Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- The center of mass of an object.
- The moment of inertia of an object.
Areas of a region on a plane.

Definition
The area of a closed, bounded region $R$ on a plane is given by

$$A = \iint_R dx \, dy.$$
Areas of a region on a plane.

Definition
The area of a closed, bounded region $R$ on a plane is given by

$$A = \int \int_R dx \, dy.$$

Remark:
- To compute the area of a region $R$ we integrate the function $f(x, y) = 1$ on that region $R$. 
Areas of a region on a plane.

Definition
The *area* of a closed, bounded region $R$ on a plane is given by

$$A = \iint_R dx\ dy.$$  

Remark:
- To compute the area of a region $R$ we integrate the function $f(x, y) = 1$ on that region $R$.
- The area of a region $R$ is computed as the volume of a 3-dimensional region with base $R$ and height equal to 1.
Areas of a region on a plane.

Example
Find the area of \( R = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2] \} \).
Areas of a region on a plane.

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Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

Solution: We express the region $R$ as an integral Type I, integrating first on vertical directions:

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \ dx.$$
Areas of a region on a plane.

Example
Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

Solution: We express the region $R$ as an integral Type I, integrating first on vertical directions:

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Areas of a region on a plane.

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A = \int_{-1}^{2} \left( y \bigg|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} \left( x + 2 - x^2 \right) dx
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$$A = \int_{x^2}^{2} \int_{-1}^{x+2} dy \, dx.$$

$$A = \int_{-1}^{2} \left( y \bigg|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} (x + 2 - x^2) \, dx = \left( \frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \bigg|_{-1}^{2}.$$
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\]

We conclude that \( A = 9/2 \).
Areas of a region on a plane.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} \, dx \, dy + \int_{1}^{4} \int_{x-2}^{2} \, dx \, dy.
\]

Verify that the result is: $A = \frac{9}{2}$. \(\triangleleft\)
Areas of a region on a plane.

Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x + 2]\} \)
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Solution: We express the region \( R \) as an
integral Type II, integrating first on
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Areas of a region on a plane.

Example

Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x+2]\} \) integrating first along horizontal directions.

Solution: We express the region \( R \) as an integral Type II, integrating first on horizontal directions:

\[
A = \iint_{R_1} d\mathbf{y} \, d\mathbf{x} + \iint_{R_2} d\mathbf{x} \, d\mathbf{y}.
\]
Areas of a region on a plane.

Example
Find the area of \( R = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2] \} \) integrating first along horizontal directions.

Solution: We express the region \( R \) as an integral Type II, integrating first on horizontal directions:

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A = \int \int_{R_1} dx \ dy + \int \int_{R_2} dx \ dy.
\]

\[
A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx \ dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx \ dy.
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Areas of a region on a plane.

**Example**
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], \ y \in [x^2, x + 2]\} \) integrating first along horizontal directions.

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Verify that the result is: \( A = 9/2. \)
Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- **Average value of a function.**
- The center of mass of an object.
- The moment of inertia of an object.
Average value of a function.

Review: The average of a single variable function.

Definition
The average of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\bar{f}$, is given by

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$
Average value of a function.

**Review:** The average of a single variable function.

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The *average* of a function \( f : [a, b] \to \mathbb{R} \) on the interval \([a, b]\), denoted by \( \bar{f} \), is given by

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\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**Definition**
The *average* of a function \( f : R \subset \mathbb{R}^2 \to \mathbb{R} \) on the region \( R \) with area \( A(R) \), denoted by \( \bar{f} \), is given by

\[
\bar{f} = \frac{1}{A(R)} \int \int_R f(x, y) \, dx \, dy.
\]
Example
Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).
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Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).
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Example
Find the average of \( f(x, y) = xy \) on the region \( R = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3] \} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \). We only need to compute \( I = \iiint_R f(x, y) \, dx \, dy \).

\[
I = \int_{x=0}^{2} \int_{y=0}^{3} xy \, dy \, dx = \int_{x=0}^{2} \left[ \frac{1}{2} y^2 \right]_{y=0}^{3} x \, dx = \int_{x=0}^{2} \frac{9}{2} x \, dx = \frac{9}{2} \left[ \frac{1}{2} x^2 \right]_{x=0}^{2} = \frac{9}{2} \left( 2^2 \right) = 9.
\]
Example
Find the average of $f(x, y) = xy$ on the region $R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\}$.

Solution: The area of the rectangle $R$ is $A(R) = 6$. We only need to compute $I = \int \int_R f(x, y) \, dx \, dy$.

$$I = \int_0^2 \int_0^3 xy \, dy \, dx$$

$$I = \frac{9}{2}.$$ Since $f = I / A(R)$, we get $f = 3/2$. $\triangleleft$
Average value of a function.

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I = \int_{0}^{2} \int_{0}^{3} xy \, dy \, dx = \int_{0}^{2} x \left( \frac{y^2}{2} \bigg|_{0}^{3} \right) \, dx
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\( I = \frac{9}{2} \)
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Find the average of \( f(x, y) = xy \) on the region
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\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \right)_0^2
\]

Since \( f = \frac{I}{A(R)} \), we get \( f = \frac{9}{6} = \frac{3}{2} \).
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Find the average of \( f(x, y) = xy \) on the region \( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \). We only need to compute \( I = \int \int_{R} f(x, y) \, dx \, dy \).

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I = \int_{0}^{2} \int_{0}^{3} xy \, dy \, dx = \int_{0}^{2} x \left( \frac{y^2}{2} \bigg|_{0}^{3} \right) \, dx = \int_{0}^{2} \frac{9}{2} x \, dx.
\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \bigg|_{0}^{2} \right) \quad \Rightarrow \quad I = 9.
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Average value of a function.

Example
Find the average of \( f(x, y) = xy \) on the region
\( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], \ y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \). We only need to compute
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I = \int \int_R f(x, y) \, dx \, dy.
\]
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\]
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I = \frac{9}{2} \left( \frac{x^2}{2} \right)_0^2 \quad \Rightarrow \quad I = 9.
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Since \( \bar{f} = I / A(R) \),
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I = \frac{9}{2} \left( \frac{x^2}{2} \right)_0^2 \Rightarrow I = 9.
\]

Since \( \bar{f} = I/A(R) \), we get \( \bar{f} = 9/6 = 3/2 \).
Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- **The center of mass of an object.**
- The moment of inertia of an object.
The center of mass of an object.

