Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- The center of mass of an object.
- The moment of inertia of an object.

Areas of a region on a plane.

Definition

The *area* of a closed, bounded region R on a plane is given by

$$A = \iint_R dx\, dy.$$

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Remark:

- To compute the area of a region R we integrate the function $f(x, y) = 1$ on that region R.
- The area of a region R is computed as the volume of a 3-dimensional region with base R and height equal to 1.

Areas of a region on a plane.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

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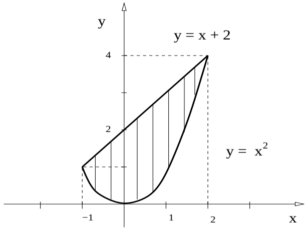

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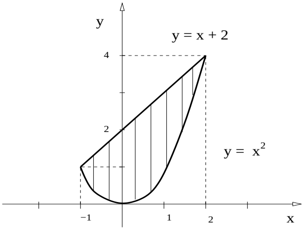

$$A = \int_{-1}^{2} \left(y \Big|_{x^2}^{x+2} \right) dx$$

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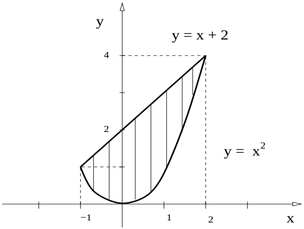

$$A = \int_{-1}^{2} \left(y\Big|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

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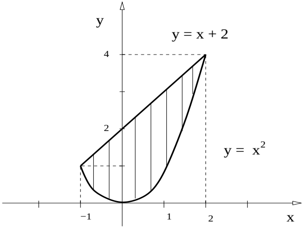

$$A = \int_{-1}^{2} \left(y\Big|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right)\Big|_{-1}^{2}.$$

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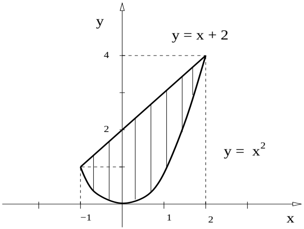

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We conclude that $A = 9/2$. $\triangleleft$

Areas of a region on a plane.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \;:\; x \in [-1, 2],\; y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

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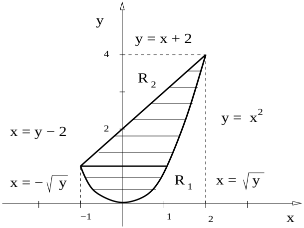

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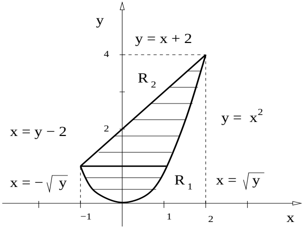

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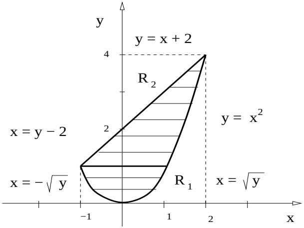

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Verify that the result is: $A = 9/2$. ◁

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- **Average value of a function.**
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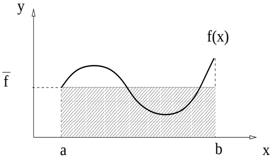
Average value of a function.

Review: The average of a single variable function.

Definition

The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b-a)} \int_a^b f(x)\, dx.$$



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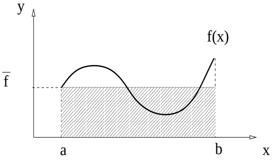

Definition

The *average* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on the region R with area $A(R)$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{A(R)} \iint_R f(x, y)\, dx\, dy.$$

Average value of a function.

Example

Find the average of $f(x, y) = xy$ on the region
$R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [0, 2], \ y \in [0, 3]\}$.

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$R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [0, 2], \ y \in [0, 3]\}$.

Solution: The area of the rectangle R is $A(R) = 6$.

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Since $\overline{f} = I/A(R)$, we get $\overline{f} = 9/6 = 3/2$. $\triangleleft$

Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
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The center of mass of an object.

