mth 235 : Preliminars

- Gabriel Nagy : off. hours: M, Tu 10 - 11:30 am
- math.msu.edu/~gnagy → teaching → mth 235 Spring 2010
- Textbook: Boyce, DiPrima 9th ed.
- We collect Homeworks. (13 out of 14)
- Three exams during semester. See dates in webpage
- common Final Exam: Tu May 4.
- Grade: The bigger of:

(1) 15 % Hw
17 % E1
17 % E2
17 % E3
34 % FE

(2) 15% Hw
34% Best two Es.
51 % FE

4.0	3.5	3.0	2.5	....
[100, 90]	(90, 85]	(85, 80]	(80, 75]	

- Lecture Notes : in webpage.
- Recitations meet tomorrow.

mth **235** **L1**

Plan:
* Overview of diff. eqs.
* Linear ODE
* Integrating factor method.

(2.1)

Read:
* Direction field

  Example 2, Sect. 1.1

* Direction field plotters in internet.

  see link in class webpage.

* Overview of differential eqs.

- A differential eq. is an equation where the unknown is a function, and both the function and its derivatives appear in the eq.

- Differential eqs. are a central part in a physical description of nature.

  * Classical Mechanics: Newton's Laws, Lagrange eqs.

  * Electromagnetism: Maxwell's eqs.

  * Quantum Mechanics: Schrödinger's eqs.

* Examples

(1) Newton's law of motion ($ma = F$)

$$\frac{d^2}{dt^2}\underline{x}(t) = \frac{1}{m}\underline{F}(t, \underline{x}(t))$$

ordinary differential eq. (ODE)

$\underline{x}(t)$ depends on only one variable, t.

(2) Wave eq. (sound propagation)

$$\frac{\partial^2}{\partial t^2}u(t,x) = v^2 \frac{\partial^2}{\partial x^2}u(t,x)$$

v : wave speed , u: air density

partial differential eq. (PDE)

$u(t,x)$ depends on more than one variable , t, x, and their derivatives appear in the eq.

* Linear ordinary differential eqs. (ODE)

Notation: Given $y : \mathbb{R} \to \mathbb{R}$, denote

$$y'(t) = \frac{dy}{dt}(t).$$

Def: [Given $f : \mathbb{R}^2 \to \mathbb{R}$, an ODE in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is the eq.

$$\boxed{y'(t) = f(t, y(t))}$$
]

Def: [An ODE is linear iff

$$\boxed{y'(t) = -a(t)\, y(t) + b(t)}$$
]

$\boxed{f(t,y) \text{ is linear in } y.}$

* Examples

(1) $$y'(t) = \underbrace{-2\; y(t) + 3}_{f(t,y) \,=\, 2y+3}$$

constant coefficients ODE.

$a(t) = 2$, $b(t) = 3$.

(2) $$y'(t) = \underbrace{-\frac{2}{t}\; y(t) + 4t}_{f(t,y) \,=\, -\frac{2y}{t} + 4t}$$

variable coefficients ODE

$a(t) = \frac{2}{t}$, $b(t) = 4t$.

* Solutions to linear ODE can be obtained using the integrating factor method.

(1) Constant coefficient case

Thrm: Given constants $a_0, b_0 \in \mathbb{R}$, $a_0 \neq 0$, the linear ODE

$$\boxed{y' = -a_0 y + b_0}$$

has infinitely many sols.

$$\boxed{y(t) = c_0 e^{-a_0 t} + \frac{b_0}{a_0}}$$

where $c_0 \in \mathbb{R}$.

<u>Proof</u>: (integrating factor method)

$$y' + a_0 y = b_0$$

$$\mu(t)\left(y' + a_0 y\right) = \mu(t)\, b_0$$

$$\left[\begin{array}{l} \mu \text{ is an integrating factor iff} \\ \boxed{\mu\,(y' + a_0\, y) = (\mu\, y)'} \end{array}\right]$$

key idea.

