Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- The center of mass of an object.
- The moment of inertia of an object.

Areas of a region on a plane.

Definition

The *area* of a closed, bounded region R on a plane is given by

$$A = \iint_R dx\, dy.$$

Remark:

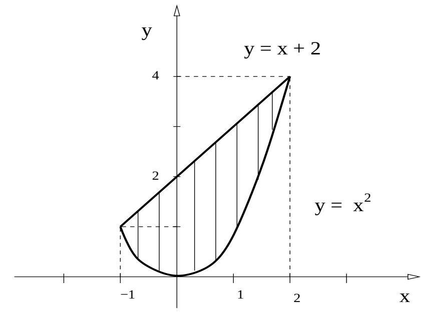
- To compute the area of a region R we integrate the function $f(x, y) = 1$ on that region R.
- The area of a region R is computed as the volume of a 3-dimensional region with base R and height equal to 1.

Areas of a region on a plane.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$.

Solution: We express the region R as an integral Type I, integrating first on vertical directions:



$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

$$A = \int_{-1}^{2} \Big(y\Big|_{x^2}^{x+2}\Big)\, dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \Big(\frac{x^2}{2} + 2x - \frac{x^3}{3}\Big)\Big|_{-1}^{2}.$$

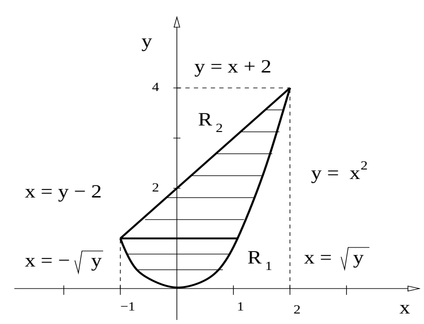
We conclude that $A = 9/2$. ◁

Areas of a region on a plane.

Example

Find the area of $R = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-1, 2], \ y \in [x^2, x + 2]\}$ integrating first along horizontal directions.

Solution: We express the region R as an integral Type II, integrating first on horizontal directions:



$$A = \iint_{R_1} dx\, dy + \iint_{R_2} dx\, dy.$$

$$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx\, dy.$$

Verify that the result is: $A = 9/2$. ◁

Area, center of mass, moments of inertia. (Sect. 15.2)

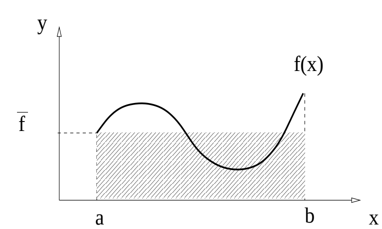
- Areas of a region on a plane.
- **Average value of a function.**
- The center of mass of an object.
- The moment of inertia of an object.

Average value of a function.

Review: The average of a single variable function.

Definition
The *average* of a function $f : [a,b] \to \mathbb{R}$ on the interval $[a,b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b-a)} \int_a^b f(x)\,dx.$$



Definition
The *average* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on the region R with area $A(R)$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{A(R)} \iint_R f(x,y)\,dx\,dy.$$

Average value of a function.

Example

Find the average of $f(x,y) = xy$ on the region $R = \{(x,y) \in \mathbb{R}^2 \; : \; x \in [0,2], \; y \in [0,3]\}$.

Solution: The area of the rectangle R is $A(R) = 6$. We only need to compute $I = \iint_R f(x,y)\,dx\,dy$.

$$I = \int_0^2 \int_0^3 xy\,dy\,dx = \int_0^2 x\Big(\frac{y^2}{2}\Big|_0^3\Big)\,dx = \int_0^2 \frac{9}{2}x\,dx.$$

$$I = \frac{9}{2}\Big(\frac{x^2}{2}\Big|_0^2\Big) \quad \Rightarrow \quad I = 9.$$

Since $\overline{f} = I/A(R)$, we get $\overline{f} = 9/6 = 3/2$. $\triangleleft$

Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
- Average value of a function.
- **The center of mass of an object.**
- The moment of inertia of an object.

The center of mass of an object.

Review: The *center of mass* of n point particles of mass m_i at the positions $\mathbf{r}_i$ in a plane, where $i = 1, \cdots, n$, is the vector $\bar{\mathbf{r}}$ given by

$$\bar{\mathbf{r}} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r}_i, \quad \text{where} \quad M = \sum_{i=1}^{n} m_i.$$

Definition

The *center of mass* of a region R in the plane, having a continuous mass distribution given by a density function $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$, is the vector $\bar{\mathbf{r}}$ given by

$$\bar{\mathbf{r}} = \frac{1}{M} \iint_R \rho(x,y) \, \langle x, y \rangle \, dx \, dy, \text{ where } M = \iint_R \rho(x,y) \, dx \, dy.$$

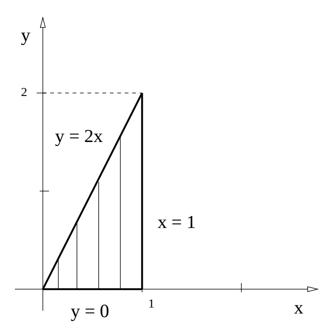
Remark: Certain gravitational effects on an extended object can be described by the gravitational force on a point particle located at the center of mass of the object.