Review: The *center of mass* of n point particles of mass m_i at the positions $\mathbf{r}_i$ in a plane, where $i = 1, \cdots, n$, is the vector $\bar{\mathbf{r}}$ given by

$$\bar{\mathbf{r}} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r}_i, \quad \text{where} \quad M = \sum_{i=1}^{n} m_i.$$

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$$\bar{\mathbf{r}} = \frac{1}{M} \iint_R \rho(x, y) \langle x, y \rangle \, dx \, dy, \text{ where } M = \iint_R \rho(x, y) \, dx \, dy.$$

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Remark: Certain gravitational effects on an extended object can be described by the gravitational force on a point particle located at the center of mass of the object.

The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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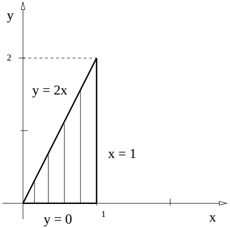

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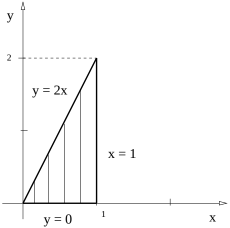

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$$M = \int_0^1 \left[x \left(y \Big|_0^{2x} \right) + \left(\frac{y^2}{2} \Big|_0^{2x} \right) \right] dx$$

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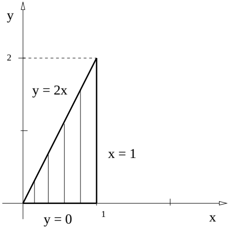

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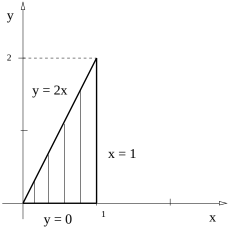

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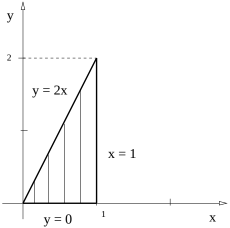

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We conclude that $M = \dfrac{4}{3}$.

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Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \frac{4}{3}$. The coordinates x and y of the center of mass are

$$\bar{r}_x = \frac{1}{M}\int_0^1\int_0^{2x}(x+y)x\,dy\,dx, \quad \bar{r}_y = \frac{1}{M}\int_0^1\int_0^{2x}(x+y)y\,dy\,dx.$$

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We compute the $\bar{r}_x$ component.

$$\bar{r}_x = \frac{3}{4}\int_0^1 \Big[x^2 \Big(y\Big|_0^{2x} \Big) + x \Big(\frac{y^2}{2}\Big|_0^{2x} \Big) \Big]\, dx$$

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We compute the $\bar{r}_x$ component.

$$\bar{r}_x = \frac{3}{4} \int_0^1 \Big[x^2 \Big(y \Big|_0^{2x} \Big) + x \Big(\frac{y^2}{2} \Big|_0^{2x} \Big) \Big] \, dx = \frac{3}{4} \int_0^1 \big[2x^3 + 2x^3 \big] \, dx,$$

so $\bar{r}_x = \frac{3}{4} x^4 \Big|_0^1$. We conclude that $\bar{r}_x = \frac{3}{4}$.

The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \dfrac{4}{3}$ and $\bar{r}_x = \dfrac{3}{4}$.

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$$\bar{r}_y = \frac{3}{4} \int_0^1 \left[x \left(\frac{y^2}{2} \Big|_0^{2x} \right) + \left(\frac{y^3}{3} \Big|_0^{2x} \right) \right] dx$$

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Therefore, the center of mass vector is $\bar{\mathbf{r}} = \left\langle \dfrac{3}{4}, \dfrac{7}{8} \right\rangle$. $\triangleleft$

The centroid of an object.