<u>Finding μ</u>

$$\mu y' + \mu a_0 y = \mu' y + \mu y'$$

$$a_0 y \mu = \mu' y$$

$$a_0 \mu = \mu'$$

$$\boxed{\frac{\mu'(t)}{\mu(t)} = a_0}$$

eq. for μ.

$$\left(\ln[\mu(t)]\right)' = (a_0 t)'$$

$$\ln[\mu(t)] = a_0 t + c_1$$

$$\mu(t) = e^{a_0 t + c_1} = e^{c_1} e^{a_0 t}$$

choose $c_1 = 0$ (we need just one solution.)

$$\boxed{\mu(t) = e^{a_0 t}}$$ integrating factor

Therefore: $\mu(y' + a_0 y) = (\mu y)'$

$$(\mu y)' = b_0 \mu$$

$$\left(y(t)\, e^{a_0 t}\right)' = b_0\, e^{a_0 t}$$

$$\int \left(y(t)\, e^{a_0 t}\right)' dt = \int b_0\, e^{a_0 t}\, dt$$

$$y(t)\, e^{a_0 t} = \frac{b_0}{a_0} e^{a_0 t} + c_0$$

$$\boxed{y(t) = c_0\, e^{-a_0 t} + \frac{b_0}{a_0}}$$

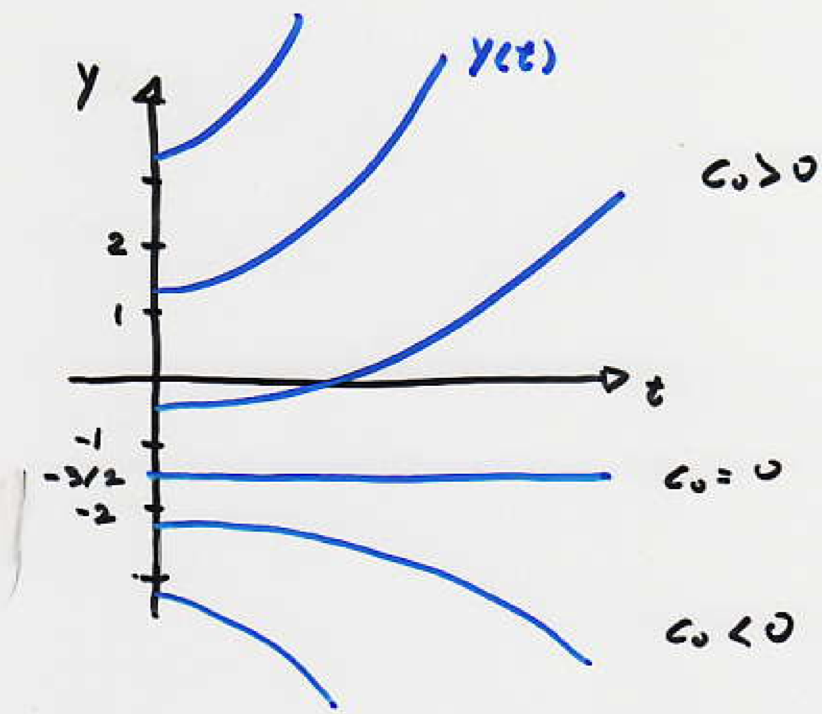
* Example : $\left[\begin{array}{l}\text{Find all functions } y \text{ sol. of} \\ y' = 2y + 3.\end{array}\right]$

Sol.

$$y' = -a_0 y + b_0 \,, \qquad \boxed{\begin{array}{l} a_0 = -2 \\ b_0 = 3 \end{array}}$$

$$y(t) = c_0 e^{-a_0 t} + \frac{b_0}{a_0}$$

$$\boxed{y(t) = c_0 e^{2t} - \frac{3}{2}} \qquad c_0 \in \mathbb{R}$$



verification

$$y' = 2 c_0 e^{2t}$$

$$c_0 e^{2t} = y + \frac{3}{2}$$

$$y' = 2y + 3 \checkmark$$

* The initial value problem (IVP)

Def: [Given constants $a_0, b_0, t_0, y_0 \in \mathbb{R}$, find $y: \mathbb{R} \to \mathbb{R}$ sol. of

$$\boxed{y' = a_0 y + b_0} \qquad (1)$$

$$\boxed{y(t_0) = y_0} \qquad (2)$$

]

IVP: differential eq. (1)
and
initial condition (2)

* [The initial condition selects only one of the solutions of (1).]

