Name: Key ID Number: ____________

TA: Please, verify ! Section: ____________

MTH 235
Practice Final Exam
April 30, 2010
120 minutes
Chptrs: 2, 3, 5,
6, 7, 10.

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. $\boxed{\textit{Box your answers.}}$

Signature: ________________________________

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	10	
7	10	
8	15	
9	10	
10	10	
11	10	
12	15	
13	15	
14	15	
15	15	
Σ	200	

$3'$

15

1. ($\cancel{10}$ points) Find the most general solution $y(x)$ of the equation

12:30

$$y' = \frac{e^x \sin(y) + 2x}{3y - e^x \cos(y)}.$$

$[3y - e^x \cos(y)]\ y' - [e^x \sin(y) + 2x] = 0$

$N = 3y - e^x \cos(y) \Rightarrow N_x = -e^x \cos(y)$

$M = -[e^x \sin(y) + 2x] \Rightarrow M_y = -e^x \cos(y)$.

3

$\phi_y = N \Rightarrow \phi_y = 3y - e^x \cos(y) \Rightarrow \phi = \frac{3}{2} y^2 - e^x \sin(y) + f(x)$

6

$\phi_x = M \qquad -e^x \sin(y) + f'(x) = \phi_x = M = -e^x \sin(y) - 2x$

$f'(x) = -2x \Rightarrow f(x) = -x^2 + c$.

$\phi = \frac{3}{2} y^2 - e^x \sin(y) - x^2 + c$

9

$$\boxed{\frac{3}{2} y^2 - e^x \sin(y) - x^2 + c = 0}$$

10

12:33

5'

2. (~~10~~ 15 points) Find the general solution $y(t)$ to the differential equation

Bernoulli method

$$t^2 y' + 2t y = y^3 \qquad \cancel{y' + \frac{2}{t}y = \frac{y^3}{t^2}}.$$

$$\frac{y'}{y^3} + \frac{2}{t}\frac{1}{y^2} = \frac{1}{t^2} \qquad v = y^{-2} \qquad v' = -2y^{-3}y'$$

$$\frac{y'}{y^3} = -\frac{1}{2}v' \qquad \Rightarrow$$

$$\Rightarrow \quad -\frac{1}{2}v' + \frac{2}{t}v = \frac{1}{t^2} \quad \Rightarrow \quad \boxed{v' - \frac{4}{t}v = -\frac{2}{t^2}}$$

4

$$\mu(t) = e^{-\int \frac{4}{t}dt} = e^{-4\ln(t)} = e^{\ln(t^{-4})} = t^{-4} \Rightarrow \boxed{\mu = t^{-4}}$$

$$t^{-4}v' - 4t^{-5}v = -2t^{-6} \quad \Rightarrow \quad (t^{-4}v)' = -2t^{-6}$$

$$t^{-4}v = (-2)\frac{1}{(-5)}t^{-5} + c$$

$$= \frac{2}{5}t^{-5} + c \quad \Rightarrow \quad v = \frac{2}{5}\frac{1}{t} + ct^4$$

$$\boxed{v = \frac{2 + 5ct^5}{5t}} \qquad y = \pm\sqrt{\frac{1}{v}}$$

8

$$\boxed{y(t) = \pm\sqrt{\frac{5t}{2 + 5ct^5}}}$$

10

3. (15 points) Find all solutions y to the equation below and leave them in implicit form,

$$y' = \frac{x^2 + 3xy + y^2}{x^2}.$$

Homogeneous eq.

