mth 235 L2

Plan:
* Separable ODE
* Definition
* Main Result
* Examples

(2.2)

* Separable ODE

Def: [Given functions $g, h : \mathbb{R} \to \mathbb{R}$, an ODE on the unknown $y : \mathbb{R} \to \mathbb{R}$ is called separable iff the ODE has the form

$$\boxed{h(y)\; y'(t) = g(t)}.$$]

* **Remark:** An ODE $y'(t) = f(t, y(t))$ is separable

iff

$$\boxed{y'(t) = \frac{g(t)}{h(y)}}$$

iff

$$\boxed{f(t, y) = \frac{g(t)}{h(y)}}$$

* **Notation:** In lecture:

$$\boxed{t, \quad y(t), \qquad -g(t) + h(y)\, y'(t) = 0}$$

In textbook:

$$\boxed{x, \quad y(x), \qquad M(x) + N(y)\, y'(x) = 0}$$

Therefore:

$$\boxed{g(t) = -M(t)}, \qquad \boxed{h(y) = N(y)}$$

* <u>Examples</u>

(1) $y' = \dfrac{t^2}{1-y^2}$ is separable, since

$$\boxed{g(t) = t^2} \quad , \quad \boxed{h(y) = 1 - y^2}$$

(2) $y' + y^2 \cos(2t) = 0$ is separable, since

$$y' = -y^2 \cos(2t)$$

$$\boxed{\frac{1}{y^2}\, y' = -\cos(2t)}$$, so;

$$\boxed{g(t) = -\cos(2t)} \quad , \quad \boxed{h(y) = \frac{1}{y^2}}$$

<u>Remark</u>: Functions g, h are Not unique;

$$\boxed{g(t) = \cos(2t)} \quad , \quad \boxed{h(y) = -\frac{1}{y^2}}$$

* Not every ODE is separable :

* Examples

(1) $y' = -\frac{2}{t} y + 4t$ is Not separable.

(2) $y' = -a(t) y + b(t)$ (linear ODE)

with $b(t) \neq 0$ is Not separable.

* Remark :

$\boxed{y' = -a(t) y}$ (linear ODE $b = 0$)

is separable, since

$\frac{1}{y} y' = -a(t)$, so :

$\boxed{g(t) = -a(t)}$, $\boxed{h(y) = \frac{1}{y}}$

* **Main Result**

Thrm. [If functions $g, h : \mathbb{R} \to \mathbb{R}$ are continuous with primitives G, H, respectively; that is,

$$\boxed{G'(t) = g(t)}, \quad \boxed{H'(u) = h(u)},$$

then, the separable ODE

$$\boxed{h(y)\, y'(t) = g(t)}$$

has infinitely many solutions, y, satisfying the equation

$$\boxed{H(y(t)) = G(t) + c}$$

with $c \in \mathbb{R}$ arbitrary.]

* **Remark:** Given g, h,

find their primitives

G, H.

* **Example**: [Find all solutions to ODE

$$y'(t) = \frac{t^2}{1 - y^2(t)} .$$]

Sol:

$$(1 - y^2)\, y' = t^2$$

$$\underbrace{g(t) = t^2} \Rightarrow \underbrace{G(t) = \frac{t^3}{3}}$$

$$\underbrace{h(u) = 1 - u^2} \Rightarrow \underbrace{H(u) = u - \frac{u^3}{3}}$$

Therefore, the sol. $y(t)$ satisfies the eq.

$$\boxed{y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c} , \quad c \in \mathbb{R} \qquad (1)$$

[The solutions $y(t)$ are given in implicit form.]

Remark: No y' appears in (1).

* Remark : Verification : Derivate on both sides of (1).

$$y'(t) - 3\,\frac{y^2(t)}{3}\,y'(t) = 3\,\frac{t^2}{3}$$

$$\boxed{(1 - y^2)\,y'(t) = t^2}$$

* <u>Proof</u> <u>of</u> <u>the</u> <u>Thrm</u>:

since $$h(y(t))\ y'(t) = g(t)$$

their primitives differ by a constant,

$$\int h(y(t))\ y'(t)\, dt = \int g(t)\, dt + c$$

Perform the substitution: $\boxed{\begin{aligned} u &= y(t) \\ du &= y'(t)\, dt \end{aligned}}$

$$\int h(u)\, du = \int g(t)\, dt + c$$

$$H(u) = G(t) + c$$

Substitute back $u = y(t)$,

$$\boxed{H(y(t)) = G(t) + c} .$$

∎

* IVP for separable eqs. are simple to solve

* <u>Example</u>: Find the sol. of IVP

$$y'(t) = \frac{t^2}{1 - y^2(t)} \qquad (1)$$

$$y(0) = 1 \qquad (2)$$

<u>Sol</u>: All solutions to (1) satisfy the eq.