The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x,y) = x + y$.

Solution:



We first compute the total mass M,

$$M = \int_0^1 \int_0^{2x} (x + y) \, dy \, dx.$$

$$M = \int_0^1 \Big[x \Big(y\Big|_0^{2x} \Big) + \Big(\frac{y^2}{2}\Big|_0^{2x} \Big) \Big] \, dx = \int_0^1 \big[2x^2 + 2x^2 \big] \, dx = 4 \, \frac{x^3}{3}\Big|_0^1.$$

We conclude that $M = \dfrac{4}{3}$.

The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \frac{4}{3}$. The coordinates x and y of the center of mass are

$$\bar{r}_x = \frac{1}{M}\int_0^1\int_0^{2x}(x+y)x\,dy\,dx, \quad \bar{r}_y = \frac{1}{M}\int_0^1\int_0^{2x}(x+y)y\,dy\,dx.$$

We compute the $\bar{r}_x$ component.

$$\bar{r}_x = \frac{3}{4}\int_0^1\Big[x^2\Big(y\Big|_0^{2x}\Big) + x\Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\Big]\,dx = \frac{3}{4}\int_0^1\big[2x^3 + 2x^3\big]\,dx,$$

so $\bar{r}_x = \frac{3}{4}\,x^4\Big|_0^1$. We conclude that $\bar{r}_x = \frac{3}{4}$.

The center of mass of an object.

Example

Find the center of mass of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The total mass is $M = \frac{4}{3}$ and $\bar{r}_x = \frac{3}{4}$.

Since $\bar{r}_y = \frac{1}{M}\int_0^1\int_0^{2x}(x+y)y\,dy\,dx$, we obtain

$$\bar{r}_y = \frac{3}{4}\int_0^1\Big[x\Big(\frac{y^2}{2}\Big|_0^{2x}\Big) + \Big(\frac{y^3}{3}\Big|_0^{2x}\Big)\Big]\,dx = \frac{3}{4}\int_0^1\Big[2x^3 + \frac{8}{3}x^3\Big]\,dx,$$

$$\bar{r}_y = \frac{3}{4}\Big[2\Big(\frac{x^4}{4}\Big|_0^1\Big) + \frac{8}{3}\Big(\frac{x^4}{4}\Big|_0^1\Big)\Big] = \frac{3}{4}\Big[\frac{1}{2} + \frac{2}{3}\Big] = \frac{3}{4}\,\frac{7}{6} \Rightarrow \bar{r}_y = \frac{7}{8}.$$

Therefore, the center of mass vector is $\bar{\mathbf{r}} = \Big\langle\frac{3}{4}, \frac{7}{8}\Big\rangle$. $\triangleleft$

The centroid of an object.

Definition

The *centroid* of a region R in the plane is the vector $\mathbf{c}$ given by

$$\mathbf{c} = \frac{1}{A(R)} \iint_R \langle x, y\rangle \, dx\, dy, \quad \text{where} \quad A(R) = \iint_R dx\, dy.$$

Remark:

- The centroid of a region can be seen as the center of mass vector of that region in the case that the mass density is constant.
- When the mass density is constant, it cancels out from the numerator and denominator of the center of mass.

The centroid of an object.

Example

Find the centroid of the triangle inside $y = 0$, $x = 1$ and $y = 2x$.

Solution: The area of the triangle is

$$A(R) = \int_0^1 \int_0^{2x} dy\, dx = \int_0^1 2x\, dx = x^2\Big|_0^1 \quad \Rightarrow \quad A(R) = 1.$$

Therefore, the centroid vector components are given by

$$c_x = \int_0^1 \int_0^{2x} x\, dy\, dx = \int_0^1 2x^2\, dx = 2\Big(\frac{x^3}{3}\Big|_0^1\Big) \quad \Rightarrow \quad c_x = \frac{2}{3}.$$

$$c_y = \int_0^1 \int_0^{2x} y\, dy\, dx = \int_0^1 \Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\, dx = \int_0^1 2x^2\, dx = 2\Big(\frac{x^3}{3}\Big|_0^1\Big)$$

so $c_y = \frac{2}{3}$. We conclude, $\mathbf{c} = \frac{2}{3}\langle 1, 1\rangle$. $\triangleleft$

Area, center of mass, moments of inertia. (Sect. 15.2)

- Areas of a region on a plane.
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The moment of inertia of an object.

Remark: The moment of inertia of an object is a measure of the resistance of the object to changes in its rotation along a particular axis of rotation.