$$y' = \frac{(x^2 + 3xy + y^2)}{x^2} \quad \frac{(1/x^2)}{(1/x^2)}$$

$$y' = \frac{1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{1} \quad ; \quad \frac{y}{x} = v$$

$$y' = 1 + 3v + v^2 \quad , \quad y = xv \Rightarrow y' = xv' + v$$

$$xv' + v = 1 + 3v + v^2 \Rightarrow xv' = 1 + 2v + v^2 = (v+1)^2$$

$$\int \frac{v'}{(1+v)^2}\, dx = \int \frac{dx}{x} + c \quad ; \quad u = 1+v \quad , \quad du = v'\, dx$$

$$\int \frac{du}{u^2} = \int \frac{dx}{x} + c \Rightarrow -\frac{1}{u} = \ln(x) + c$$

$$\frac{1}{1+v} = -\ln(x) - c \Rightarrow 1+v = \frac{-1}{\ln(x)+c}$$

$$\frac{y}{x} = \frac{-1}{\ln(x)+c} - 1 \Rightarrow \boxed{y(x) = \frac{-x}{\ln(x)+c} - x}$$

$\overset{9'}{\overset{15}{}}$

4. ~~3~~. (~~10~~ 15 points) Find the general solution $y(t)$ to the differential equation

$$2y'' + y' = t + 2\sin(t).$$

$y_h = e^{rt}$ $\quad 2r^2 + r = 0 \Rightarrow r(2r+1) = 0 \Rightarrow \boxed{r_1 = 0 \quad r_2 = -\frac{1}{2}}$

$$\boxed{y_h = c_1 + c_2 e^{-t/2}}$$

+3

$y_{p_1} = k_1 t + k_0$

k_0: sol. homog.

$\longrightarrow \quad y_{p_1} = k_1 t^2 + k_0 t$

$y_{p_1}' = 2k_1 t + k_0$

$y_{p_1}'' = 2k_1$

$2(2k_1) + (2k_1 t + k_0) = t$

$(4k_1 + k_0) + 2k_1 t = t \Rightarrow \boxed{k_1 = \frac{1}{2}}$, $4k_1 + k_0 = 0 \Rightarrow \frac{4}{2} + k_0 = 0$

$\boxed{k_0 = -2}$

$$\boxed{y_{p_1} = \frac{t^2}{2} - 2t}$$

+4

$$y_{p_2}(t) = k_1 \cos(t) + k_2 \sin(t)$$

$$y'_{p_2} = -k_1 \sin(t) + k_2 \cos(t)$$

$$y''_{p_2} = -k_1 \cos(t) - k_2 \sin(t)$$

$$-2\left(k_1 \cos(t) + k_2 \sin(t)\right) - k_1 \sin(t) + k_2 \cos(t) = 2 \sin(t)$$

$$(-2k_1 + k_2) \cos(t) + (-2k_2 - k_1) \sin(t) = 2 \sin(t)$$

$$\left.\begin{array}{l} 2k_1 = k_2 \\ 2k_2 + k_1 = -2 \end{array}\right] \Rightarrow 4k_1 + k_1 = -2 \Rightarrow \boxed{k_1 = -\frac{2}{5}}$$

$$\boxed{k_2 = -\frac{4}{5}}$$

$$\boxed{y_{p_2} = -\frac{2}{5} \cos(t) - \frac{4}{5} \sin(t)}$$

+3

$$\boxed{y(t) = c_1 + c_2 e^{-t/2} + \frac{t^2}{2} - 2t - \frac{2}{5}\left(\cos(t) + 2\sin(t)\right)}$$

$\overset{15}{5}$ $\cancel{4}$

5. ($\cancel{10}$ 15 points) Find the general solution $y(t)$ to the equation

$$y'' + 2y' + y = \frac{e^{-t}}{t}.$$

$y_h = e^{rt}$ $\quad r^2 + 2r + 1 = 0 \Rightarrow r_\pm = \frac{-2 \pm \sqrt{4-4}}{2}$

$\boxed{y_{h_1} = e^{-t}}$; $\boxed{y_{h_2} = t e^{-t}}$ $\quad \boxed{r_\pm = -1}$