Definition

The *centroid* of a region R in the plane is the vector $\mathbf{c}$ given by

$$\mathbf{c} = \frac{1}{A(R)} \iint_R \langle x, y \rangle \, dx \, dy, \quad \text{where} \quad A(R) = \iint_R dx \, dy.$$

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Remark:

- The centroid of a region can be seen as the center of mass vector of that region in the case that the mass density is constant.

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- The centroid of a region can be seen as the center of mass vector of that region in the case that the mass density is constant.
- When the mass density is constant, it cancels out from the numerator and denominator of the center of mass.

The centroid of an object.

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so $c_y = \dfrac{2}{3}$.

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so $c_y = \frac{2}{3}$. We conclude, $\mathbf{c} = \frac{2}{3}\langle 1, 1\rangle$. ◁

Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- The center of mass of an object.
- **The moment of inertia of an object.**

The moment of inertia of an object.

Remark: The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.

The moment of inertia of an object.

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Definition

The *moment of inertia* about the x-axis and the y-axis of a region R in the plane having mass density $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$ are given by, respectively,

$$I_x = \iint_R y^2 \rho(x, y)\, dx\, dy, \qquad I_y = \iint_R x^2 \rho(x, y)\, dx\, dy.$$

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If M denotes the total mass of the region, then the *radii of gyration* about the x-axis and the y-axis are given by

$$R_x = \sqrt{I_x / M} \qquad R_y = \sqrt{I_y / M}.$$

The moment of inertia of an object.

Example

Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The moment of inertia I_x is given by

$$I_x = \int_0^1 \int_0^{2x} x^2 (x + y)\, dy\, dx$$

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Solution: The moment of inertia I_x is given by

$$I_x = \int_0^1 \int_0^{2x} x^2(x + y)\, dy\, dx = \int_0^1 \left[x^3 \left(y \Big|_0^{2x} \right) + x^2 \left(\frac{y^2}{2} \Big|_0^{2x} \right) \right] dx$$

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$$I_x = \int_0^1 4x^4\, dx$$

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$$I_x = \int_0^1 4x^4\,dx = 4\left(\frac{x^5}{5}\Big|_0^1 \right) \quad \Rightarrow \quad I_x = \frac{4}{5}.$$

Since the mass of the region is $M = 4/3$, the radius of gyration along the x-axis is $R_x = \sqrt{I_x/M}$

The moment of inertia of an object.

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Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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$$I_x = \int_0^1 4x^4\,dx = 4\left(\frac{x^5}{5}\Big|_0^1\right) \quad \Rightarrow \quad I_x = \frac{4}{5}.$$

Since the mass of the region is $M = 4/3$, the radius of gyration along the x-axis is $R_x = \sqrt{I_x/M} = \sqrt{\frac{4}{5}\,\frac{3}{4}}$,

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Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

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Since the mass of the region is $M = 4/3$, the radius of gyration along the x-axis is $R_x = \sqrt{I_x/M} = \sqrt{\frac{4}{5}\,\frac{3}{4}}$, that is, $R_x = \sqrt{\frac{3}{5}}$. $\triangleleft$

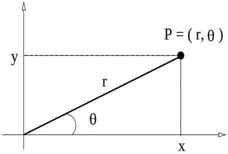
Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.

Review: Polar coordinates.

Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



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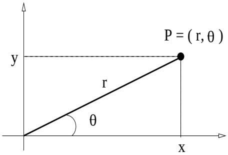

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Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point $P = (r, \theta)$ in the first quadrant are given by

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

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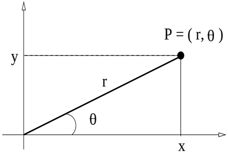

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The Cartesian coordinates of a point $P = (r, \theta)$ *in the first quadrant are given by*

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ *in the first quadrant are given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- **Double integrals in disk sections.**
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.
- Double integrals in arbitrary regions.

Double integrals on disk sections.