* [Read Thrm 1.4 in L.N.]

* **Example:** Find $y : \mathbb{R} \to \mathbb{R}$ sol. of IVP

$$y' = 2y + 3 \qquad (1)$$
$$y(0) = 1 \qquad (2)$$

Sol:

All solutions of (1) are

$$\boxed{y(t) = c_0 \, e^{2t} - \frac{3}{2}}$$

The initial condition (2) selects only one solution.

$$1 = y(0) = c_0 - \frac{3}{2} \implies \boxed{c_0 = \frac{5}{2}}$$

$$\boxed{y(t) = \frac{5}{2} \, e^{2t} - \frac{3}{2}}$$

* Linear ODE: Variable coefficients

- Integration factor method.

Thrm: Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$, and constants $t_0, y_0 \in \mathbb{R}$
The IVP

$$\boxed{y'(t) = -a(t)\, y(t) + b(t)} \qquad (1)$$

$$\boxed{y(t_0) = y_0} \qquad (2)$$

has the unique sol.

$$\boxed{y(t) = \frac{1}{\mu(t)} \left[y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \right],}$$

where

$$\mu(t) = e^{A(t)}, \quad A(t) = \int_{t_0}^{t} a(s)\, ds.$$

* $\mu(t)$: Integration factor

* See the proof in L.N.

* The Thrm includes the constant coeffs.

If $\boxed{a(t) = a_0}$, $\boxed{b(t) = b_0}$

and $\boxed{t_0 = 0}$, then

$$A(t) = \int_0^t a_0 \, ds \Rightarrow \boxed{A(t) = a_0 t}$$

$$\mu(t) = e^{A(t)} \Rightarrow \boxed{\mu(t) = e^{a_0 t}}$$

$$\int_{t_0}^t \mu(s)\, b(s)\, ds = \int_0^t e^{a_0 s} b_0 \, ds$$

$$= \frac{b_0}{a_0} \left(e^{at} - 1 \right)$$

$$y(t) = e^{-a_0 t} \left[y_0 + \frac{b_0}{a_0} e^{at} - \frac{b_0}{a_0} \right]$$

$$\boxed{y(t) = \underbrace{\left(y_0 - \frac{b_0}{a_0} \right)}_{c_0} e^{-a_0 t} + \frac{b_0}{a_0}}$$

constant coeff. case.

* Example :
$$\left[\begin{array}{ll} \text{Find } y:\mathbb{R}\to\mathbb{R} \text{ sol. of IVP} & \\ t\,y' + 2y = 4t^2 & (1) \\ y(1) = 2 & (2) \end{array}\right]$$

Sol:

(1) $\Rightarrow$ $$y' = -\frac{2}{t}\,y + 4t$$

$$a(t) = \frac{2}{t}, \qquad b(t) = 4t.$$

(2) $\Rightarrow$ $$t_0 = 1, \qquad y_0 = 2.$$

$$A(t) = \int_{t_0}^{t} a(s)\,ds = \int_1^t \frac{2}{s}\,ds$$

$$A(t) = 2\left[\ln(t) - \ln(1)\right]$$

$$= 2\ln(t)$$

$$\boxed{A(t) = \ln(t^2)}$$

* The integrating factor μ :

$$\mu(t) = e^{A(t)} = e^{\ln(t^2)} \Rightarrow$$

$$\boxed{\mu(t) = t^2}$$

$$y(t) = \frac{1}{\mu(t)} \left[y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \right]$$

$$= \frac{1}{t^2} \left[2 + \int_1^t s^2 \, 4s \, ds \right]$$

$$= \frac{1}{t^2} \left[2 + 4 \int_1^t s^3 \, ds \right]$$

$$= \frac{1}{t^2} \left[2 + 4 \, \frac{1}{4} \left(s^4 \Big|_1^t \right) \right]$$

$$= \frac{1}{t^2} \left[2 + t^4 - 1 \right]$$

$$\boxed{y(t) = \frac{1}{t^2} + t^2}$$

(verify the solution)