3

$u_1' = -\frac{y_2 g}{W_{12}}$ $\quad u_2' = \frac{y_1 g}{W_{12}}$

$W_{12} = \begin{vmatrix} e^{-t} & -e^{-t} \\ te^{-t} & (e^{-t} - te^{-t}) \end{vmatrix} = e^{-t}[e^{-t} - \cancel{t e^{-t}}] + \cancel{t e^{-t}} e^{-t}$

$\boxed{W_{12} = e^{-2t}}$ $\quad \boxed{g(t) = \frac{1}{t} e^{-t}}$

3

$u_1' = -\ t e^{-t} \frac{e^{-t}}{t} \frac{1}{e^{-2t}} = -1 \Rightarrow \boxed{u_1 = t}$

$u_2' = e^{-t} \frac{e^{-t}}{t} \frac{1}{e^{-2t}} = \frac{1}{t} \Rightarrow \boxed{u_2 = \ln(t)}$

3

$\tilde{y}_p = t e^{-t} + \ln(t)\, t e^{-t} \Rightarrow \boxed{y_p = t \ln(t)\, e^{-t}}$

$\boxed{y(t) = (c_1 + c_2 t) e^{-t} + t \ln(t)\, e^{-t}}$

1

2'

6. (10 points) Find the general solution $y(x)$ of

$$4x^2 y'' + 8x y' + y = 0, \qquad x > 0.$$

$y(x) = x^r$

$4r(r-1) + 8r + 1 = 0$

$4r^2 - 4r + 8r + 1 = 0$

$4r^2 + 4r + 1 = 0 \Rightarrow r_{\pm} = \dfrac{-4 \pm \sqrt{16-16}}{8}$

$\boxed{y(x) = c_1 x^{-\frac{1}{2}} + c_2 \ln(x)\, x^{-\frac{1}{2}}}$ 5

$\boxed{r_{\pm} = -\frac{1}{2}}$ 5

7. (10 points) Find all singular points of the equation below and determine which of those are regular singular points, where

$$x(x-2)^2\, y'' + 3(x+2)\, y' + (x-3)\, y = 0.$$

$P(x)\; y'' + Q(x)\; y' + R(x)\; y = 0$

$P(x) = x(x-2)^2$ $\qquad P(x)=0 \Rightarrow$ $\boxed{\begin{matrix} x_0 = 0 \\ x_1 = 2 \end{matrix}}$, Singular Points.

$Q(x) = 3(x+2)$

$R(x) = (x-3)$

$x_0 = 0$

$$\frac{x\, Q(x)}{P(x)} = \frac{x\, 3(x+2)}{x(x-2)^2} \underset{x\to 0}{\longrightarrow} \frac{6}{4} = \frac{3}{2} \quad ✓$$

$$\frac{x^2 R(x)}{P(x)} = \frac{x^2 (x-3)}{x(x-2)^2} \underset{x\to 0}{\longrightarrow} 0 \quad ✓$$

$\boxed{x_0 = 0 \text{ is a Regular-singular point.}}$

$x_1 = 2$

$$\frac{(x-2)\, Q(x)}{P(x)} = \frac{(x-2)\, 3(x+2)}{x(x-2)^2} = \frac{3(x+2)}{x(x-2)} \underset{x\to 2}{\longrightarrow} \pm\infty$$

$\boxed{X_1 = 2 \text{ is a NOT regular-Singular.}}$

8. (15 points) Use power series centered at $x_0 = 1$ to look for two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$-3xy'' + 2y' + y = 0. \qquad (1)$$

(a) Find the recurrence relation for the power series coefficients.

(b) Find the first ~~three~~ TWO non-zero terms of the power series for each of the linearly independent solutions y_1 and y_2.

