Review for Exam 2.

- Sections 13.1, 13.3. 14.1-14.7.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Section 14.7

Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:
(a) $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y\rangle = \langle 0, 0\rangle$, then,

$$x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x-4) = 0.$$

There are two solutions, $x = 0 \Rightarrow y = 0$, and $x = 4 \Rightarrow y = 8$.
That is, there are two critical points, $(0, 0)$ and $(4, 8)$.

Section 14.7

Example

(a) Find all the critical points of $f(x, y) = 12xy - 2x^3 - 3y^2$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

Solution:

(b) Recalling $\nabla f(x, y) = \langle 12y - 6x^2, 12x - 6y \rangle$, we compute

$$f_{xx} = -12x, \quad f_{yy} = -6, \quad f_{xy} = 12.$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144 \left(\frac{x}{2} - 1 \right),$$

Since $D(0, 0) = -144 < 0$, the point $(0, 0)$ is a saddle point of f.

Since $D(4, 8) = 144(2 - 1) > 0$, and $f_{xx}(4, 8) = (-12)4 < 0$, the point $(4, 8)$ is a local maximum of f. ◁

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4} y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

We start finding the critical points inside the triangular region.

$$\nabla f(x, y) = \left\langle y - 2, x - \frac{1}{2} y \right\rangle = \langle 0, 0 \rangle, \quad \Rightarrow \quad y = 2, \quad y = 2x.$$

The solution is $(1, 2)$. This point is outside in the triangular region given by the problem, so there is no critical point inside the region.

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

We now find the candidates for absolute maximum and minimum on the borders of the triangular region. We first record the boundary vertices:

$$\begin{aligned} (0,0) &\quad \Rightarrow \quad f(0,0) = 2, \\ (1,0) &\quad \Rightarrow \quad f(1,0) = 0, \\ (0,2) &\quad \Rightarrow \quad f(0,2) = 1. \end{aligned}$$

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

- The horizontal side of the triangle, $y = 0$, $x \in (0, 1)$. Since

  $$g(x) = f(x, 0) = 2 - 2x, \quad \Rightarrow \quad g'(x) = -2 \neq 0.$$

  there are no candidates in this part of the boundary.
- The vertical side of the triangle is $x = 0$, $y \in (0, 2)$. Then,

  $$g(y) = f(0, y) = 2 - \frac{1}{4}y^2, \quad \Rightarrow \quad g'(y) = -\frac{1}{2}y = 0,$$

  so $y = 0$ and we recover the point $(0, 0)$.

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

- The hypotenuse of the triangle $y = 2 - 2x$, $x \in (0, 1)$. Then,

$$\begin{aligned} g(x) = f(x, 2-2x) &= 2 + x(2-2x) - 2x - \frac{1}{4}(2-2x)^2, \\ &= 2 + 2x - 2x^2 - 2x - (x^2 - 2x + 1), \\ &= 1 + 2x - 3x^2. \end{aligned}$$

Then, $g'(x) = 2 - 6x = 0$ implies $x = \frac{1}{3}$, hence $y = \frac{4}{3}$. The candidate is $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Section 14.7

Example

Find the absolute maximum and absolute minimum of $f(x, y) = 2 + xy - 2x - \frac{1}{4}y^2$ in the closed triangular region with vertices given by $(0, 0)$, $(1, 0)$, and $(0, 2)$.

Solution:

- Recall that we have obtained a candidate point $\left(\frac{1}{3}, \frac{4}{3}\right)$. We evaluate f at this point,

$$f\left(\frac{1}{3}, \frac{4}{3}\right) = 2 + \frac{4}{9} - \frac{2}{3} - \frac{1}{4}\frac{16}{9} = \frac{4}{3}.$$

Recalling that $f(0, 0) = 2$, $f(1, 0) = 0$, and $f(0, 2) = 1$, the absolute maximum is at $(0, 0)$, and the minimum is at $(1, 0)$. $\triangleleft$

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.8, 2.3)$.

Solution:

(a) $L(x, y) = f_x(-3, 2)\,(x + 3) + f_y(-3, 2)\,(y - 2) + f(-3, 2)$.