**Review:** The *center of mass* of $n$ point particles of mass $m_i$ at the positions $r_i$ in a plane, where $i = 1, \cdots, n$, is the vector $\bar{r}$ given by

$$
\bar{r} = \frac{1}{M} \sum_{i=1}^{n} m_i r_i, \quad \text{where} \quad M = \sum_{i=1}^{n} m_i.
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\mathbf{r} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r}_i,
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where \( M = \sum_{i=1}^{n} m_i \).

**Definition**

The *center of mass* of a region \( R \) in the plane, having a continuous mass distribution given by a density function \( \rho : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \), is the vector \( \mathbf{r} \) given by

\[
\mathbf{r} = \frac{1}{M} \iint_{R} \rho(x, y) \langle x, y \rangle \, dx \, dy,
\]

where \( M = \iint_{R} \rho(x, y) \, dx \, dy \).

Remark: Certain gravitational effects on an extended object can be described by the gravitational force on a point particle located at the center of mass of the object.
The center of mass of an object.

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Remark: Certain gravitational effects on an extended object can be described by the gravitational force on a point particle located at the center of mass of the object.
The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$. 

Solution:

We first compute the total mass $M$,

$$M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx.$$

We conclude that $M = \frac{4}{3}$. 

$$
\begin{align*}
\int_0^1 \left[ \int_0^{2x} x \, dy + \int_0^{2x} y \, dy \right] \, dx &= \int_0^1 \left[ x \int_0^{2x} \, dy + \frac{1}{2} y^2 \bigg|_0^{2x} \right] \, dx \\
&= \int_0^1 \left[ x \cdot 2x + \frac{1}{2} (2x)^2 \right] \, dx \\
&= \int_0^1 \left[ 2x^2 + 2x^2 \right] \, dx \\
&= \int_0^1 4x^2 \, dx \\
&= \left[ \frac{4x^3}{3} \right]_0^1 \\
&= \frac{4}{3}
\end{align*}
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$$M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx.$$

$$M = \int_0^1 \left[ x \left( y \big|_0^{2x} \right) + \left( \frac{y^2}{2} \big|_0^{2x} \right) \right] \, dx.$$
The center of mass of an object.

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Find the center of mass of the triangle with boundaries \( y = 0, \quad x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

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We first compute the total mass \( M \),

\[
M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx.
\]

\[
M = \left[ \int_0^1 \left( x \left( y \bigg|_0^{2x} \right) + \left( \frac{y^2}{2} \right) \bigg|_0^{2x} \right) \right] \, dx = \int_0^1 \left[ 2x^2 + 2x^2 \right] \, dx
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M = \int_0^1 \left[ x \left( y \bigg|_0^{2x} \right) + \left( \frac{y^2}{2} \bigg|_0^{2x} \right) \right] \, dx = \int_0^1 \left[ 2x^2 + 2x^2 \right] \, dx = 4 \frac{x^3}{3} \bigg|_0^1.
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The center of mass of an object.

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Solution: The total mass is \( M = \frac{4}{3} \).
The center of mass of an object.

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Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \frac{4}{3}$. The coordinates $x$ and $y$ of the center of mass are

$$\bar{r}_x = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)x \, dy \, dx,$$

$$\bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)y \, dy \, dx.$$
The center of mass of an object.

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Find the center of mass of the triangle with boundaries \( y = 0, \ x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

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\bar{r}_x = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)x \ dy \ dx, \quad \bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)y \ dy \ dx.
\]

We compute the \( \bar{r}_x \) component.

\[
\bar{r}_x = \frac{3}{4} \int_0^1 \left[ x^2 \left( y \right|_0^{2x} \right) + x \left( \frac{y^2}{2} \right|_0^{2x} \right] \ dx
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The center of mass of an object.

**Example**

Find the center of mass of the triangle with boundaries \( y = 0, \ x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

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\[
\bar{r}_x = \frac{3}{4} \int_0^1 \left[ x^2 \left( y \bigg|_{0}^{2x} \right) + x \left( \frac{y^2}{2} \bigg|_{0}^{2x} \right) \right] \, dx = \frac{3}{4} \int_0^1 \left[ 2x^3 + 2x^3 \right] \, dx,
\]
The center of mass of an object.

Example
Find the center of mass of the triangle with boundaries \( y = 0, \) \( x = 1 \) and \( y = 2x, \) and mass density \( \rho(x, y) = x + y. \)

Solution: The total mass is \( M = \frac{4}{3}. \) The coordinates \( x \) and \( y \) of the center of mass are

\[
\bar{r}_x = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)x \, dy \, dx, \quad \bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)y \, dy \, dx.
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We compute the \( \bar{r}_x \) component.

\[
\bar{r}_x = \frac{3}{4} \int_0^1 \left[ x^2 \left( y \bigg|_{0}^{2x} \right) + x \left( \frac{y^2}{2} \bigg|_{0}^{2x} \right) \right] \, dx = \frac{3}{4} \int_0^1 \left[ 2x^3 + 2x^3 \right] \, dx,
\]

so \( \bar{r}_x = \frac{3}{4} x^4 \bigg|_0^1. \) We conclude that \( \bar{r}_x = \frac{3}{4}. \)
The center of mass of an object.

Example
Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \frac{4}{3}$ and $\bar{r}_x = \frac{3}{4}$. 
The center of mass of an object.

Example
Find the center of mass of the triangle with boundaries \( y = 0, \ x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

Solution: The total mass is \( M = \frac{4}{3} \) and \( \bar{r}_x = \frac{3}{4} \).

Since \( \bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y) y \, dy \, dx \),
The center of mass of an object.

Example
Find the center of mass of the triangle with boundaries \( y = 0, x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

Solution: The total mass is \( M = \frac{4}{3} \) and \( \bar{r}_x = \frac{3}{4} \).

Since \( \bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y) y \ dy \ dx \), we obtain

\[
\bar{r}_y = \frac{3}{4} \int_0^1 \left[ x \left( \frac{y^2}{2} \big|_0^{2x} \right) + \left( \frac{y^3}{3} \big|_0^{2x} \right) \right] \ dx
\]
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Find the center of mass of the triangle with boundaries \( y = 0, \ x = 1 \) and \( y = 2x \), and mass density \( \rho(x, y) = x + y \).

Solution: The total mass is \( M = \frac{4}{3} \) and \( \bar{r}_x = \frac{3}{4} \).

Since \( \bar{r}_y = \frac{1}{M} \int_0^1 \int_0^{2x} (x + y)y \, dy \, dx \), we obtain

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\bar{r}_y = \frac{3}{4} \int_0^1 \left[ x \left( \frac{y^2}{2} \biggr|_0^{2x} \right) + \left( \frac{y^3}{3} \biggr|_0^{2x} \right) \right] \, dx = \frac{3}{4} \int_0^1 \left[ 2x^3 + \frac{8}{3}x^3 \right] \, dx,
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Therefore, the center of mass vector is \( \mathbf{\bar{r}} = \left\langle \frac{3}{4}, \frac{7}{8} \right\rangle \).
The centroid of an object.