(a) $x_0 = 1$ is a regular point of (1).

$$\boxed{y = \sum_{n=0}^{\infty} a_n (x-1)^n} \Rightarrow \boxed{y' = \sum_{n=0}^{\infty} n a_n (x-1)^{n-1}}$$

$$\boxed{y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}}$$

$$x y'' = x \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$= (x-1+1) \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$\boxed{x y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}}$$

$$\boxed{\begin{aligned}&\sum_{n=1}^{\infty} (-3) n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} (-3) n(n-1) a_n (x-1)^{n-2} \\ &+ \sum_{n=1}^{\infty} 2 n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0\end{aligned}}$$

$$\left[\sum_{n=0}^{\infty} (-3)(n+1)(n)\, a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} (-3)(n+2)(n+1)\, a_{n+2} (x-1)^n \right.$$

$$\left. + \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0 \right]$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)n\, a_{n+1} - 3(n+2)(n+1)\, a_{n+2} + 2(n+1)\, a_{n+1} + a_n \right] (x-1)^n = 0$$

$$-3(n+2)(n+1)\, a_{n+2} - 3(n+1)n\, a_{n+1} + 2(n+1)\, a_{n+1} + a_n = 0$$

$$-3(n+2)(n+1)\, a_{n+2} + (n+1)(-3n+2)\, a_{n+1} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n+1)(-3n+2)\, a_{n+1} + a_n}{3(n+2)(n+1)}}$$

Recurrence Relation

$n \geq 0.$

$n=0 \qquad a_2 = \dfrac{2a_1 + a_0}{6} \quad \Rightarrow \quad \boxed{a_2 = \tfrac{1}{3}a_1 + \tfrac{1}{6}a_0}$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + \left(\tfrac{1}{3}a_1 + \tfrac{1}{6}a_0\right)x^2 + \cdots$$

$$\boxed{\begin{aligned} y(x) &= a_0\left[1 + \tfrac{1}{6}x^2 + \cdots\right] \\ &\quad + a_1\left[x + \tfrac{1}{3}x^2 + \cdots\right] \end{aligned}}$$

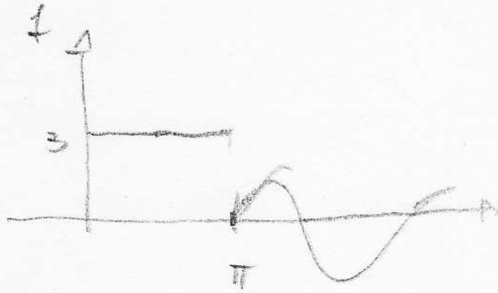
$$\boxed{y_1(x) = 1 + \tfrac{1}{6}x^2 + \cdots}$$

$$\boxed{y_2(x) = x + \tfrac{1}{3}x^2 + \cdots}$$

41

9. ~~7~~. (~~10~~ points) Find the Laplace transform of the function f given by

$$f(t) = \begin{cases} 3 & \text{if } 0 \leqslant t < \pi, \\ \sin(t-\pi) & \text{if } \quad t \geqslant \pi. \end{cases}$$



$f(t) = 3\,[u(t) - u(t-\pi)]$

$+\ u(t-\pi)\sin(t-\pi)$

2

$\mathcal{L}[f] = 3\left(\mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)]\right) + \mathcal{L}[u(t-\pi)\sin(t-\pi)]$

$= 3\left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right) + e^{-\pi s}\frac{1}{(s^2+1)}$

$$\boxed{\mathcal{L}[f] = \frac{3}{s}(1-e^{-\pi s}) + \frac{e^{-\pi s}}{s^2+1}}$$

3

4'

10. 8. (10 points) Find an explicit expression (that is, without using convolutions) of the inverse Laplace transform of the function F given by

$$F(s) = \frac{e^{-\pi s}}{(s^2+1)(s^2+4)}.$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{a}{(s^2+1)} + \frac{b}{(s^2+4)} \Rightarrow$$

$$\Rightarrow \quad 1 = a(s^2+4) + b(s^2+1)$$

$$= a s^2 + 4a + b s^2 + b$$

$$= (a+b)s^2 + (4a+b)$$

$$\rightarrow \begin{cases} a+b=0 \\ 4a+b=1 \end{cases} \Rightarrow 3a = 1$$

$$\boxed{a = \frac{1}{3}} \qquad \boxed{b = -\frac{1}{3}}$$

5

$$F(s) = \frac{e^{-\pi s}}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{e^{-\pi s}}{3}\left[\frac{1}{s^2+1} - \frac{1}{2}\,\frac{2}{s^2+4}\right]$$