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ *is continuous in the region*

$$R = \{(r,\theta) \in \mathbb{R}^2 : r \in [r_0, r_1],\ \theta \in [\theta_0, \theta_1]\}$$

where $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 2\pi$*, then the double integral of function* f *in that region can be expressed in polar coordinates as follows,*

$$\iint_R f\, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r,\theta)\, r\, dr\, d\theta.$$

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Remark:

- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.

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- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
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- Notice the extra factor r on the right-hand side.

Double integrals on disk sections.

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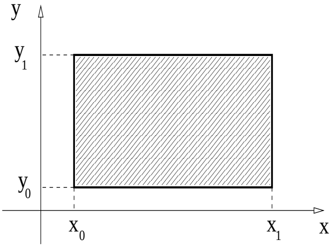

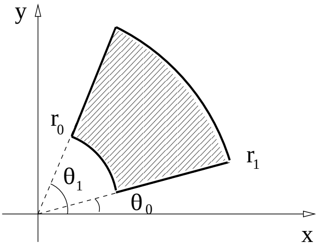

$$x_0 \leqslant x \leqslant x_1, \qquad 0 \leqslant r_0 \leqslant r \leqslant r_1,$$

$$y_0 \leqslant y \leqslant y_1, \qquad 0 \leqslant \theta_0 \leqslant \theta \leqslant \theta_1 \leqslant 2\pi.$$

Double integrals on disk sections.

Example

Find the area of an arbitrary circular section
$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [r_0, r_1],\ \theta \in [\theta_0, \theta_1]\}$.
Evaluate that area in the particular case of a disk with radius R.

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In that case we reobtain the usual formula $A = \pi R^2$. $\triangleleft$

Double integrals on disk sections.

Example

Find the integral of $f(r,\theta) = r^2\cos(\theta)$ in the disk
$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [0,1],\ \theta \in [0,\pi/4]\}$.

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We conclude that $\iint_R f\,dA = \sqrt{2}/8$. $\triangleleft$

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Theorem

If the function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

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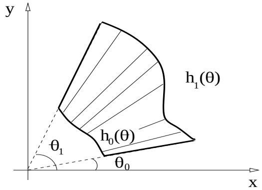

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Completing the square in x we obtain

$$\Big(x - \frac{1}{2}\Big)^2 + y^2 = \Big(\frac{1}{2}\Big)^2.$$

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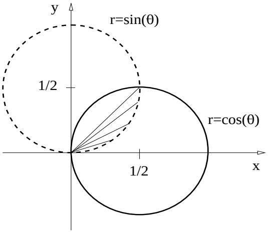

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Also works: $A = \int_0^{\pi/4}\int_0^{\sin(\theta)} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\cos(\theta)} r\,dr\,d\theta.$

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Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $f(x, y)$ represents the function values in Cartesian coordinates, then holds

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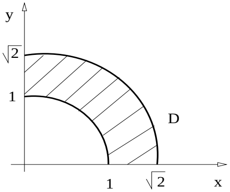

Changing Cartesian integrals into polar integrals.

Example

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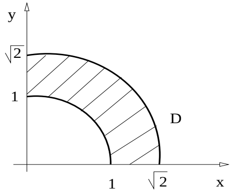

The region is

$$D = \left\{(r,\theta) \in \mathbb{R}^2 \ : \ 0 \leqslant \theta \leqslant \frac{\pi}{2},\ 1 \leqslant r \leqslant \sqrt{2}\right\}.$$

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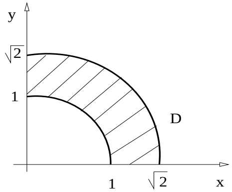

The region is

$$D = \Big\{(r,\theta) \in \mathbb{R}^2 \ : \ 0 \leqslant \theta \leqslant \frac{\pi}{2}, \ 1 \leqslant r \leqslant \sqrt{2}\Big\}.$$

$$\iint_D f(r,\theta)\,dA = \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\big(1+\sin^2(\theta)\big)\,r\,dr\,d\theta,$$

$$= \Big[\int_0^{\pi/2}\big(1+\sin^2(\theta)\big)\,d\theta\Big]\Big[\int_1^{\sqrt{2}} r^3\,dr\Big],$$

