(t) \Rightarrow \boxed{b(t) = r\left(2 + \sin(2t)\right)}$$

IVP

$$\begin{cases} Q'(t) = -a_0 \, Q(t) + b(t) \\ Q(0) = 0. \end{cases}$$

$$Q(t) = \frac{1}{\mu(t)} \int_0^t \mu(s) \, b(s) \, ds$$

$$\mu(t) = e^{a_0 t}$$

$$\boxed{Q(t) = r \, e^{-a_0 t} \int_0^t e^{a_0 s} \left(2 + \sin(2s)\right) ds}$$

integration tables.