$$f(t) = \frac{1}{3}\left[u(t-\pi)\sin(t-\pi) - \frac{1}{2}u(t-\pi)\sin[2(t-\pi)]\right]$$

$$\boxed{f(t) = \frac{1}{3}u(t-\pi)\left[\sin(t-\pi) - \frac{1}{2}\sin[2(t-\pi)]\right]}$$

5

11. (10 points) Given a continuous but otherwise arbitrary function g, use the Laplace transform method to find the solution y to the initial value problem

$$y'' + 4y' + 8y = g(t), \qquad y(0) = y'(0) = 0.$$

Express the solution y in terms of appropriate convolutions with function g.

$$(s^2 + 4s + 8)\ \mathcal{L}[y] = \mathcal{L}[g]$$

$$\mathcal{L}[y] = \frac{1}{(s^2 + 4s + 8)}\ \mathcal{L}[g]$$

$$s^2 + 4s + 8 = 0 \Rightarrow s_{\pm} = \frac{-4 \pm \sqrt{16 - 32}}{2} \rightarrow \text{complex roots.}$$

$$s^2 + 4s + 8 = s^2 + 2(2s) + 4 - 4 + 8 = (s+2)^2 + 4$$

$$\mathcal{L}[y] = \frac{1}{[(s+2)^2 + 4]}\ \mathcal{L}[g] = \frac{1}{2}\ \frac{2}{[(s+2)^2+4]}\ \mathcal{L}[g]$$

$$\mathcal{L}[y] = \frac{1}{2}\ \mathcal{L}[e^{-2t}\sin(2t)]\ \mathcal{L}[g(t)]$$

$$\boxed{y(t) = \frac{1}{2} \int_0^t e^{-2\tau} \sin(2\tau)\ g(t-\tau)\, d\tau}$$

12. (15 points) Find the general solution $\boldsymbol{x}$ to the ~~initial value problem~~ equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$

Sketch a phase portrait with few solution trajectories.

$$P(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-2) + 3 = \lambda^2 - 4 + 3 = \lambda^2 - 1$$

$$P(\lambda) = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{\pm} = \pm 1}$$

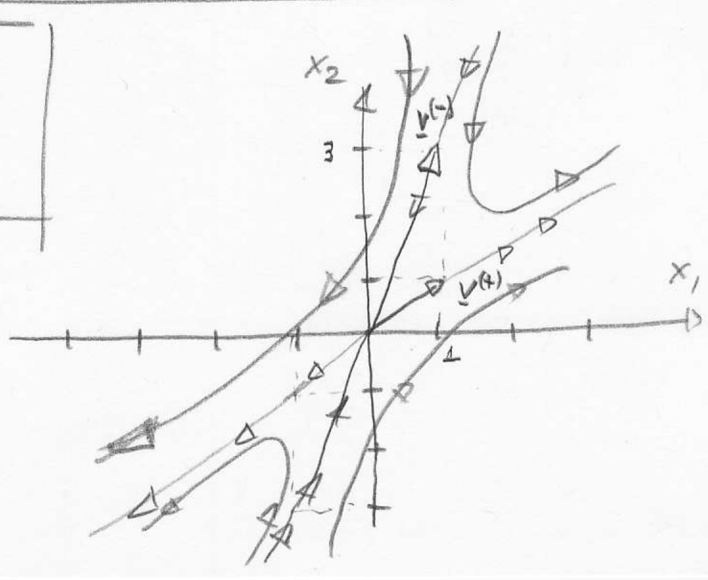
$\boxed{\lambda_+ = 1}$, $A - I = \begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow$