Definition

The *moment of inertia* about the x-axis and the y-axis of a region R in the plane having mass density $\rho : R \subset \mathbb{R}^2 \to \mathbb{R}$ are given by, respectively,

$$I_x = \iint_R y^2 \rho(x, y)\, dx\, dy, \qquad I_y = \iint_R x^2 \rho(x, y)\, dx\, dy.$$

If M denotes the total mass of the region, then the *radii of gyration* about the x-axis and the y-axis are given by

$$R_x = \sqrt{I_x / M} \qquad R_y = \sqrt{I_y / M}.$$

The moment of inertia of an object.

Example

Find the moment of inertia and the radius of gyration about the x-axis of the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, and mass density $\rho(x, y) = x + y$.

Solution: The moment of inertia I_x is given by

$$I_x = \int_0^1 \int_0^{2x} x^2(x+y)\,dy\,dx = \int_0^1 \Big[x^3\Big(y\Big|_0^{2x}\Big) + x^2\Big(\frac{y^2}{2}\Big|_0^{2x}\Big)\Big]\,dx$$

$$I_x = \int_0^1 4x^4\,dx = 4\Big(\frac{x^5}{5}\Big|_0^1\Big) \quad \Rightarrow \quad I_x = \frac{4}{5}.$$

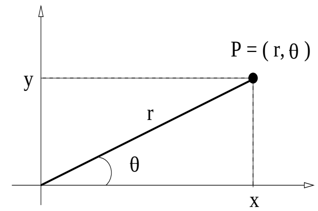
Since the mass of the region is $M = 4/3$, the radius of gyration along the x-axis is $R_x = \sqrt{I_x/M} = \sqrt{\frac{4}{5}\,\frac{3}{4}}$, that is, $R_x = \sqrt{\frac{3}{5}}$. $\lhd$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.

Review: Polar coordinates.

Definition
The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



Theorem (Cartesian-polar transformations)
The Cartesian coordinates of a point $P = (r, \theta)$ in the first quadrant are given by

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ in the first quadrant are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- **Double integrals in disk sections.**
- Double integrals in arbitrary regions.
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.
- Double integrals in arbitrary regions.

Double integrals on disk sections.

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

$$R = \{(r, \theta) \in \mathbb{R}^2 : r \in [r_0, r_1],\ \theta \in [\theta_0, \theta_1]\}$$

where $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 2\pi$, then the double integral of function f in that region can be expressed in polar coordinates as follows,

$$\iint_R f\, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r,\theta)\, r\, dr\, d\theta.$$

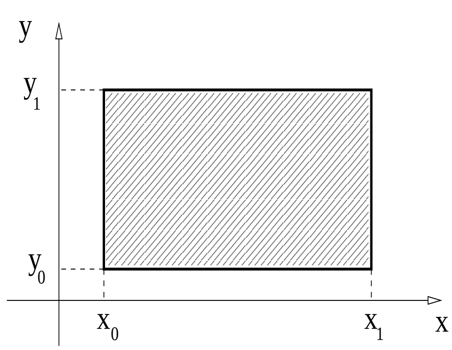
Remark:

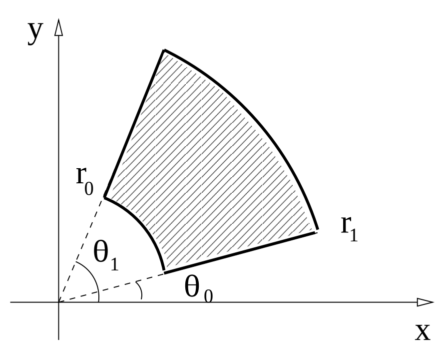
- Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.
- The boundaries of both domains are given by a coordinate equal constant.
- Notice the extra factor r on the right-hand side.

Double integrals on disk sections.

Remark:

Disk sections in polar coordinates are analogous to rectangular sections in Cartesian coordinates.





$$x_0 \leqslant x \leqslant x_1, \qquad 0 \leqslant r_0 \leqslant r \leqslant r_1,$$

$$y_0 \leqslant y \leqslant y_1, \qquad 0 \leqslant \theta_0 \leqslant \theta \leqslant \theta_1 \leqslant 2\pi.$$

Double integrals on disk sections.

Example

Find the area of an arbitrary circular section
$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [r_0, r_1], \ \theta \in [\theta_0, \theta_1]\}$.
Evaluate that area in the particular case of a disk with radius R.

Solution:

$$A = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} (r\,dr)\,d\theta = \int_{\theta_0}^{\theta_1} \frac{1}{2}\big[(r_1)^2 - (r_0)^2\big]\,d\theta$$

we obtain: $A = \frac{1}{2}\big[(r_1)^2 - (r_0)^2\big](\theta_1 - \theta_0)$.

The case of a disk is: $\theta_0 = 0$, $\theta_1 = 2\pi$, $r_0 = 0$ and $r_1 = R$.

In that case we reobtain the usual formula $A = \pi R^2$. $\lhd$

Double integrals on disk sections.

Example

Find the integral of $f(r,\theta) = r^2\cos(\theta)$ in the disk
$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [0,1], \ \theta \in [0, \pi/4]\}$.