Definition

The *centroid* of a region $R$ in the plane is the vector $\mathbf{c}$ given by

$$
\mathbf{c} = \frac{1}{A(R)} \iint_R \langle x, y \rangle \, dx \, dy,
$$

where $A(R) = \iint_R dx \, dy$.

**Remark:**

▶ The centroid of a region can be seen as the center of mass vector of that region in the case that the mass density is constant.

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Find the centroid of the triangle inside $y = 0$, $x = 1$ and $y = 2x$. 

Solution:
The area of the triangle is

$$A(R) = \int_0^1 \int_0^{2x} dy \, dx = \int_0^1 2x \, dx = x^2 \bigg|_0^1 = 1.$$ 

Therefore, the centroid vector components are given by

$$c_x = \int_0^1 \int_0^{2x} x \, dy \, dx = \int_0^1 2x^2 \, dx = 2 \left( \frac{x^3}{3} \right) \bigg|_0^1 = \frac{2}{3},$$

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We conclude,

$$c = \frac{2}{3} \langle 1, 1 \rangle.$$ 

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We conclude, $\mathbf{c} = \frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{j}$. \hfill \Box
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so \( c_y = \frac{2}{3} \). We conclude, \( \mathbf{c} = \frac{2}{3}(1, 1) \). \( \triangle \)
Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- The center of mass of an object.
- The moment of inertia of an object.
The moment of inertia of an object.

**Remark:** The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.
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**Definition**

The *moment of inertia* about the $x$-axis and the $y$-axis of a region $R$ in the plane having mass density $\rho : R \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are given by, respectively,

$$I_x = \iint_{R} y^2 \rho(x, y) \, dx \, dy, \quad I_y = \iint_{R} x^2 \rho(x, y) \, dx \, dy.$$
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If $M$ denotes the total mass of the region, then the radii of gyration about the $x$-axis and the $y$-axis are given by

$$R_x = \sqrt{I_x/M} \quad R_y = \sqrt{I_y/M}.$$
The moment of inertia of an object.

Example
Find the moment of inertia and the radius of gyration about the $x$-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution:
The moment of inertia $I_x$ is given by

$$
I_x = \int_0^1 \int_0^{2x} x^2 (x + y) \, dy \, dx
$$

$$
= \int_0^1 \left[ x^3 + \frac{1}{2} x^2 y \right]_0^{2x} \, dx
$$

$$
= \int_0^1 4x^4 \, dx
$$

$$
= \frac{4}{5} x^5 \bigg|_0^1 = \frac{4}{5}
$$

Since the mass of the region is $M = \frac{4}{3}$, the radius of gyration along the $x$-axis is $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{4}{3}}$, that is, $R_x = \sqrt{\frac{3}{5}}$. ▷
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Find the moment of inertia and the radius of gyration about the $x$-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The moment of inertia $I_x$ is given by

$$I_x = \int_0^1 \int_0^{2x} x^2(x + y) \, dy \, dx = \int_0^1 \left[ x^3 \left( y \bigg|_0^{2x} \right) + x^2 \left( \frac{y^2}{2} \bigg|_0^{2x} \right) \right] \, dx$$

$$I_x = \int_0^1 4x^4 \, dx = 4 \left( \frac{x^5}{5} \bigg|_0^1 \right) \quad \Rightarrow \quad I_x = \frac{4}{5}.$$ 

Since the mass of the region is $M = \frac{4}{3}$, the radius of gyration along the $x$-axis is $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{4}{5} \frac{3}{4}}$, that is, $R_x = \sqrt{\frac{3}{5}}$. ◁
Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.
Review: Polar coordinates.

Definition
The *polar coordinates* of a point \( P \in \mathbb{R}^2 \) is the ordered pair \((r, \theta)\) defined by the picture.
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**Theorem (Cartesian-polar transformations)**
*The Cartesian coordinates of a point \( P = (r, \theta) \) in the first quadrant are given by*

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x = r \cos(\theta), \quad y = r \sin(\theta).
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\[ r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \left( \frac{y}{x} \right). \]
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- **Double integrals in disk sections.**
- Double integrals in arbitrary regions.
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Double integrals on disk sections.

**Theorem**

*If* $f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ *is continuous in the region*

$$R = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\}$$

*where* $0 \leq \theta_0 \leq \theta_1 \leq 2\pi$, *then the double integral of function* $f$ *in that region can be expressed in polar coordinates as follows,*

$$\int\int_{R} f \, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r, \theta) \, r \, dr \, d\theta.$$
Double integrals on disk sections.

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$$\int\int_R f \, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r, \theta) \, r \, dr \, d\theta.$$

Remark:

- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
Double integrals on disk sections.

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- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
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- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
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- Notice the extra factor \( r \) on the right-hand side.
Double integrals on disk sections.

Remark:
Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.

\[ x_0 \leq x \leq x_1, \]
\[ y_0 \leq y \leq y_1, \]
\[ 0 \leq r_0 \leq r \leq r_1, \]
\[ 0 \leq \theta_0 \leq \theta \leq \theta_1 \leq 2\pi. \]
Double integrals on disk sections.

Example
Find the area of an arbitrary circular section
\( R = \{ (r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1] \} \).
Evaluate that area in the particular case of a disk with radius \( R \).
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Solution:

\[
A = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} (r \ dr) \ d\theta
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Find the area of an arbitrary circular section

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Evaluate that area in the particular case of a disk with radius \( R \).

Solution:

\[
A = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} (r \, dr) \, d\theta = \int_{\theta_0}^{\theta_1} \frac{1}{2} \left[ (r_1)^2 - (r_0)^2 \right] \, d\theta
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The case of a disk is: \( \theta_0 = 0, \ \theta_1 = 2\pi, \ r_0 = 0 \) and \( r_1 = R \).
In that case we reobtain the usual formula \( A = \pi R^2 \). \( \triangleq \)
Double integrals on disk sections.

Example

Find the integral of $f(r, \theta) = r^2 \cos(\theta)$ in the disk
$R = \{(r, \theta) \in \mathbb{R}^2 : r \in [0, 1], \ \theta \in [0, \pi/4]\}$.