$\Rightarrow \quad v_1 = v_2 \quad \Rightarrow \quad \boxed{\underline{v}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_+ = 1}$

$\boxed{\lambda_- = -1}$, $A + I = \begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow$

$\Rightarrow \quad 3v_1 = v_2 \quad \Rightarrow \quad \boxed{\underline{v}^{(-)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \lambda_- = -1}$

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}}$$



13. (~~10~~ 15 points) Find the solution $\boldsymbol{x}$ to the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}.$$

$$P(\lambda) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (\lambda+9)(\lambda-3) + 36 = \lambda^2 + 9\lambda - 3\lambda - 27 + 36$$

$$P(\lambda) = \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda_\pm = \frac{-6 \pm \sqrt{36-36}}{2} = -3$$

$$\boxed{\lambda_\pm = -3}$$

$$A + 3I = \begin{bmatrix} 3+3 & -18 \\ 2 & -9+3 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$v_1 = 3v_2 \Rightarrow \boxed{\underline{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \lambda = -3}$$

$$(A + 3I)\,\underline{w} = \underline{v} \Rightarrow \left[\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -6 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow 2w_1 = 6w_2 + 1$ choosing $w_2 = 0 \Rightarrow$ $\boxed{\underline{w} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}$

$\boxed{w_1 = \frac{1}{2}}$

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) e^{-3t}}$$

I.C. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underset{\sim}{x}(0) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(0-\frac{1}{2})} \begin{bmatrix} 0 & -1/2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (-2) \begin{bmatrix} -3/2 \\ 7 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \end{bmatrix}}$$

$$\boxed{\underset{\sim}{x}(t) = 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} - 14 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) e^{-3t}}$$

14. (15 points) Find every positive eigenvalue λ and nonzero function y, solutions of the boundary value problem

$$y'' - 4y' + 4y = -\lambda y, \qquad y(0) = 0, \quad y(3) = 0.$$

$$y'' - 4y' + (4+\lambda)\, y = 0, \qquad y(x) = e^{rx}.$$

$$P(r) = r^2 - 4r + (4+\lambda) = 0 \Rightarrow r_\pm = \frac{4 \pm \sqrt{16 - 4(4+\lambda)}}{2}$$

$$r_\pm = \frac{4 \pm \sqrt{-4\lambda}}{2}; \qquad \lambda > 0, \quad \lambda = \mu^2.$$

$$r_\pm = \frac{4 \pm 2\mu i}{2} \Rightarrow \boxed{r_\pm = 2 \pm i\mu}$$

$$\boxed{y(x) = c_1 e^{2x} \cos(\mu x) + c_2 e^{2x} \sin(\mu x)}$$

B.C. $\quad 0 = y(0) = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\boxed{y(x) = c_2 e^{2x} \sin(\mu x)}$$