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$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y, \ 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: $$\iint_D f(r, \theta) dA = \left[\int_0^{\pi/2} \big(1 + \sin^2(\theta)\big) \, d\theta \right] \left[\int_1^{\sqrt{2}} r^3 dr \right].$$

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$$\iint_D f(r,\theta)dA = \left[\left(\theta\Big|_0^{\pi/2}\right) + \int_0^{\pi/2} \frac{1}{2}\left(1 - \cos(2\theta)\right) d\theta\right] \frac{1}{4}\left(r^4\Big|_1^{\sqrt{2}}\right)$$

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We conclude: $\displaystyle\iint_D f(r,\theta)\,dA = \frac{9}{16}\pi.$ ◁

Changing Cartesian integrals into polar integrals.

Example

Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain
$D = \{(r, \theta) \in R^2 \ : \ 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant r \leqslant 2\}$.

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Solution: Since $f(r\cos(\theta), r\sin(\theta)) = e^{-r^2}$, the double integral is

Changing Cartesian integrals into polar integrals.

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Substituting $u = r^2$, hence $du = 2r\,dr$, we obtain

$$\iint_D f(x,y)\,dx\,dy = \frac{1}{2} \int_0^{\pi} \int_0^4 e^{-u} du\,d\theta = \frac{1}{2} \int_0^{\pi} \Big(-e^{-u} \Big|_0^4 \Big)\,d\theta;$$

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We conclude: $\displaystyle\iint_D f(x,y)\,dx\,dy = \frac{\pi}{2}\Big(1 - \frac{1}{e^4}\Big)$. ◁

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- **Computing volumes using double integrals.**

Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Computing volumes using double integrals.

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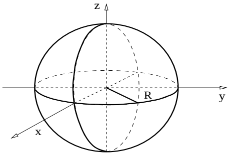
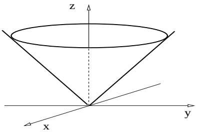
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Let us first draw the sets that form the volume we are interested to compute.





$$z = \pm\sqrt{1 - r^2},$$

$$z = r.$$

Computing volumes using double integrals.

Example

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Solution: The integration region can be decomposed as follows:

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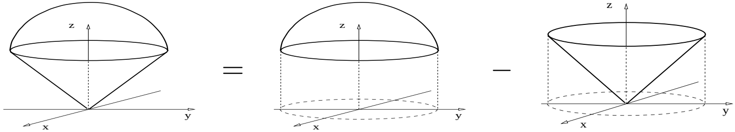

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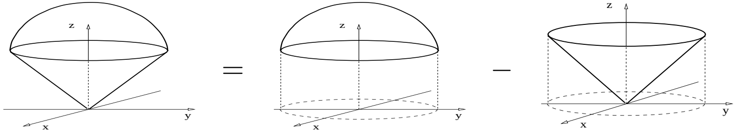

The volume we are interested to compute is:

$$V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2}\,(r dr) d\theta - \int_0^{2\pi} \int_0^{r_0} r\,(r dr) d\theta.$$

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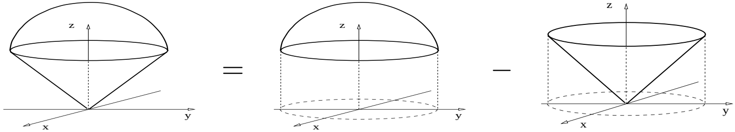

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We need to find r_0, the intersection of the cone and the sphere.

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$$V = 2\pi \Big[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr) - \int_0^{1/\sqrt{2}} r\,(r\,dr) \Big].$$

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We conclude: $V = \dfrac{\pi}{3}\big(2 - \sqrt{2}\big)$. ◁

Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- Volume on a region in space.

Triple integrals in rectangular boxes.