Solution:

$$\iint_R f\,dA = \int_0^{\pi/4} \int_0^1 r^2 \cos(\theta)(r\,dr)\,d\theta,$$

$$\iint_R f\,dA = \int_0^{\pi/4} \Big(\frac{r^4}{4}\Big|_0^1\Big)\cos(\theta)\,d\theta = \frac{1}{4}\sin(\theta)\Big|_0^{\pi/4}.$$

We conclude that $\iint_R f\,dA = \sqrt{2}/8$. $\lhd$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- **Double integrals in arbitrary regions.**
- Changing Cartesian integrals into polar integrals.
- Computing volumes using double integrals.

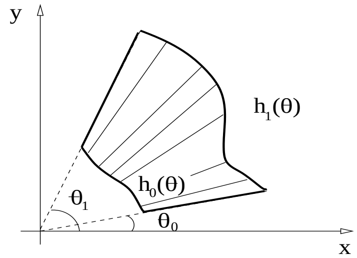
Double integrals in arbitrary regions.

Theorem
If the function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in the region

$$R = \big\{(r,\theta) \in \mathbb{R}^2 \ : \ r \in [h_0(\theta), h_1(\theta)], \ \theta \in [\theta_0, \theta_1]\big\}.$$

where $0 \leqslant h_0(\theta) \leqslant h_1(\theta)$ are continuous functions defined on an interval $[\theta_0, \theta_1]$, then the integral of function f in R is given by

$$\iint_R f(r,\theta)\, dA = \int_{\theta_0}^{\theta_1} \int_{h_0(\theta)}^{h_1(\theta)} f(r,\theta) r \, dr \, d\theta.$$



Double integrals in arbitrary regions.

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first show that these curves are actually circles.

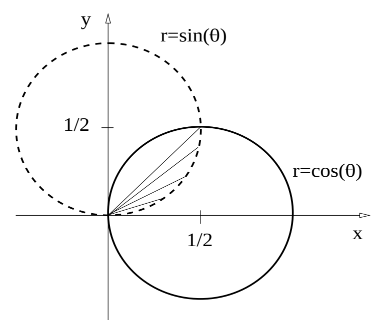
$$r = \cos(\theta) \quad \Leftrightarrow \quad r^2 = r\cos(\theta) \quad \Leftrightarrow \quad x^2 + y^2 = x.$$

Completing the square in x we obtain

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$

Analogously, $r = \sin(\theta)$ is the circle

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$



Double integrals in arbitrary regions.

Example

Find the area of the region bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: $A = 2\int_0^{\pi/4}\int_0^{\sin(\theta)} r\,dr\,d\theta = 2\int_0^{\pi/4} \frac{1}{2}\sin^2(\theta)\,d\theta;$

$$A = \int_0^{\pi/4} \frac{1}{2}\big[1 - \cos(2\theta)\big]\,d\theta = \frac{1}{2}\Big[\Big(\frac{\pi}{4} - 0\Big) - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4}\Big];$$

$$A = \frac{1}{2}\Big[\frac{\pi}{4} - \Big(\frac{1}{2} - 0\Big)\Big] = \frac{\pi}{8} - \frac{1}{4} \quad \Rightarrow \quad A = \frac{1}{8}(\pi - 2).$$

◁

Also works: $A = \int_0^{\pi/4}\int_0^{\sin(\theta)} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\cos(\theta)} r\,dr\,d\theta.$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
- Double integrals in disk sections.
- Double integrals in arbitrary regions.
- **Changing Cartesian integrals into polar integrals.**
- Computing volumes using double integrals.

Changing Cartesian integrals into polar integrals.

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $f(x, y)$ represents the function values in Cartesian coordinates, then holds

$$\iint_D f(x, y)\, dx\, dy = \iint_D f(r\cos(\theta), r\sin(\theta)) r\, dr\, d\theta.$$

Example

Compute the integral of $f(x, y) = x^2 + 2y^2$ on
$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y, \quad 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: First, transform Cartesian into polar coordinates:
$x = r\cos(\theta)$, $y = r\sin(\theta)$. Since $f(x, y) = (x^2 + y^2) + y^2$,

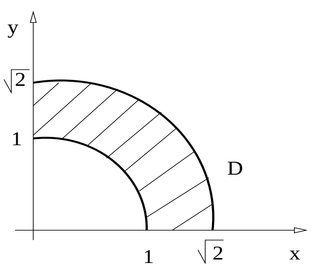
$$f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\sin^2(\theta).$$

Changing Cartesian integrals into polar integrals.

Example

Compute the integral of $f(x,y) = x^2 + 2y^2$ on
$D = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant y, \ 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: We computed: $f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\sin^2(\theta)$.



The region is

$$D = \left\{(r,\theta) \in \mathbb{R}^2 \ : \ 0 \leqslant \theta \leqslant \frac{\pi}{2}, \ 1 \leqslant r \leqslant \sqrt{2}\right\}.$$

$$\iint_D f(r,\theta)dA = \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\big(1+\sin^2(\theta)\big)\,r\,dr d\theta,$$

$$= \left[\int_0^{\pi/2}\big(1+\sin^2(\theta)\big)\,d\theta\right]\left[\int_1^{\sqrt{2}} r^3 dr\right],$$

Changing Cartesian integrals into polar integrals.