Solution:

$$\int\int_R f \, dA = \int_{\pi/4}^0 \int_0^1 r^2 \cos(\theta) \, (r \, dr) \, d\theta,$$

$$\int\int_R f \, dA = \int_{\pi/4}^0 \left. \frac{r^4}{4} \right|_0^1 \cos(\theta) \, d\theta = \frac{1}{4} \sin(\theta) \bigg|_{\pi/4}^0 = \frac{\sqrt{2}}{8}.$$
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\]
\[
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\]
\[
\int \int_{R} f \ dA = \int_{0}^{\pi/4} \frac{\cos(\theta)}{4} \ d\theta
\]
\[
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Find the integral of \( f(r, \theta) = r^2 \cos(\theta) \) in the disk 
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Solution:

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\int \int_R f \, dA = \int_0^{\pi/4} \int_0^1 r^2 \cos(\theta)(r \, dr) \, d\theta,
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\int \int_R f \, dA = \int_0^{\pi/4} \left( \frac{r^4}{4} \bigg|_0^1 \right) \cos(\theta) \, d\theta = \frac{1}{4} \sin(\theta) \bigg|_0^{\pi/4}.
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- Review: Polar coordinates.
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- **Double integrals in arbitrary regions.**
- Changing Cartesian integrals into polar integrals.
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**Theorem**

*If the function* \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) *is continuous in the region*

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R = \{(r, \theta) \in \mathbb{R}^2 : r \in [h_0(\theta), h_1(\theta)], \ \theta \in [\theta_0, \theta_1]\}.
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*where* \( 0 \leq h_0(\theta) \leq h_1(\theta) \) *are continuous functions defined on an interval* \([\theta_0, \theta_1]\), *then the integral of function* \( f \) *in* \( R \) *is given by*

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\int\int_R f(r, \theta) \, dA = \int_{\theta_0}^{\theta_1} \int_{h_0(\theta)}^{h_1(\theta)} f(r, \theta) r \, dr \, d\theta.
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Double integrals in arbitrary regions.

Example
Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$. 

Solution:
We first show that these curves are actually circles.

$r = \cos(\theta) 
\iff r^2 = r \cos(\theta) 
\iff x^2 + y^2 = x.$

Completing the square in $x$ we obtain 

$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$

Analogously, 

$r = \sin(\theta)$ is the circle 

$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$
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Example

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Solution: $A = 2 \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta$
Double integrals in arbitrary regions.

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Solution: $A = 2 \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta = 2 \int_0^{\pi/4} \frac{1}{2} \sin^2(\theta) \, d\theta$;
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$A = \int_{0}^{\pi/4} \frac{1}{2} [1 - \cos(2\theta)] \ d\theta$
Double integrals in arbitrary regions.

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Find the area of the region bounded by the curves \( r = \cos(\theta) \) and \( r = \sin(\theta) \).

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\[ A = \int_{0}^{\pi/4} \frac{1}{2} \left[ 1 - \cos(2\theta) \right] d\theta = \frac{1}{2} \left[ \left( \frac{\pi}{4} - 0 \right) - \frac{1}{2} \sin(2\theta) \right]_{0}^{\pi/4}; \]

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\[ \triangle \]

Also works: \[ A = \int_0^{\pi/4} \int_0^{\sin(\theta)} r \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{\cos(\theta)} r \, dr \, d\theta. \]
Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
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Theorem
If \( f : D \subset \mathbb{R}^2 \to \mathbb{R} \) is a continuous function, and \( f(x, y) \) represents the function values in Cartesian coordinates, then holds

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\int\int_D f(x, y) \, dx \, dy = \int\int_D f(r \cos(\theta), r \sin(\theta)) r \, dr \, d\theta.
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Example

Compute the integral of $f(x, y) = x^2 + 2y^2$ on $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\}$. 

Solution: First, transform Cartesian into polar coordinates: $x = r \cos(\theta), \ y = r \sin(\theta)$. Since $f(x, y) = x^2 + 2y^2$, $f(r \cos(\theta), r \sin(\theta)) = r^2 + 2r^2 \sin^2(\theta)$. 

Changing Cartesian integrals into polar integrals.

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Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\} \).

**Solution:** First, transform Cartesian into polar coordinates: \( x = r \cos(\theta), \ y = r \sin(\theta) \). Since \( f(x, y) = (x^2 + y^2) + y^2 \),
Changing Cartesian integrals into polar integrals.

**Theorem**

If $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function, and $f(x, y)$ represents the function values in Cartesian coordinates, then holds

$$
\int \int_D f(x, y) \, dx \, dy = \int \int_D f(r \cos(\theta), r \sin(\theta))r \, dr \, d\theta.
$$

**Example**

Compute the integral of $f(x, y) = x^2 + 2y^2$ on $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\}$.

**Solution:** First, transform Cartesian into polar coordinates:

$x = r \cos(\theta), \ y = r \sin(\theta)$. Since $f(x, y) = (x^2 + y^2) + y^2$,

$$
f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta).
$$
Changing Cartesian integrals into polar integrals.

Example
Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \; 0 \leq x, \; 1 \leq x^2 + y^2 \leq 2\} \).

Solution: We computed: \( f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta) \).
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Solution: We computed: \( f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta) \).

The region is
\( D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \frac{\pi}{2}, \ 1 \leq r \leq \sqrt{2}\} \).
Changing Cartesian integrals into polar integrals.

**Example**

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on 
\[ D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2 \} \].

**Solution:** We computed: 
\[ f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \sin^2(\theta). \]

The region is
\[ D = \left\{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \frac{\pi}{2}, \ 1 \leq r \leq \sqrt{2} \right\}. \]

\[
\int_{D} \int f(r, \theta) \, dA = \int_{0}^{\pi/2} \int_{1}^{\sqrt{2}} r^2 (1 + \sin^2(\theta)) \, r \, dr \, d\theta,
\]
\[
= \left[ \int_{0}^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_{1}^{\sqrt{2}} r^3 \, dr \right],
\]
Changing Cartesian integrals into polar integrals.

Example
Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on 
\( D = \{ (x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2 \} \).

Solution: 
\[
\int \int_D f(r, \theta) dA = \left[ \int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_1^{\sqrt{2}} r^3 \, dr \right].
\]
Changing Cartesian integrals into polar integrals.

Example

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on
\[ D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\}. \]

Solution:
\[
\int\int_D f(r, \theta) \, dA = \left[ \int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_1^{\sqrt{2}} r^3 \, dr \right].
\]
\[
\int\int_D f(r, \theta) \, dA = \left[ \left. \theta \right|_0^{\pi/2} \right] + \int_0^{\pi/2} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \right] \frac{1}{4} \left[ \left. r^4 \right|_1^{\sqrt{2}} \right].
\]

We conclude:
\[
\int\int_D f(r, \theta) \, dA = \frac{9}{16} \pi.
\]
Changing Cartesian integrals into polar integrals.