Definition

The *triple integral* of a function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ in the rectangular box $R = [\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$ is the number

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z.$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$, $z_k^* \in [z_k, z_{k+1}]$ are sample points, while $\{x_i\}$, $\{y_j\}$, $\{z_k\}$, with $i, j, k = 0, \cdots, n$, are partitions of the intervals $[\hat{x}_0, \hat{x}_1]$, $[\hat{y}_0, \hat{y}_1]$, $[\hat{z}_0, \hat{z}_1]$, respectively, and

$$\Delta x = \frac{(\hat{x}_1 - \hat{x}_0)}{n}, \quad \Delta y = \frac{(\hat{y}_1 - \hat{y}_0)}{n}, \quad \Delta z = \frac{(\hat{z}_1 - \hat{z}_0)}{n}.$$

Triple integrals in rectangular boxes.

Remark:

- A finite sum S_n below is called a Riemann sum, where

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z.$$

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- Then holds $\displaystyle\iiint_R f(x, y, z)\, dx\, dy\, dz = \lim_{n\to\infty} S_n.$

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- Then holds $\displaystyle\iiint_R f(x,y,z)\,dx\,dy\,dz = \lim_{n\to\infty} S_n.$

Theorem (Fubini)

If function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the rectangle $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then holds

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x,y,z)\,dz\,dy\,dx.$$

Furthermore, the integral above can be computed integrating the variables x, y, z in any order.

Triple integrals in rectangular boxes.

Review: The Riemann sums and their limits.

Single variable functions in $[\hat{x}_0, \hat{x}_1]$:

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Triple integrals in rectangular boxes.

Example

Compute the integral of $f(x, y, z) = xyz^2$ on the domain $R = [0, 1] \times [0, 2] \times [0, 3]$.

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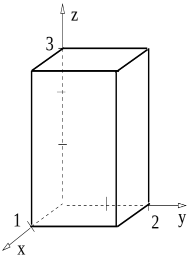

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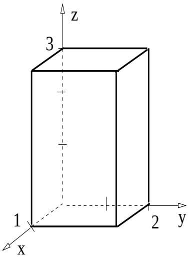

The integral we need to compute is

$$\iiint_R f \, dv = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx,$$

where we denoted $dv = dx\,dy\,dz$.

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We conclude: $\displaystyle\iiint_R f\,dv = 9.$ $\lhd$

Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- **Triple integrals in arbitrary domains.**
- Volume on a region in space.

Triple integrals in arbitrary domains.

Theorem

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ *is continuous in the domain*

$$D = \{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ *and* $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ *are continuous, then the triple integral of the function* f *in the region* D *is given by*

$$\iiint_D f\, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

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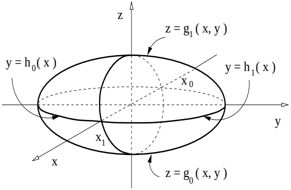
Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



Triple integrals in Cartesian coordinates (Sect. 15.4)

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Volume on a region in space.

Remark: The volume of a bounded, closed region $D \in \mathbb{R}^3$ is

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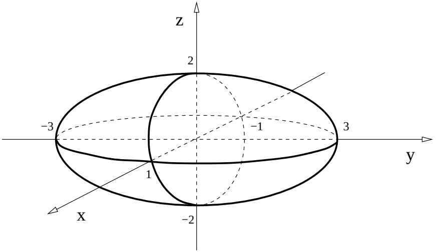

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Solution: The functions $z = g_1$ and $z = g_0$ are, respectively,

$$z = 2\sqrt{1 - x^2 - \frac{y^2}{3^2}}, \quad z = -2\sqrt{1 - x^2 - \frac{y^2}{3^2}}.$$

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The limits on integration in x are ± 1. We conclude:

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx. \qquad \triangleleft$$

Volume on a region in space.