Example

Compute the integral of $f(x,y) = x^2 + 2y^2$ on
$D = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant y, \ 0 \leqslant x, \quad 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Solution: $\displaystyle\iint_D f(r,\theta)dA = \left[\int_0^{\pi/2}\big(1+\sin^2(\theta)\big)\,d\theta\right]\left[\int_1^{\sqrt{2}} r^3 dr\right].$

$$\iint_D f(r,\theta)dA = \left[\left(\theta\Big|_0^{\pi/2}\right) + \int_0^{\pi/2}\frac{1}{2}\big(1-\cos(2\theta)\big)\,d\theta\right]\frac{1}{4}\left(r^4\Big|_1^{\sqrt{2}}\right)$$

$$\iint_D f(r,\theta)dA = \left[\frac{\pi}{2}+\frac{1}{2}\left(\theta\Big|_0^{\pi/2}\right)-\frac{1}{4}\left(\sin(2\theta)\Big|_0^{\pi/2}\right)\right]\frac{3}{4} = \left[\frac{\pi}{2}+\frac{\pi}{4}\right]\frac{3}{4}.$$

We conclude: $\displaystyle\iint_D f(r,\theta)dA = \frac{9}{16}\pi.$ ◁

Changing Cartesian integrals into polar integrals.

Example

Integrate $f(x, y) = e^{-(x^2+y^2)}$ on the domain
$D = \{(r, \theta) \in R^2 \; : \; 0 \leqslant \theta \leqslant \pi, \; 0 \leqslant r \leqslant 2\}$.

Solution: Since $f(r\cos(\theta), r\sin(\theta)) = e^{-r^2}$, the double integral is

$$\iint_D f(x, y)\, dx\, dy = \int_0^{\pi} \int_0^{2} e^{-r^2} r\, dr\, d\theta.$$

Substituting $u = r^2$, hence $du = 2r\, dr$, we obtain

$$\iint_D f(x, y)\, dx\, dy = \frac{1}{2} \int_0^{\pi} \int_0^{4} e^{-u} du\, d\theta = \frac{1}{2} \int_0^{\pi} \Big(-e^{-u} \Big|_0^4 \Big)\, d\theta;$$

We conclude: $\displaystyle\iint_D f(x, y)\, dx\, dy = \frac{\pi}{2}\Big(1 - \frac{1}{e^4}\Big).$ $\triangleleft$

Double integrals in polar coordinates (Sect. 15.3)

- Review: Polar coordinates.
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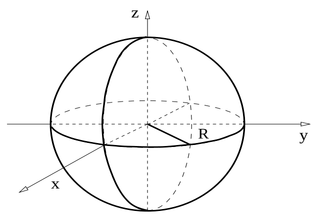
- Changing Cartesian integrals into polar integrals.
- **Computing volumes using double integrals.**

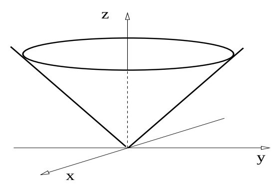
Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Let us first draw the sets that form the volume we are interested to compute.





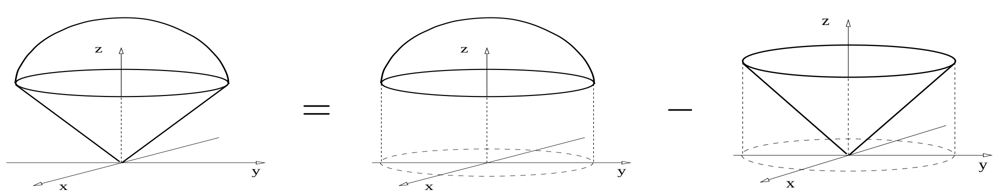
$$z = \pm\sqrt{1 - r^2}, \qquad z = r.$$

Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: The integration region can be decomposed as follows:



The volume we are interested to compute is:

$$V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2}\,(r dr) d\theta - \int_0^{2\pi} \int_0^{r_0} r\,(r dr) d\theta.$$

We need to find r_0, the intersection of the cone and the sphere.

Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: We find r_0, the intersection of the cone and the sphere.

$$\sqrt{1 - r_0^2} = r_0 \quad \Leftrightarrow \quad 1 - r_0^2 = r_0^2 \quad \Leftrightarrow \quad 2r_0^2 = 1;$$

that is, $r_0 = 1/\sqrt{2}$. Therefore

$$V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr)d\theta - \int_0^{2\pi} \int_0^{1/\sqrt{2}} r\,(r\,dr)d\theta.$$

$$V = 2\pi\Big[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr) - \int_0^{1/\sqrt{2}} r\,(r\,dr)\Big].$$

Computing volumes using double integrals.