Since $f_x(x, y) = 2\cos(2x + 3y)$ and $f_y(x, y) = 3\cos(2x + 3y)$,

$$\begin{aligned} f_x(-3, 2) &= 2\cos(-6 + 6) = 2, \\ f_y(-3, 2) &= 3\cos(-6 + 6) = 3, \\ f(-3, 2) &= \sin(-6 + 6) + 1 = 1. \end{aligned}$$

the linear approximation is $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

Section 14.6

Example

(a) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sin(2x + 3y) + 1$ at the point $(-3, 2)$.

(b) Use the approximation above to estimate the value of $f(-2.8, 2.3)$.

Solution:

(b) Recall: $L(x, y) = 2(x + 3) + 3(y - 2) + 1$.

Now, the linear approximation of $f(-2.8, 2.3)$ is $L(-2.8, 2.3)$, and

$$\begin{aligned} L(-2.8, 2.3) &= 2(-2.8 + 3) + 3(2.3 - 2) + 2 \\ &= 2(0.2) + 3(0.3) + 1 \\ &= 2.3. \end{aligned}$$

We conclude $L(-2.8, 2.3) = 2.3$. ◁

Section 14.5

Example

(a) Find the gradient of $f(x,y,z)=\sqrt{x+2yz}$.

(b) Find the directional derivative of f at $(0,2,1)$ in the direction given by $\langle 0,3,4\rangle$.

(c) Find the maximum rate of change of f at the point $(0,2,1)$.

Solution:

(a) $\nabla f(x,y,z)=\dfrac{1}{2\sqrt{x+2yz}}\langle 1,2z,2y\rangle$.

(b) We evaluate the gradient above at $(0,2,1)$,

$$\nabla f(0,2,1)=\frac{1}{2\sqrt{0+4}}\langle 1,2,4\rangle=\frac{1}{4}\langle 1,2,4\rangle.$$

Section 14.5

Example

(a) Find the gradient of $f(x,y,z)=\sqrt{x+2yz}$.

(b) Find the directional derivative of f at $(0,2,1)$ in the direction given by $\langle 0,3,4\rangle$.

(c) Find the maximum rate of change of f at the point $(0,2,1)$.

Solution:

(b) Recall: $\nabla f(0,2,1)=\frac{1}{4}\langle 1,2,4\rangle$.
We now need a unit vector parallel to $\langle 0,3,4\rangle$,

$$\mathbf{u}=\frac{1}{\sqrt{9+16}}\langle 0,3,4\rangle=\frac{1}{5}\langle 0,3,4\rangle.$$

Then, $D_uf(0,2,1)=\frac{1}{4}\langle 1,2,4\rangle\cdot\frac{1}{5}\langle 0,3,4\rangle=\frac{1}{20}(6+16)=\frac{11}{10}$.
Therefore, $D_uf(0,2,1)=11/10$.

Section 14.5

Example

(a) Find the gradient of $f(x,y,z) = \sqrt{x+2yz}$.

(b) Find the directional derivative of f at $(0,2,1)$ in the direction given by $\langle 0,3,4\rangle$.

(c) Find the maximum rate of change of f at the point $(0,2,1)$.

Solution:

(c) The maximum rate of change of f at a point is the magnitude of its gradient at that point, that is,

$$|\nabla f(0,2,1)| = \frac{1}{4}|\langle 1,2,4\rangle| = \frac{1}{4}\sqrt{1+4+16} = \frac{\sqrt{21}}{4}.$$

Therefore, the maximum rate of change of the function f at the point $(0,2,1)$ is given by $\sqrt{21}/4$. $\triangleleft$

Section 14.3

Example

Find any value of the constant a such that the function $f(x,y) = e^{-ax}\cos(y) - e^{-y}\cos(x)$ is solution of Laplace's equation $f_{xx} + f_{yy} = 0$.