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Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

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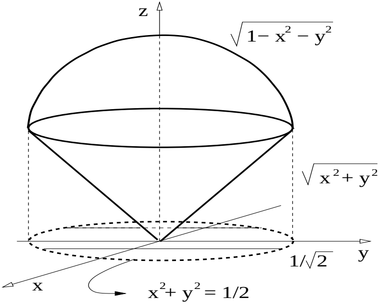

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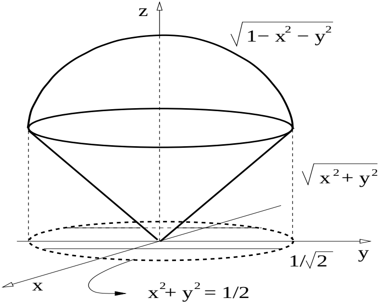

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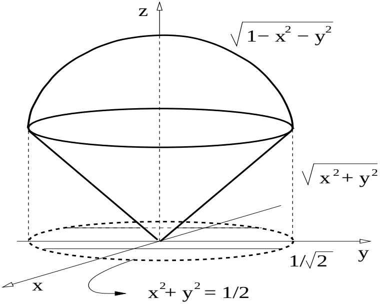

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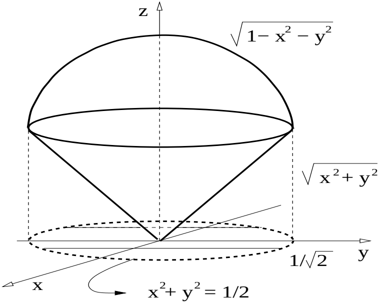

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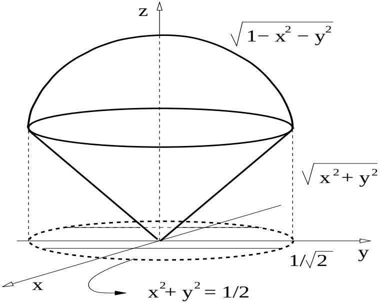

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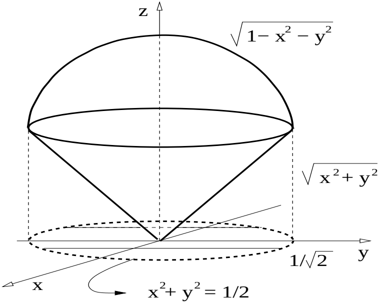

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(The polar radius at the intersection cone-sphere was $r_0 = 1/\sqrt{2}$.)

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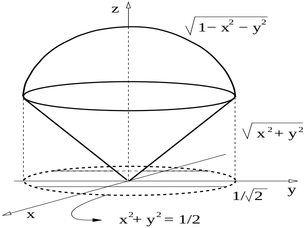

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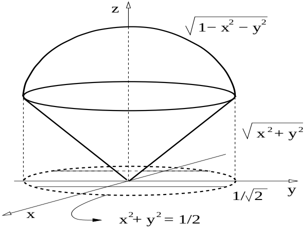

The y-top of the disk is,

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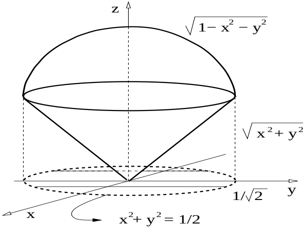

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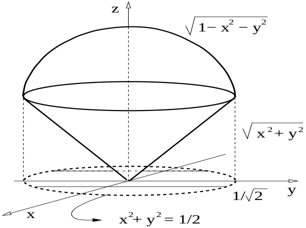

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We conclude: $V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx.$ $\triangleleft$

Volume on a region in space.

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

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Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

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The region is given by the first octant and below the plane

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This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.

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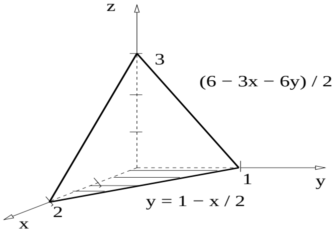

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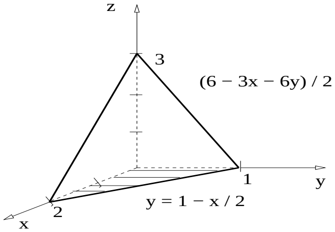

In z the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$.