Example

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: $V = 2\pi\Big[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\,(r\,dr) - \int_0^{1/\sqrt{2}} r\,(r\,dr)\Big].$

Use the substitution $u = 1 - r^2$, so $du = -2r\,dr$. We obtain,

$$V = 2\pi\Big[\frac{1}{2}\int_{1/2}^{1} u^{1/2}\,du - \frac{1}{3}r^3\Big|_0^{1/\sqrt{2}}\Big],$$

$$V = 2\pi\Big[\frac{1}{2}\,\frac{2}{3}u^{3/2}\Big|_{1/2}^{1} - \frac{1}{3}\,\frac{1}{2^{3/2}}\Big] = \frac{2\pi}{3}\Big[1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}}\Big],$$

We conclude: $V = \frac{\pi}{3}\left(2 - \sqrt{2}\right)$. $\triangleleft$

Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- Volume on a region in space.

Triple integrals in rectangular boxes.

Definition

The *triple integral* of a function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ in the rectangular box $R = [\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$ is the number

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z.$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$, $z_k^* \in [z_k, z_{k+1}]$ are sample points, while $\{x_i\}$, $\{y_j\}$, $\{z_k\}$, with $i, j, k = 0, \cdots, n$, are partitions of the intervals $[\hat{x}_0, \hat{x}_1]$, $[\hat{y}_0, \hat{y}_1]$, $[\hat{z}_0, \hat{z}_1]$, respectively, and

$$\Delta x = \frac{(\hat{x}_1 - \hat{x}_0)}{n}, \quad \Delta y = \frac{(\hat{y}_1 - \hat{y}_0)}{n}, \quad \Delta z = \frac{(\hat{z}_1 - \hat{z}_0)}{n}.$$

Triple integrals in rectangular boxes.

Remark:

- A finite sum S_n below is called a Riemann sum, where

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\, \Delta x \Delta y \Delta z.$$

- Then holds $\iiint_R f(x,y,z)\, dx\, dy\, dz = \lim_{n\to\infty} S_n$.

Theorem (Fubini)

If function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the rectangle $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then holds

$$\iiint_R f(x,y,z)\, dx\, dy\, dz = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x,y,z)\, dz\, dy\, dx.$$

Furthermore, the integral above can be computed integrating the variables x, y, z in any order.

Triple integrals in rectangular boxes.

Review: The Riemann sums and their limits.

Single variable functions in $[\hat{x}_0, \hat{x}_1]$:

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x = \int_{\hat{x}_0}^{\hat{x}_1} f(x) dx.$$

Two variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1]$: (Fubini)

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x \Delta y = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} f(x,y)\, dy\, dx.$$

Three variable functions in $[\hat{x}_0, \hat{x}_1] \times [\hat{y}_0, \hat{y}_1] \times [\hat{z}_0, \hat{z}_1]$: (Fubini)

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\Delta x \Delta y \Delta z = \int_{\hat{x}_0}^{\hat{x}_1} \int_{\hat{y}_0}^{\hat{y}_1} \int_{\hat{z}_0}^{\hat{z}_1} f(x,y,z)\, dz\, dy\, dx.$$

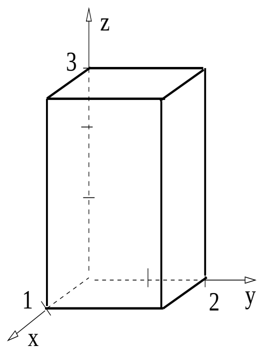
Triple integrals in rectangular boxes.

Example

Compute the integral of $f(x,y,z) = xyz^2$ on the domain $R = [0,1] \times [0,2] \times [0,3]$.

Solution: It is useful to sketch the integration region first:

$$R = \{(x,y,z) \in \mathbb{R}^3 : \ x \in [0,1], \ y \in [0,2], \ z \in [0,3]\}.$$



The integral we need to compute is

$$\iiint_R f \, dv = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx,$$

where we denoted $dv = dx \, dy \, dz$.

Triple integrals in rectangular boxes.

Example

Compute the integral of $f(x,y,z) = xyz^2$ on the domain $R = [0,1] \times [0,2] \times [0,3]$.

Solution: $\displaystyle\iiint_R f \, dV = \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx.$

We have chosen a particular integration order. (Recall: Since the region is a rectangle, integration limits are simple to interchange.)

$$\iiint_R f \, dv = \int_0^1 \int_0^2 xy \Big(\frac{z^3}{3} \Big|_0^3 \Big) \, dy \, dx = \frac{27}{3} \int_0^1 \int_0^2 xy \, dy \, dx.$$

$$\iiint_R f \, dv = 9 \int_0^1 x \Big(\frac{y^2}{2} \Big|_0^2 \Big) \, dx = 18 \int_0^1 x \, dx = 9.$$

We conclude: $\displaystyle\iiint_R f \, dv = 9.$ ◁

Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- **Triple integrals in arbitrary domains.**
- Volume on a region in space.

Triple integrals in arbitrary domains.