Example

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on 
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\} \).

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\[
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\]

\[
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\]

\[
\iint_D f(r, \theta) \, dA = \left[ \frac{\pi}{2} + \frac{1}{2} \left. \theta \right|_0^{\pi/2} \right] - \frac{1}{4} \left. \sin(2\theta) \right|_0^{\pi/2} \right] \frac{3}{4}
\]

We conclude:
\[
\iint_D f(r, \theta) \, dA = \frac{9}{16} \pi.
\]
\( \triangledown \)
Changing Cartesian integrals into polar integrals.

Example

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, \ 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2\} \).

Solution:
\[
\int \int_D f(r, \theta) \, dA = \left[ \int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta \right] \left[ \int_1^\sqrt{2} r^3 \, dr \right].
\]

\[
\int \int_D f(r, \theta) \, dA = \left[ (\theta \bigg|_0^{\pi/2}) + \int_0^{\pi/2} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \right] \frac{1}{4} (r^4 \bigg|_1^{\sqrt{2}})
\]

\[
\int \int_D f(r, \theta) \, dA = \left[ \frac{\pi}{2} + \frac{1}{2} (\theta \bigg|_0^{\pi/2}) - \frac{1}{4} (\sin(2\theta) \bigg|_0^{\pi/2}) \right] \frac{3}{4} = \left[ \frac{\pi}{2} + \frac{\pi}{4} \right] \frac{3}{4}.
\]
Changing Cartesian integrals into polar integrals.

**Example**

Compute the integral of \( f(x, y) = x^2 + 2y^2 \) on 
\( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y, 0 \leq x, \ 1 \leq x^2 + y^2 \leq 2 \} \).

**Solution:**

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\]

\[
\int\int_D f(r, \theta) \, dA = \left[ (\theta \bigg|_0^{\pi/2}) + \int_0^{\pi/2} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \right] \frac{1}{4} (r^4 \bigg|_1^{\sqrt{2}})
\]

\[
\int\int_D f(r, \theta) \, dA = \left[ \frac{\pi}{2} + \frac{1}{2} (\theta \bigg|_0^{\pi/2}) - \frac{1}{4} (\sin(2\theta) \bigg|_0^{\pi/2}) \right] \frac{3}{4} = \left[ \frac{\pi}{2} + \frac{\pi}{4} \right] \frac{3}{4}.
\]

We conclude:

\[
\int\int_D f(r, \theta) \, dA = \frac{9}{16}\pi.
\]
Changing Cartesian integrals into polar integrals.

Example
Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain $D = \{(r, \theta) \in R^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\}$. 

\[
\begin{align*}
\text{Solution:} & \\
& \text{Since } f(r \cos(\theta), r \sin(\theta)) = e^{-r^2}, \text{ the double integral is} \\
& \int \int_D f(x, y) \, dx \, dy = \int_0^\pi \int_0^2 e^{-r^2} r \, dr \, d\theta. \\
& \text{Substituting } u = r^2, \text{ hence } du = 2r \, dr, \text{ we obtain} \\
& \int \int_D f(x, y) \, dx \, dy = \frac{1}{2} \int_0^\pi \int_0^4 e^{-u} \, du \, d\theta; \\
& \text{We conclude:} \\
& \int \int_D f(x, y) \, dx \, dy = \frac{\pi}{2} \left(1 - e^{-4}\right).
\end{align*}
\]
Example

Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain $D = \{(r, \theta) \in R^2 : 0 \leq \theta \leq \pi, 0 \leq r \leq 2\}$.

Solution: Since $f(r \cos(\theta), r \sin(\theta)) = e^{-r^2}$, the double integral is
Changing Cartesian integrals into polar integrals.

Example
Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain $D = \{(r, \theta) \in \mathbb{R}^2 : \theta \leq \pi, 0 \leq r \leq 2\}$.

Solution: Since $f(r \cos(\theta), r \sin(\theta)) = e^{-r^2}$, the double integral is

$$
\int \int_D f(x, y) \, dx \, dy = \int_0^\pi \int_0^2 e^{-r^2} r \, dr \, d\theta.
$$
Changing Cartesian integrals into polar integrals.

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Integrate \( f(x, y) = e^{-(x^2+y^2)} \) on the domain 
\( D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\} \).

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\[
\int\int_D f(x, y) \, dx \, dy = \frac{1}{2} \int_0^\pi \int_0^4 e^{-u} \, du \, d\theta
\]
Changing Cartesian integrals into polar integrals.

Example
Integrate \( f(x, y) = e^{-(x^2+y^2)} \) on the domain \( D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\} \).

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Substituting \( u = r^2 \), hence \( du = 2r \, dr \), we obtain

\[
\iint_D f(x, y) \, dx \, dy = \frac{1}{2} \int_0^\pi \int_0^4 e^{-u} \, du \, d\theta = \frac{1}{2} \int_0^\pi \left(-e^{-u}\right|_0^4) \, d\theta;
\]
Changing Cartesian integrals into polar integrals.

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Integrate \( f(x, y) = e^{-(x^2+y^2)} \) on the domain \( D = \{ (r, \theta) \in \mathbb{R}^2 : 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2 \} \).

Solution: Since \( f(r \cos(\theta), r \sin(\theta)) = e^{-r^2} \), the double integral is

\[
\int\int_{D} f(x, y) \, dx \, dy = \int_{0}^{\pi} \int_{0}^{2} e^{-r^2} r \, dr \, d\theta.
\]

Substituting \( u = r^2 \), hence \( du = 2r \, dr \), we obtain

\[
\int\int_{D} f(x, y) \, dx \, dy = \frac{1}{2} \int_{0}^{\pi} \int_{0}^{4} e^{-u} \, du \, d\theta = \frac{1}{2} \int_{0}^{\pi} \left( -e^{-u} \right|_{0}^{4} \) \, d\theta;
\]

We conclude:

\[
\int\int_{D} f(x, y) \, dx \, dy = \frac{\pi}{2} \left( 1 - \frac{1}{e^4} \right). \]
Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.
Computing volumes using double integrals.

Example
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$. 
Computing volumes using double integrals.

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: Let us first draw the sets that form the volume we are interested to compute.

\[
z = \pm \sqrt{1 - r^2}, \quad z = r.
\]
Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: The integration region can be decomposed as follows:
Computing volumes using double integrals.

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Computing volumes using double integrals.