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$$

The initial condition (2) implies

$$y(0) - \frac{y^3(0)}{3} = 0 + c$$

$$1 - \frac{1}{3} = c \quad \Rightarrow \quad \boxed{c = \frac{2}{3}}$$

$$\boxed{y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + \frac{2}{3}}$$

Implicit form.

* Example: Find the sol. of IVP

$$y'(t) + y^2(t)\cos(2t) = 0 \qquad (1)$$

$$y(0) = 1 \qquad (2)$$

Sol: Eq. (1) is separable, since

$$\boxed{\frac{1}{y^2}\, y'(t) = -\cos(2t)}$$

So:

$$g(t) = -\cos(2t) \implies G(t) = -\frac{1}{2}\sin(2t)$$

$$h(u) = \frac{1}{u^2} \implies H(u) = -\frac{1}{u}$$

All solutions of eq. (1) satisfy the eq.

$$\boxed{-\frac{1}{y(t)} = -\frac{1}{2}\sin(2t) + c}\ , \quad c \in \mathbb{R}$$

Implicit form

$$\frac{1}{y(t)} = \frac{\sin(2t) - 2c}{2}$$

$$\boxed{y(t) = \frac{2}{\sin(2t) - 2c}}$$

Solutions in explicit form.

The initial condition $y(0) = 1$ implies

$$1 = y(0) = \frac{2}{0 - 2c} = -\frac{1}{c} \Rightarrow \boxed{c = -1}$$

$$\boxed{y(t) = \frac{2}{\sin(2t) + 2}}$$

explicit form.

* **Example:** Find the sol. of IVP

$$y'(t) = \frac{2-t}{1+y(t)} \quad (1)$$

$$y(0) = 1 \quad (2)$$

Sol: Eq. (1) is separable with

$$\boxed{g(t) = 2-t} \quad , \quad \boxed{h(u) = 1+u}$$

Their primitives are:

$$g(t) = 2-t \quad \Rightarrow \quad G(t) = 2t - \frac{t^2}{2}$$

$$h(u) = 1+u \quad \Rightarrow \quad H(u) = u + \frac{u^2}{2}$$

All sols. $y(t)$ of (1) satisfy the eq.

$$\boxed{y(t) + \frac{y^2(t)}{2} = 2t - \frac{t^2}{2} + c} \quad c \in \mathbb{R}$$

The initial condition $y(0)=1$ implies

$$y(0) + \frac{y^2(0)}{2} = 0 - 0 + c$$

$$1 + \frac{1}{2} = c \quad \Rightarrow \quad \boxed{c = \frac{3}{2}}.$$

Therefore:

$$\frac{y^2(t)}{2} + y(t) = -\frac{t^2}{2} + 2t + \frac{3}{2}$$

equivalently:

$$\boxed{y^2(t) + 2\,y(t) = -t^2 + 4t + 3}$$

Implicit form.

* Sometimes the solution can be given in explicit form :

$$y^2 + 2y + (t^2 - 4t - 3) = 0$$

$y(t)$ is the root of the polynomial.

$$y_\pm(t) = \frac{-2 \pm \sqrt{4 - 4(t^2 - 4t - 3)}}{2}$$

$$= -1 \pm \sqrt{1 - (t^2 - 4t - 3)}$$

$$\boxed{y_\pm(t) = -1 \pm \sqrt{-t^2 + 4t + 4}}$$

Two possibilities : $y_+(t)$, $y_-(t)$

* Only $y_+(t)$ satisfies the initial condition:

$$y_+(0) = -1 + \sqrt{4} = 1$$ ✓

$$y_-(0) = -1 - \sqrt{4} = -3$$

The solution is

$$\boxed{y(t) = -1 + \sqrt{-t^2 + 4t + 4}}$$

Explicit form.