Theorem

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain

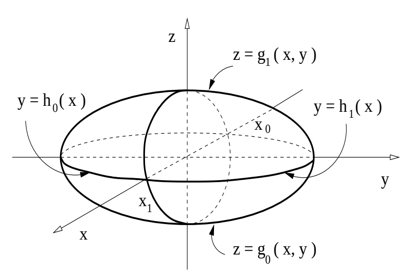
$$D = \{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f\, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



Triple integrals in Cartesian coordinates (Sect. 15.4)

- Triple integrals in rectangular boxes.
- Triple integrals in arbitrary domains.
- **Volume on a region in space.**

Volume on a region in space.

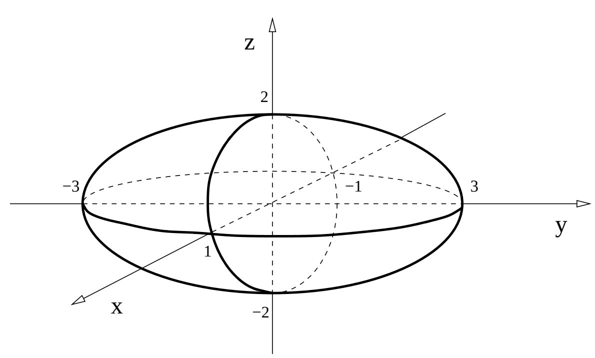
Remark: The volume of a bounded, closed region $D \in \mathbb{R}^3$ is

$$V = \iiint_D dv.$$

Example

Find the integration limits needed to compute the volume of the ellipsoid $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$.

Solution: We first sketch the integration domain.



Volume on a region in space.

Example

Find the integration limits needed to compute the volume of the ellipsoid $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$.

Solution: The functions $z = g_1$ and $z = g_0$ are, respectively,

$$z = 2\sqrt{1 - x^2 - \frac{y^2}{3^2}}, \quad z = -2\sqrt{1 - x^2 - \frac{y^2}{3^2}}.$$

The functions $y = h_1$ and $y = h_0$ are defined on $z = 0$, and are given by, respectively, $y = 3\sqrt{1 - x^2}$ and $y = -3\sqrt{1 - x^2}$.

The limits on integration in x are ± 1. We conclude:

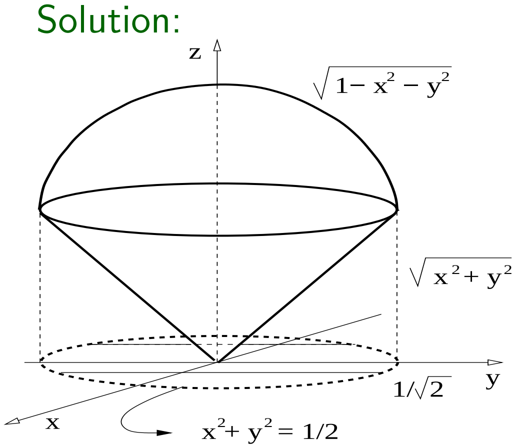
$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx. \qquad \triangleleft$$

Volume on a region in space.

Example

Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution:



The top surface is the sphere,

$$z = \sqrt{1 - x^2 - y^2}.$$

The bottom surface is the cone,

$$z = \sqrt{x^2 + y^2}.$$

The limits on y are obtained projecting the 3-dimensional figure onto the plane $z = 0$. We obtain the disk $x^2 + y^2 = 1/2$.

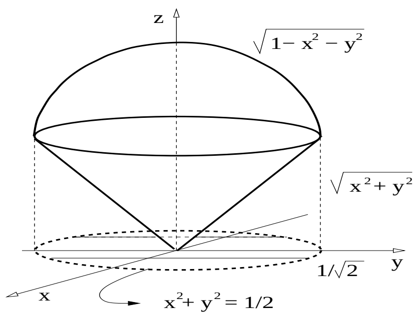
(The polar radius at the intersection cone-sphere was $r_0 = 1/\sqrt{2}$.)

Volume on a region in space.

Example

Use Cartesian coordinates to find the integration limits needed to compute the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: Recall: $z = \sqrt{1 - x^2 - y^2}$, $z = \sqrt{x^2 + y^2}$.



The y-top of the disk is,

$$y = \sqrt{1/2 - x^2}.$$

The y-bottom of the disk is,

$$y = -\sqrt{1/2 - x^2}.$$

We conclude: $V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$ ◁

Volume on a region in space.

Example

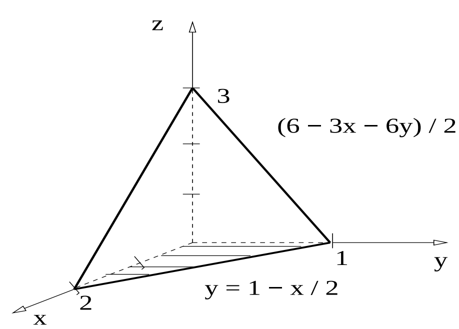
Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution:

The region is given by the first octant and below the plane

$$3x + 6y + 2z = 6.$$

This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.