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: The integration region can be decomposed as follows:

\[
V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2} (rdr) d\theta - \int_0^{2\pi} \int_0^{r_0} r (rdr) d\theta.
\]
Computing volumes using double integrals.

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: The integration region can be decomposed as follows:

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V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2} \, (rdr) \, d\theta - \int_0^{2\pi} \int_0^{r_0} r \, (rdr) \, d\theta.
\]

We need to find \( r_0 \), the intersection of the cone and the sphere.
Computing volumes using double integrals.

Example
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: We find $r_0$, the intersection of the cone and the sphere.
Computing volumes using double integrals.

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Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: We find \( r_0 \), the intersection of the cone and the sphere.

\[
\sqrt{1 - r_0^2} = r_0
\]
Computing volumes using double integrals.

**Example**
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

**Solution:** We find \( r_0 \), the intersection of the cone and the sphere.

\[
\sqrt{1 - r_0^2} = r_0 \quad \Leftrightarrow \quad 1 - r_0^2 = r_0^2
\]
Computing volumes using double integrals.

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: We find \( r_0 \), the intersection of the cone and the sphere.

\[
\sqrt{1 - r_0^2} = r_0 \quad \Leftrightarrow \quad 1 - r_0^2 = r_0^2 \quad \Leftrightarrow \quad 2r_0^2 = 1;
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that is, \( r_0 = 1/\sqrt{2} \).
Computing volumes using double integrals.

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that is, \( r_0 = 1/\sqrt{2} \). Therefore

\[
V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \, dr) \, d\theta - \int_0^{2\pi} \int_0^{1/\sqrt{2}} r \, (r \, dr) \, d\theta.
\]
Computing volumes using double integrals.

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Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: We find $r_0$, the intersection of the cone and the sphere.

$$\sqrt{1 - r_0^2} = r_0 \iff 1 - r_0^2 = r_0^2 \iff 2r_0^2 = 1;$$

that is, $r_0 = 1/\sqrt{2}$. Therefore

$$V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \, dr) d\theta - \int_0^{2\pi} \int_0^{1/\sqrt{2}} r (r \, dr) d\theta.$$
Computing volumes using double integrals.

Example
Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: \( V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \, dr) - \int_0^{1/\sqrt{2}} r \, (r \, dr) \right] \).
Computing volumes using double integrals.

Example

Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

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Use the substitution \( u = 1 - r^2 \), so \( du = -2r \, dr \).
Computing volumes using double integrals.

Example
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \ dr) - \int_0^{1/\sqrt{2}} r \ (r \ dr) \right]$. Use the substitution $u = 1 - r^2$, so $du = -2r \ dr$. We obtain,

$$V = 2\pi \left[ \frac{1}{2} \int_{1/2}^1 u^{1/2} \ du - \frac{1}{3} r^3 \Big|_0^{1/\sqrt{2}} \right],$$
Computing volumes using double integrals.

Example
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \ dr) - \int_0^{1/\sqrt{2}} r \ (r \ dr) \right]$. Use the substitution $u = 1 - r^2$, so $du = -2r \ dr$. We obtain,

$$V = 2\pi \left[ \int_0^{1/\sqrt{2}} u^{1/2} \ du - \frac{1}{3} r^3 \bigg|_0^{1/\sqrt{2}} \right],$$

$$V = 2\pi \left[ \int_0^{1/2} \frac{1}{2} u^{3/2} \bigg|_0^{1/\sqrt{2}} - \frac{1}{3} \frac{1}{2^{3/2}} \right]$$
Example

Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: \( V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1 - r^2} (r \, dr) - \int_0^{1/\sqrt{2}} r (r \, dr) \right] \).

Use the substitution \( u = 1 - r^2 \), so \( du = -2r \, dr \). We obtain,

\[
V = 2\pi \left[ \frac{1}{2} \int_{1/2}^{1} u^{1/2} \, du - \frac{1}{3} r^3 \bigg|_0^{1/\sqrt{2}} \right],
\]

\[
V = 2\pi \left[ \frac{1}{2} \left( \frac{2}{3} u^{3/2} \right) \bigg|_{1/2}^{1} - \frac{1}{3} \frac{1}{2^{3/2}} \right] = \frac{2\pi}{3} \left[ 1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}} \right],
\]
Example

Find the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

Solution: \( V = 2\pi \left[ \int_0^{1/\sqrt{2}} \sqrt{1-r^2} \, (r \, dr) - \int_0^{1/\sqrt{2}} r \, (r \, dr) \right] \).

Use the substitution \( u = 1 - r^2 \), so \( du = -2r \, dr \). We obtain,

\[
V = 2\pi \left[ \frac{1}{2} \int_{1/2}^{1} u^{1/2} \, du - \frac{1}{3} r^3 \bigg|_0^{1/\sqrt{2}} \right],
\]

\[
V = 2\pi \left[ \frac{1}{2} \frac{2}{3} u^{3/2} \bigg|_{1/2}^{1} - \frac{1}{3} \frac{1}{23/2} \right] = \frac{2\pi}{3} \left[ 1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}} \right],
\]

We conclude: \( V = \frac{\pi}{3} \left( 2 - \sqrt{2} \right) \). \( \triangle \)
Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- Volume on a region in space.
Triple integrals in rectangular boxes.

Definition
The *triple integral* of a function \( f : R \subset \mathbb{R}^3 \to \mathbb{R} \) in the rectangular box \( R = [\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1] \) is the number

\[
\int\int\int_R f(x, y, z) \, dx \, dy \, dz = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z.
\]

where \( x_i^* \in [x_i, x_{i+1}] \), \( y_j^* \in [y_j, y_{j+1}] \), \( z_k^* \in [z_k, z_{k+1}] \) are sample points, while \( \{x_i\}, \{y_j\}, \{z_k\} \), with \( i, j, k = 0, \ldots, n \), are partitions of the intervals \( [\hat{x}_0, \hat{x}_1], [\hat{y}_0, \hat{y}_1], [\hat{z}_0, \hat{z}_1] \), respectively, and

\[
\Delta x = \frac{(\hat{x}_1 - \hat{x}_0)}{n}, \quad \Delta y = \frac{(\hat{y}_1 - \hat{y}_0)}{n}, \quad \Delta z = \frac{(\hat{z}_1 - \hat{z}_0)}{n}.
\]
Triple integrals in rectangular boxes.

Remark:

- A finite sum $S_n$ below is called a Riemann sum, where

\[
S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z.
\]
Triple integrals in rectangular boxes.