Volume on a region in space.

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

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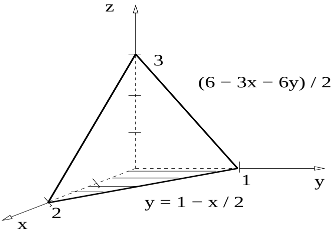

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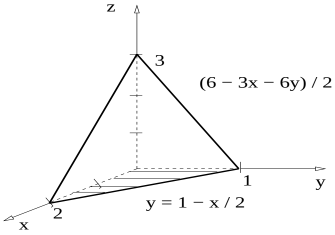

We conclude: $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

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$$\begin{aligned}
V &= 3 \int_0^2 \int_0^{1-x/2} \Big(1 - \frac{x}{2} - y\Big)\, dy\, dx, \\
&= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(y\Big|_0^{(1-x/2)}\Big) - \Big(\frac{y^2}{2}\Big|_0^{(1-x/2)}\Big)\Big]\, dx, \\
&= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(1 - \frac{x}{2}\Big) - \frac{1}{2}\Big(1 - \frac{x}{2}\Big)^2\Big]\, dx.
\end{aligned}$$

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$$V = 3 \int_0^2 \int_0^{1-x/2} \Big(1 - \frac{x}{2} - y\Big)\, dy\, dx,$$

$$= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(y\Big|_0^{(1-x/2)}\Big) - \Big(\frac{y^2}{2}\Big|_0^{(1-x/2)}\Big)\Big]\, dx,$$

$$= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(1 - \frac{x}{2}\Big) - \frac{1}{2}\Big(1 - \frac{x}{2}\Big)^2\Big]\, dx.$$

We only need to compute: $V = \frac{3}{2} \int_0^2 \Big(1 - \frac{x}{2}\Big)^2 dx$.

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Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: Recall: $V = \dfrac{3}{2}\int_0^2 \Big(1 - \dfrac{x}{2}\Big)^2\, dx$.

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We conclude: $V = 1$. ◁

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

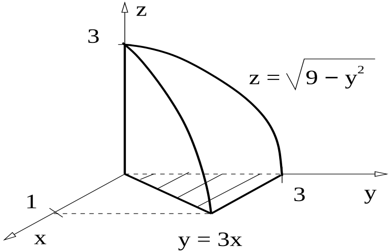
Solution:

The upper surface is

$$z = \sqrt{9 - y^2},$$

the bottom surface is

$$z = 0.$$



The y coordinate is bounded below by the line $y = 3x$ and above by $y = 3$. (Because of the cylinder equation at $z = 0$.)

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Solution: Recall: $0 \leqslant z \leqslant \sqrt{9 - y^2}$ and $3x \leqslant y \leqslant 3$.
Since $f = z$, we obtain

$$\iiint_D f\,dv = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx,$$

$$= \int_0^1 \int_{3x}^3 \Big(\frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} \Big)\,dy\,dx,$$

$$= \frac{1}{2} \int_0^1 \int_{3x}^3 (9 - y^2)dy\,dx,$$

$$= \frac{1}{2} \int_0^1 \Big[27(1-x) - \Big(\frac{y^3}{3}\Big|_{3x}^{3} \Big) \Big] dx.$$

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Therefore,

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Substitute $u = 1 - x$, then $du = -dx$, so,

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Solution:

$$\int\int\int_D f\,dv = \frac{9}{2}\int_0^1 (3u - u^3)du,$$
$$= \frac{9}{2}\Big[\frac{3}{2}\Big(u^2\Big|_0^1\Big) - \frac{1}{4}\Big(u^4\Big|_0^1\Big)\Big],$$
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$$= \frac{9}{2}\Big(\frac{3}{2} - \frac{1}{4}\Big).$$

We conclude $\iiint_D f\,dv = \frac{45}{8}$. $\triangleleft$