Solution:

$$f_x = -ae^{-ax}\cos(y) + e^{-y}\sin(x), \quad f_y = -e^{-ax}\sin(y) + e^{-y}\cos(x),$$

$$f_{xx} = a^2e^{-ax}\cos(y) + e^{-y}\cos(x), \quad f_{yy} = -e^{-ax}\cos(y) - e^{-y}\cos(x),$$

then

$$\begin{aligned} f_{xx} + f_{yy} &= \left[a^2e^{-ax}\cos(y) + e^{-y}\cos(x)\right] \\ &\quad + \left[-e^{-ax}\cos(y) - e^{-y}\cos(x)\right], \\ &= (a^2-1)e^{-ax}\cos(y). \end{aligned}$$

Function f is solution of $f_{xx} + f_{yy} = 0$ iff $a = \pm 1$. $\triangleleft$

Section 14.2

Example

Compute the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{2x^2+3y^2}$.

Solution:

Since $x^2 \leqslant 2x^2+3y^2$, that is, $\dfrac{x^2}{2x^2+3y^2} \leqslant 1$, the non-negative function $f(x,y) = \dfrac{x^2\sin^2(y)}{2x^2+3y^2}$ satisfies the bounds

$$0 \leqslant f(x,y) \leqslant \sin^2(y).$$

Since $\lim_{y\to 0}\sin^2(y) = 0$, the Sandwich Theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{2x^2+3y^2} = 0.$$

◁

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2)\right\rangle$ with respect to its arc length measured from $t=1$ in the direction of increasing t.

Solution:

We first compute the arc length function. We start with the derivative

$$\mathbf{r}'(t) = \langle 3t\cos(t^2), 4t, -3\sin(t^2)\rangle,$$

We now need its magnitude,

$$\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{9t^2\cos^2(t^2) + 16t^2 + 9\sin^2(t^2)},\\
&= \sqrt{9t^2+16t^2},\\
&= \sqrt{9+16}\,t,\\
&= 5t.
\end{aligned}$$

Section 13.3

Example

Reparametrize the curve $\mathbf{r}(t) = \left\langle \frac{3}{2}\sin(t^2), 2t^2, \frac{3}{2}\cos(t^2) \right\rangle$ with respect to its arc length measured from $t = 1$ in the direction of increasing t.

Solution:

Recall: $|\mathbf{r}'(t)| = 5t$. Then, the arc length function is

$$s(t) = \int_1^t 5\tau \, d\tau = \frac{5}{2}\Big(\tau^2\Big|_1^t\Big) = \frac{5}{2}(t^2 - 1).$$

Inverting this function for t^2, we obtain $t^2 = \frac{2}{5}s + 1$. The reparametrization of $\mathbf{r}(t)$ is given by

$$\mathbf{r}(s) = \left\langle \frac{3}{2}\sin\Big(\frac{2}{5}s + 1\Big), 2\Big(\frac{2}{5}s + 1\Big), \frac{3}{2}\cos\Big(\frac{2}{5}s + 1\Big) \right\rangle.$$

◁

Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- Fubini Theorem on rectangular domains.
- Examples.

Next class:

- Double integrals over non-rectangular regions.
- Fubini Theorem on non-rectangular domains.
- Finding the limits of integration.

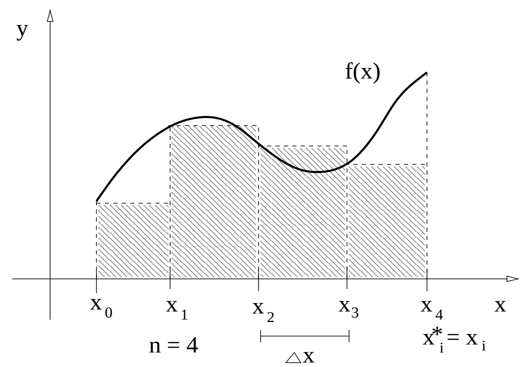
Review: Integral of a single variable function.