Remark:

- A finite sum $S_n$ below is called a Riemann sum, where

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x^*_i, y^*_j, z^*_k) \Delta x \Delta y \Delta z.$$ 

- Then holds

$$\int \int \int_{R} f(x, y, z) \, dx \, dy \, dz = \lim_{n \to \infty} S_n.$$
Triple integrals in rectangular boxes.

Remark:

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$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z.$$  

- Then holds $\int\int\int_R f(x, y, z) \, dx \, dy \, dz = \lim_{n \to \infty} S_n.$

Theorem (Fubini)

*If function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the rectangle $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then holds*

$$\int\int\int_R f(x, y, z) \, dx \, dy \, dz = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) \, dz \, dy \, dx.$$  

*Furthermore, the integral above can be computed integrating the variables $x, y, z$ in any order.*
Triple integrals in rectangular boxes.

**Review:** The Riemann sums and their limits.

Single variable functions in $[\hat{x}_0, \hat{x}_1]$:

$$
\lim_{n \to \infty} \sum_{i=0}^{n} f(x_i^*) \Delta x = \int_{\hat{x}_0}^{\hat{x}_1} f(x) \, dx.
$$
Review: The Riemann sums and their limits.

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Two variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1]$: (Fubini)

$$\lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} f(x, y) \, dy \, dx.$$
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Three variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$: (Fubini)

$$\lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} \int_{\hat{z}_0}^{\hat{z}_1} f(x, y, z) dz \ dy \ dx.$$
Example

Compute the integral of $f(x, y, z) = xyz^2$ on the domain $R = [0, 1] \times [0, 2] \times [0, 3]$. 

Solution:

It is useful to sketch the integration region first:

The integral we need to compute is

$$\int\int\int_R f\, dv = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz\, dy\, dx,$$

where we denoted $dv = dx\, dy\, dz$. 

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\[
\int_0^1 \int_0^2 xy \left( \frac{z^3}{3} \right)_0^3 \, dy \, dx
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We conclude:

\[
\int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx = 9
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\int \int \int_R f \, dv = \int_0^1 \int_0^2 xy \left( \frac{z^3}{3} \right) \bigg|_0^3 \, dy \, dx = \frac{27}{3} \int_0^1 \int_0^2 xy \, dy \, dx.
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We conclude: $\iiint_R f \, dv = 9$. \[\triangleq\]
Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- **Triple integrals in arbitrary domains.**
- Volume on a region in space.
Triple integrals in arbitrary domains.

**Theorem**

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain $D = \{ x \in [x_0, x_1], \ y \in [h_0(x), h_1(x)], \ z \in [g_0(x, y), g_1(x, y)] \}$, where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function $f$ in the region $D$ is given by

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Example
In the case that $D$ is an ellipsoid, the figure represents the graph of functions $g_1, g_0$ and $h_1, h_0$. 
Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
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- Volume on a region in space.
Volume on a region in space.

**Remark:** The volume of a bounded, closed region $D \in \mathbb{R}^3$ is

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Find the integration limits needed to compute the volume of the ellipsoid $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$. 

We first sketch the integration domain.
Volume on a region in space.

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Solution: The functions \( z = g_1 \) and \( z = g_0 \) are, respectively,

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z = 2\sqrt{1 - x^2 - \frac{y^2}{3^2}}, \quad z = -2\sqrt{1 - x^2 - \frac{y^2}{3^2}}.
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The limits on integration in \(x\) are \(\pm 1\). We conclude:

\[
V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz \, dy \, dx.
\]

\(\triangle\)
Volume on a region in space.

Example
Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.
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Solution:

The top surface is the sphere,

\[ z = \sqrt{1 - x^2 - y^2}. \]
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The limits on $y$ are obtained projecting the 3-dimensional figure onto the plane $z = 0$. 

The polar radius at the intersection cone-sphere was $r_0 = 1/\sqrt{2}$. 

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Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Recall: $z = \sqrt{1 - x^2 - y^2}, z = \sqrt{x^2 + y^2}$. 
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Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

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The \( y \)-top of the disk is,

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y = \sqrt{1/2 - x^2}.
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We conclude:

\[
V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2 - x^2}}^{\sqrt{1/2 - x^2}} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{x^2+y^2}} dz \ dy \ dx.
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Example
Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).
Volume on a region in space.

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Compute the volume of the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 6y + 2z \leq 6$.

Solution:
The region is given by the first octant and below the plane

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This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$. 
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In $z$ the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$. 
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Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

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At \( z = 0 \) the projection of the region is the triangle \( x \geq 0, \ y \geq 0, \) and \( x + 2y \leq 2 \).
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Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

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In \( y \) the limits are \( y = 1 - x/2 \) and \( y = 0 \).
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Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

Solution: In \( z \) the limits are \( z = \frac{(6 - 3x - 6y)}{2} \) and \( z = 0 \).

At \( z = 0 \) the projection of the region is the triangle \( x \geq 0, \ y \geq 0, \) and \( x + 2y \leq 2 \).

In \( y \) the limits are \( y = 1 - x/2 \) and \( y = 0 \).

We conclude: \( V = \int_{0}^{2} \int_{0}^{1-x/2} \int_{0}^{3-3y-3x/2} dz \ dy \ dx \).
Volume on a region in space.

Example
Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

Solution: Recall: \( V = \int_{0}^{2} \int_{0}^{1-x/2} \int_{0}^{3-3y-3x/2} dz \ dy \ dx \).
Volume on a region in space.

Example

Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

Solution: Recall: 

\[
V = \iiint_{0}^{2} \int_{0}^{1-x/2} \int_{0}^{3-3y-3x/2} dz \, dy \, dx.
\]

\[
V = 3 \iiint_{0}^{2} \int_{0}^{1-x/2} \left(1 - \frac{x}{2} - y\right) dy \, dx,
\]

\[
= 3 \int_{0}^{2} \left[ \left(1 - \frac{x}{2}\right) \left(y \bigg|_{0}^{(1-x/2)}\right) - \left(\frac{y^2}{2}\bigg|_{0}^{(1-x/2)}\right) \right] \, dx,
\]

\[
= 3 \int_{0}^{2} \left[ \left(1 - \frac{x}{2}\right) \left(1 - \frac{x}{2}\right) - \frac{1}{2} \left(1 - \frac{x}{2}\right)^2 \right] \, dx.
\]
Volume on a region in space.

Example
Compute the volume of the region given by \( x \geq 0, \ y \geq 0, \ z \geq 0 \) and \( 3x + 6y + 2z \leq 6 \).