In z the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$.

Volume on a region in space.

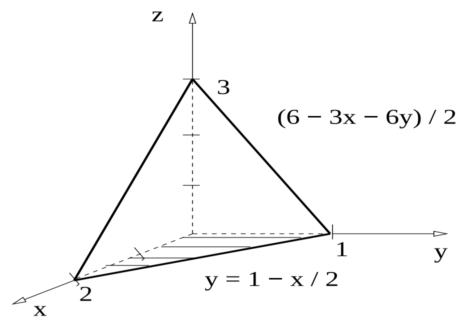
Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: In z the limits are $z = (6 - 3x - 6y)/2$ and $z = 0$.

At $z = 0$ the projection of the region is the triangle $x \geqslant 0$, $y \geqslant 0$, and $x + 2y \leqslant 2$.

In y the limits are $y = 1 - x/2$ and $y = 0$.



We conclude: $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

Volume on a region in space.

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: Recall: $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

$$V = 3 \int_0^2 \int_0^{1-x/2} \Big(1 - \frac{x}{2} - y\Big)\, dy\, dx,$$

$$= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(y\Big|_0^{(1-x/2)}\Big) - \Big(\frac{y^2}{2}\Big|_0^{(1-x/2)}\Big)\Big]\, dx,$$

$$= 3 \int_0^2 \Big[\Big(1 - \frac{x}{2}\Big)\Big(1 - \frac{x}{2}\Big) - \frac{1}{2}\Big(1 - \frac{x}{2}\Big)^2\Big]\, dx.$$

We only need to compute: $V = \frac{3}{2} \int_0^2 \Big(1 - \frac{x}{2}\Big)^2 dx.$

Volume on a region in space.

Example

Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$.

Solution: Recall: $V = \dfrac{3}{2} \displaystyle\int_0^2 \Big(1 - \frac{x}{2}\Big)^2 dx$.

Substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = -3 \int_1^0 u^2 du = 3 \int_0^1 u^2\, du = 3\Big(\frac{u^3}{3}\Big|_0^1\Big)$$

We conclude: $V = 1$. $\lhd$

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

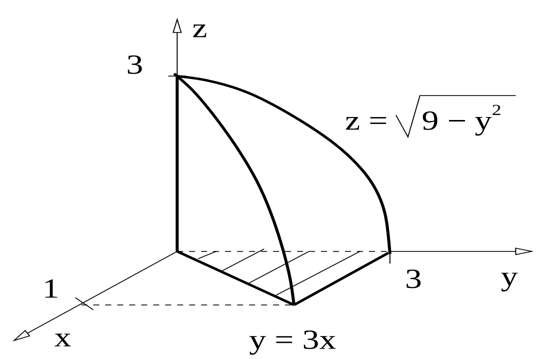
Solution:

The upper surface is

$$z = \sqrt{9 - y^2},$$

the bottom surface is

$$z = 0.$$



The y coordinate is bounded below by the line $y = 3x$ and above by $y = 3$. (Because of the cylinder equation at $z = 0$.)

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution: Recall: $0 \leqslant z \leqslant \sqrt{9 - y^2}$ and $3x \leqslant y \leqslant 3$.
Since $f = z$, we obtain

$$\iiint_D f\,dv = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx,$$
$$= \int_0^1 \int_{3x}^3 \Big(\frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} \Big)\,dy\,dx,$$
$$= \frac{1}{2} \int_0^1 \int_{3x}^3 (9 - y^2)dy\,dx,$$
$$= \frac{1}{2} \int_0^1 \Big[27(1-x) - \Big(\frac{y^3}{3}\Big|_{3x}^3\Big)\Big]\,dx.$$

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution: Recall: $\iiint_D f\,dv = \frac{1}{2} \int_0^1 \Big[27(1-x) - \Big(\frac{y^3}{3}\Big|_{3x}^3\Big)\Big]\,dx.$
Therefore,

$$\iiint_D f\,dv = \frac{1}{2} \int_0^1 \Big[27(1-x) - 9(1-x)^3\Big]\,dx,$$
$$= \frac{9}{2} \int_0^1 \Big[3(1-x) - (1-x)^3\Big]\,dx.$$

Substitute $u = 1 - x$, then $du = -dx$, so,

$$\iiint_D f\,dv = \frac{9}{2} \int_0^1 (3u - u^3)du.$$

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the first octant and bounded by $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$.

Solution:

$$\int\int\int_D f\,dv = \frac{9}{2}\int_0^1 (3u - u^3)du,$$
$$= \frac{9}{2}\Big[\frac{3}{2}\Big(u^2\Big|_0^1\Big) - \frac{1}{4}\Big(u^4\Big|_0^1\Big)\Big],$$
$$= \frac{9}{2}\Big(\frac{3}{2} - \frac{1}{4}\Big).$$

We conclude $\iiint_D f\,dv = \frac{45}{8}$. ◁