Definition

The definite integral of a function $f : [a, b] \to \mathbb{R}$, in the interval $[a, b]$ is the number

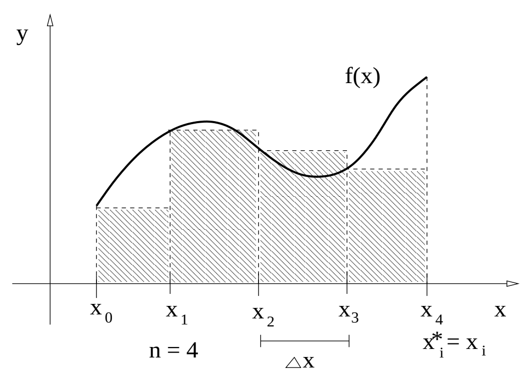
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\,\Delta x.$$

where $x_i^* \in [x_i, x_{i+1}]$ is called a sample point, while $\{x_i\}$ is a partition in $[a, b]$, $i = 0, \cdots, n$, and with $x_i = a + i\Delta x$, and $\Delta x = \dfrac{(b-a)}{n}$.

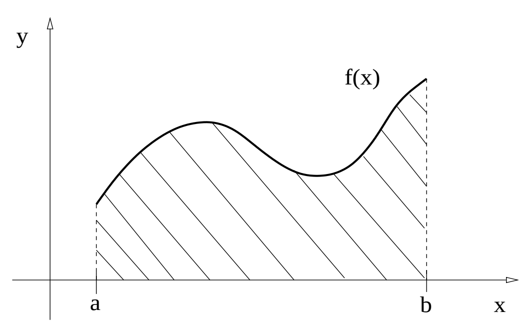


The integral as an area.

The sum $S_n = \sum_{i=0}^{n} f(x_i^*)\,\Delta x$ is called a Riemann sum. Then, $\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$.



The integral $\int_a^b f(x)\,dx$ is the area in between the graph of f and the horizontal axis.



Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- **Double integral on rectangles.**
- Fubini Theorem on rectangular domains.
- Examples.

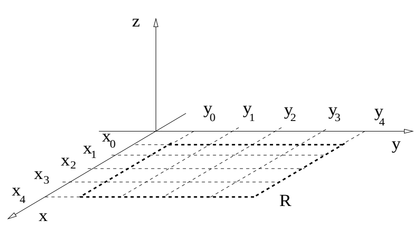
Double integrals on rectangles

Definition

The *double integral* of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ in the rectangle $R = [a, b] \times [c, d]$ is the number

$$\iint_R f(x, y)\, dx\, dy = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y.$$

where $x_i^* \in [x_i, x_{i+1}]$, $y_j^* = [y_j, y_{j+1}]$, are sample points, while $\{x_i\}$ and $\{y_j\}$, $i, j = 0, \cdots, n$ are partitions of the intervals $[a, b]$ and $[c, d]$, and $\Delta x = \dfrac{(b-a)}{n}$, $\Delta y = \dfrac{(d-c)}{n}$.



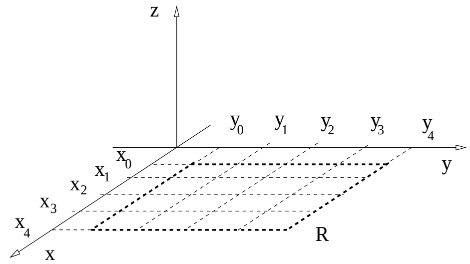
The double integral as a volume.

The sum

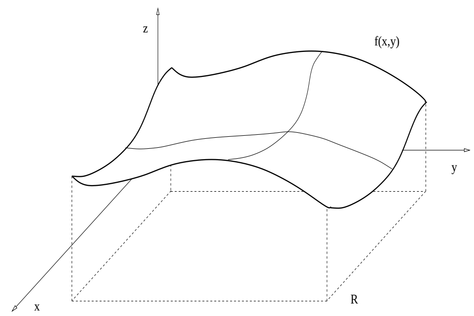
$S_n = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x \Delta y$ is

called a Riemann sum. Then,

$\iint_R f(x,y)\,dx\,dy = \lim_{n\to\infty} S_n.$



The integral $\iint_R f(x,y)\,dx\,dy$ is the volume above R and below the graph of f.