Solution: Recall: \( V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} \ dz \ dy \ dx \).

\[
V = 3 \int_0^2 \int_0^{1-x/2} (1 - \frac{x}{2} - y) \ dy \ dx,
\]

\[
= 3 \int_0^2 \left[ (1 - \frac{x}{2}) \left( y \bigg|_0^{(1-x/2)} \right) - \left( \frac{y^2}{2} \right) \right] \ dx,
\]

\[
= 3 \int_0^2 \left[ (1 - \frac{x}{2}) \left( 1 - \frac{x}{2} \right) - \frac{1}{2} \left( 1 - \frac{x}{2} \right)^2 \right] \ dx.
\]

We only need to compute: \( V = \frac{3}{2} \int_0^2 \left( 1 - \frac{x}{2} \right)^2 \ dx \).
Volume on a region in space.

Example
Compute the volume of the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 6y + 2z \leq 6$.

Solution: Recall: $V = \frac{3}{2} \int_{0}^{2} \left(1 - \frac{x}{2}\right)^2 \, dx$. 
Example
Compute the volume of the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 6y + 2z \leq 6$.

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Substitute $u = 1 - x/2$, then $du = -dx/2$, so
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$$V = -3 \int_1^0 u^2 du$$
Volume on a region in space.

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Substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = -3 \int_{1}^{0} u^2 du = 3 \int_{0}^{1} u^2 du$$
Volume on a region in space.

Example

Compute the volume of the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 6y + 2z \leq 6$.

Solution: Recall: $V = \frac{3}{2} \int_0^2 \left(1 - \frac{x}{2}\right)^2 \, dx$.

Substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = -3 \int_1^0 u^2 \, du = 3 \int_0^1 u^2 \, du = 3 \left( \frac{u^3}{3} \right|_0^1$$
Volume on a region in space.

Example
Compute the volume of the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 6y + 2z \leq 6$.

Solution: Recall: $V = \frac{3}{2} \int_{0}^{2} \left(1 - \frac{x}{2}\right)^{2} dx$.

Substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = -3 \int_{1}^{0} u^{2} du = 3 \int_{0}^{1} u^{2} du = 3 \left(\frac{u^{3}}{3}\right|_{0}^{1}$$

We conclude: $V = 1$. \triangleleft
Example
Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, 3x \leq y, 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution:
The upper surface is
\[
z = \sqrt{9 - y^2},
\]
the bottom surface is
\[
z = 0.
\]
The \( y \) coordinate is bounded below by the line \( y = 3x \) and above by \( y = 3 \). (Because of the cylinder equation at \( z = 0 \).)
Triple integrals in arbitrary domains.

Example
Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, \ 3x \leq y, \ 0 \leq z \) and \( y^2 + z^2 \leq 9. \)

Solution: Recall: \( 0 \leq z \leq \sqrt{9 - y^2} \) and \( 3x \leq y \leq 3. \)
Triple integrals in arbitrary domains.

Example

Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, 3x \leq y, 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution: Recall: \( 0 \leq z \leq \sqrt{9 - y^2} \) and \( 3x \leq y \leq 3 \).
Since \( f = z \), we obtain

\[
\int \int \int_D f \, dv = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx,
\]

\[
= \int_0^1 \int_{3x}^3 \left( \frac{z^2}{2} \bigg|_0^{\sqrt{9-y^2}} \right) \, dy \, dx,
\]

\[
= \frac{1}{2} \int_0^1 \int_{3x}^3 (9 - y^2) \, dy \, dx,
\]

\[
= \frac{1}{2} \int_0^1 \left[ 27(1 - x) - \left( \frac{y^3}{3} \right|_{3x}^3 \right) \right) \, dx.
\]
Triple integrals in arbitrary domains.

Example

Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, 3x \leq y, 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution: Recall: \[
\iiint_D f \, dv = \frac{1}{2} \int_0^1 \left[ 27(1 - x) - \left( \frac{y^3}{3} \bigg|_3^{3x} \right) \right] \, dx.
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Triple integrals in arbitrary domains.

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Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, 3x \leq y, 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution: Recall: \[
\iiint_{D} f \, dv = \frac{1}{2} \int_{0}^{1} \left[ 27(1 - x) - \left( \frac{y^3}{3} \right)_{3x} \right] \, dx.
\]

Therefore,

\[
\iiint_{D} f \, dv = \frac{1}{2} \int_{0}^{1} \left[ 27(1 - x) - 9(1 - x)^3 \right] \, dx,
\]

\[
= \frac{9}{2} \int_{0}^{1} \left[ 3(1 - x) - (1 - x)^3 \right] \, dx.
\]
Triple integrals in arbitrary domains.

Example

Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x \), \( 3x \leq y \), \( 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution: Recall: 
\[
\begin{align*}
\iiint_D f \, dv &= \frac{1}{2} \int_0^1 \left[ 27(1 - x) - \left( \frac{y^3}{3} \right) \right] \, dx.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\iiint_D f \, dv &= \frac{1}{2} \int_0^1 \left[ 27(1 - x) - 9(1 - x)^3 \right] \, dx, \\
&= \frac{9}{2} \int_0^1 \left[ 3(1 - x) - (1 - x)^3 \right] \, dx.
\end{align*}
\]

Substitute \( u = 1 - x \), then \( du = -dx \), so,
\[
\begin{align*}
\iiint_D f \, dv &= \frac{9}{2} \int_0^1 (3u - u^3) \, du.
\end{align*}
\]
Example
Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, \ 3x \leq y, \ 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution:

\[
\iiint_D f \, dv = \frac{9}{2} \int_0^1 (3u - u^3) \, du,
\]

\[
= \frac{9}{2} \left[ \frac{3}{2} (u^2) \bigg|_0^1 - \frac{1}{4} (u^4) \bigg|_0^1 \right],
\]

\[
= \frac{9}{2} \left( \frac{3}{2} - \frac{1}{4} \right).
\]
Triple integrals in arbitrary domains.

Example

Compute the triple integral of \( f(x, y, z) = z \) in the first octant and bounded by \( 0 \leq x, 3x \leq y, 0 \leq z \) and \( y^2 + z^2 \leq 9 \).

Solution:

\[
\int \int \int_{D} f \, dv = \frac{9}{2} \int_{0}^{1} (3u - u^3) du,
\]

\[
= \frac{9}{2} \left[ \frac{3}{2} (u^2 \Bigr|_{1}^{0}) - \frac{1}{4} (u^4 \Bigr|_{1}^{0}) \right],
\]

\[
= \frac{9}{2} \left( \frac{3}{2} - \frac{1}{4} \right).
\]

We conclude \( \int \int \int_{D} f \, dv = \frac{45}{8} \). \( \triangle \)