$$0 = y(3) = c_2 e^{6} \sin(\mu 3); \qquad c_2 \neq 0 \Rightarrow$$

$$\Rightarrow \sin(\mu 3) = 0 \Rightarrow \boxed{\mu 3 = n\pi} \qquad n \geq 1$$

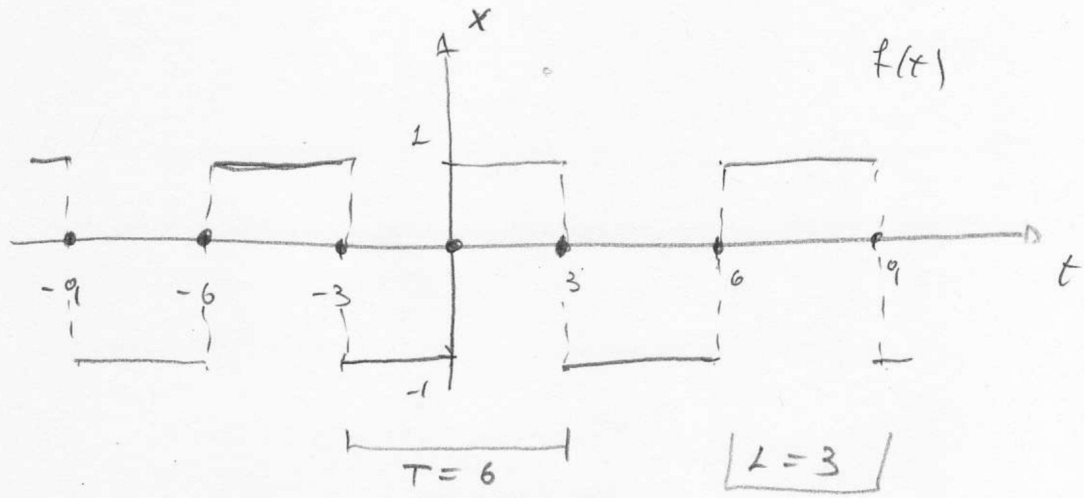
$$\boxed{\mu = \frac{n\pi}{3}} \qquad \boxed{\lambda_n = \left(\frac{n\pi}{3}\right)^2} \qquad \boxed{y_n(x) = e^{2x} \sin\left(\frac{n\pi}{3} x\right)}$$

15. (15 points) Consider the function $f(x) = -1$, defined for $-3 < x < 0$.

(a) Sketch the graph of the odd periodic extension of period $T = 6$ of the function f above. Sketch the graph of this extension for at least three periods.

(b) Determine the Fourier series of this odd periodic extension of f.

(a)



$$\boxed{L = 3}$$

(b) f is odd $\Rightarrow$ $\boxed{a_n = 0}$ $\quad n \geq 0$

$$b_n = \frac{1}{3} \int_{-3}^{3} f(x) \sin\left(\frac{n\pi}{3} x\right) dx$$

$$= \frac{2}{3} \int_0^3 f(x) \sin\left(\frac{n\pi}{3} x\right) dx$$

$$= \frac{2}{3} \int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left(\frac{-3}{n\pi}\right) \cos\left(\frac{n\pi x}{3}\right)\Big|_0^3$$

$$\boxed{b_n = -\frac{2}{n\pi} \left[\cos(n\pi) - 1\right]}$$

$$\boxed{b_n = -\frac{2}{n\pi}\left[(-1)^n - 1\right]}$$

$$\boxed{b_{2k} = 0} \qquad b_{2k+1} = -\frac{2}{(2k+1)\pi}(-2)$$

$$\boxed{b_{2k+1} = \frac{4}{(2k+1)\pi}} \quad \boxed{k \geq 1}$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\sin\left(\frac{(2n+1)\pi}{3}x\right)}$$

You are allowed to use the Laplace transform table on page 317 in the textbook.

Nevertheless, this is a short list of Laplace transforms and Laplace transform properties that could be useful for the exam. We use the notation $\mathcal{L}[f(t)] = F(s)$.

$f(t) = e^{at}$	$F(s) = \dfrac{1}{s-a}$	$s > \max\{a, 0\}$,
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$,
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2+a^2}$	$s > 0$,
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2+a^2}$	$s > 0$,
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2-a^2}$	$s > 0$,
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2-a^2}$	$s > 0$,
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > \max\{a, 0\}$,
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2+b^2}$	$s > \max\{a, 0\}$,
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{s-a}{(s-a)^2+b^2}$	$s > \max\{a, 0\}$.

The following Laplace transforms could also be useful, where u denotes the step function at $t = 0$, and δ the Dirac delta generalized function:

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad \mathcal{L}[\delta(t-c)] = e^{-cs}.$$

The following relations could also be useful:

$$\begin{aligned} e^{-cs}\,\mathcal{L}\big[f(t)\big] &= \mathcal{L}\big[u(t-c)\,f(t-c)\big], \\ \mathcal{L}\big[e^{ct}\,f(t)\big] &= F(s-c), \\ \mathcal{L}\big[f^{(n)}(t)\big] &= s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0). \end{aligned}$$