Double integrals (Sect. 15.1)

- Review: Integral of a single variable function.
- Double integral on rectangles.
- **Fubini Theorem on rectangular domains.**
- Examples.

Fubini Theorem on rectangular domains.

Theorem

If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ *is continuous in* $R = [x_0, x_1] \times [y_0, y_1]$, *then*

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1} \Big[\int_{x_0}^{x_1} f(x,y)\,dx\Big]\,dy,$$
$$= \int_{x_0}^{x_1} \Big[\int_{y_0}^{y_1} f(x,y)\,dy\Big]\,dx.$$

Remark: Fubini's Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.

Notation: The double integral is also written as

$$\iint_R f(x,y)\,dx\,dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x,y)\,dx\,dy.$$

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: Since $x \in [0,2]$ and $y \in [1,3]$,

$$\iint_R f(x,y)\,dx\,dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3)dx\,dy$$
$$= \int_1^3 \Big[\int_0^2 (xy^2 + 2x^2y^3)dx\Big]\,dy.$$

We compute the interior integral in x first, keeping y constant. After that we compute the integral in y.

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: We compute the integral in x first, keeping y constant.

$$\iint_R f(x,y)\,dx\,dy = \int_1^3 \Big[\int_0^2 (xy^2 + 2x^2y^3)dx\Big]\,dy$$
$$= \int_1^3 \Big[\frac{y^2}{2}\Big(x^2\Big|_0^2\Big) + \frac{2y^3}{3}\Big(x^3\Big|_0^2\Big)\Big]\,dy,$$
$$= \int_1^3 \Big[2y^2 + \frac{16}{3}y^3\Big]\,dy.$$

We now compute the integral in y,

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: We now compute the integral in y,

$$\iint_R f(x,y)\,dx\,dy = \int_1^3 \Big[2y^2 + \frac{16}{3}y^3\Big]\,dy,$$
$$= 2\,\frac{y^3}{3}\Big|_1^3 + \frac{16}{3}\,\frac{y^4}{4}\Big|_1^3.$$

We conclude: $\iint_R f(x,y)\,dx\,dy = 2\,\frac{26}{3} + \frac{4}{3}80 = 372/3.$ $\triangleleft$

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in y, then in x.

Solution:

$$\iint_R f(x,y)\,dx\,dy = \int_1^3 \int_0^2 (xy^2 + 2x^2y^3)dx\,dy$$
$$= \int_0^2 \Big[\int_1^3 (xy^2 + 2x^2y^3)dy\Big]\,dx.$$

$$\iint_R f(x,y)\,dx\,dy = \int_0^2 \Big[\frac{x}{3}\Big(y^3\Big|_1^3\Big) + \frac{2x^2}{4}\Big(y^4\Big|_1^3\Big)\Big]\,dx.$$

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = xy^2 + 2x^2y^3$, and $R = [0,2] \times [1,3]$. Integrate first in x, then in y.

Solution: $\iint_R f(x,y)\,dx\,dy = \int_0^2 \Big[\frac{x}{3}\Big(y^3\Big|_1^3\Big) + \frac{2x^2}{4}\Big(y^4\Big|_1^3\Big)\Big]\,dx.$

$$\iint_R f(x,y)\,dx\,dy = \int_0^2 \Big[\frac{26}{3}\,x + 40\,x^2\Big]\,dx$$
$$= \frac{26}{3}\,\frac{x^2}{2}\Big|_0^2 + 40\,\frac{x^3}{3}\Big|_0^2$$
$$= \frac{26}{3}\,(2) + 40\,\frac{8}{3} \Rightarrow \iint_R f(x,y)\,dx\,dy = 372/3.$$

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$, and $R = [1,4] \times [1,2]$.

Solution: We choose to first integrate in y and then in x.

$$\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_1^4 \int_1^2 \Big(\frac{x}{y} + \frac{y}{x}\Big)\,dy\,dx,\\
&= \int_1^4 \Big[\int_1^2 \Big(\frac{x}{y} + \frac{y}{x}\Big)\,dy\Big]\,dx,\\
&= \int_1^4 \Big[x\Big(\ln(y)\Big|_1^2\Big) + \frac{1}{x}\Big(\frac{y^2}{2}\Big|_1^2\Big)\Big]\,dx,\\
&= \int_1^4 \Big[\ln(2)\,x + \frac{3}{2}\,\frac{1}{x}\Big]\,dx.
\end{aligned}$$

Fubini Theorem on rectangular domains.

Example

Use Fubini's Theorem to compute the double integral $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$, and $R = [1,4] \times [1,2]$.

Solution: We compute the integral in x,

$$\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \int_1^4 \Big[\ln(2)\,x + \frac{3}{2}\,\frac{1}{x}\Big]\,dx\\
&= \ln(2)\Big(\frac{x^2}{2}\Big|_1^4\Big) + \frac{3}{2}\Big(\ln(x)\Big|_1^4\Big),\\
&= \frac{15}{2}\ln(2) + \frac{3}{2}\ln(4),\\
&= \Big(\frac{15}{2} + 3\Big)\ln(2).
\end{aligned}$$

We conclude: $\iint_R f(x,y)\,dx\,dy = (21/2)\ln(2)$. $\lhd$

A particular case of Fubini's Theorem.

Corollary

If the continuous function $f : R \subset R^2 \to \mathbb{R}$ satisfies that $f(x,y) = g(x)h(y)$, then the double integral of function f in the rectangle $R = [x_0, x_1] \times [y_0, y_1]$ is given by

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} g(x)h(y)dy\, dx = \Big(\int_{x_0}^{x_1} g(x)dx\Big)\Big(\int_{y_0}^{y_1} h(y)dy\Big).$$

Remark: In the case that $f(x,y)$ is a product of two functions g, h, with $g(x)$ and $h(y)$, then the double integral of f is also a product of the integral of g times the integral of h.

A particular case of Fubini's Theorem.

Example

Compute the double integral of $f(x,y) = \dfrac{1+x^2}{1+y^2}$, in the rectangular region $R = [0,2] \times [0,1]$.

Solution:

$$\begin{aligned}
\iint_R f(x,y)\, dx\, dy &= \int_0^2 \int_0^1 \frac{1+x^2}{1+y^2} dy\, dx, \\
&= \Big[\int_0^2 (1+x^2)dx\Big]\Big[\int_0^1 \frac{1}{1+y^2} dy\Big], \\
&= \Big(x\Big|_0^2 + \frac{1}{3}x\Big|_0^2\Big)\Big(\arctan(y)\Big|_0^1\Big), \\
&= \frac{\pi}{4}\Big(2 + \frac{8}{3}\Big).
\end{aligned}$$

We conclude $\displaystyle\iint_R f(x,y)\, dx\, dy = (14/3)(\pi/4) = (7/6)\pi.$ $\lhd$

Double integrals on regions (Sect. 15.1)

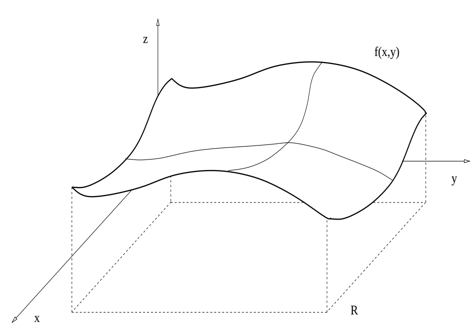
- Review: Fubini's Theorem on rectangular domains.
- Fubini's Theorem on non-rectangular domains.
  - Type I: Domain functions $y(x)$.
  - Type II: Domain functions $x(y)$.
- Finding the limits of integration.

Review: Fubini's Theorem on rectangular domains.

Theorem
If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx,$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Remark: Fubini's Theorem: It is simple to switch the order of integration in double integrals of continuous functions on a rectangle.



Double integrals on regions (Sect. 15.1)

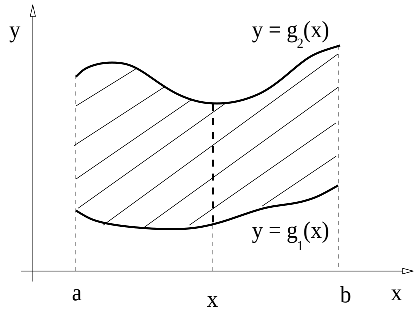
- Review: Fubini's Theorem on rectangular domains.
- **Fubini's Theorem on non-rectangular domains.**
  - **Type I: Domain functions $y(x)$.**
  - Type II: Domain functions $x(y)$.
- Finding the limits of integration.

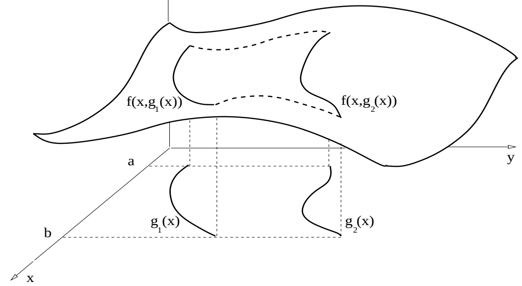
Fubini's Theorem on Type I domains, $y(x)$.

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type I):
If $D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a, b]$, then

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$





Double integrals on regions (Sect. 15.1)

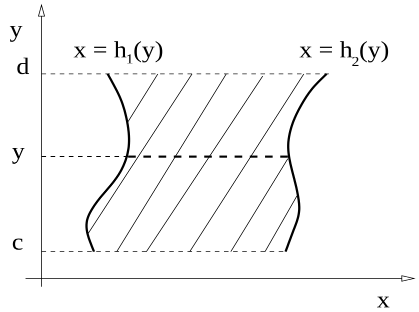
- Review: Fubini's Theorem on rectangular domains.
- **Fubini's Theorem on non-rectangular domains.**
  - Type I: Domain functions $y(x)$.
  - **Type II: Domain functions $x(y)$.**
- Finding the limits of integration.

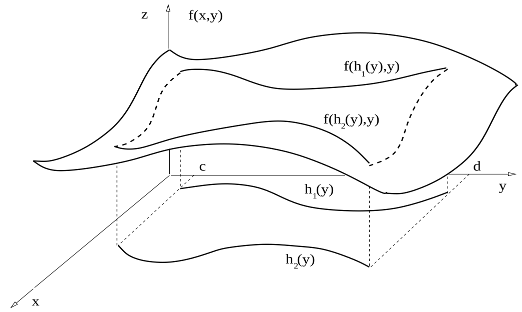
Fubini's Theorem on Type II domains, $x(y)$.

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold (Type II):
If $D = \{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c, d]\}$, with h_1, h_2 continuous functions on $[c, d]$, then

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$





Summary: Fubini's Theorem on non-rectangular domains.

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold:

(a) *(Type I) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ y \in [g_1(x), g_2(x)]\}$, with g_1, g_2 continuous functions on $[a,b]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) *(Type II) If $D = \{(x,y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c,d]\}$, with h_1, h_2 continuous functions on $[c,d]$, then*

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \ :\ 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

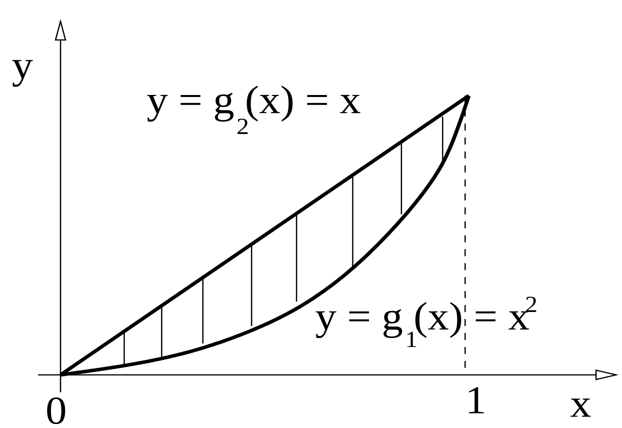
Solution:

This is a Type I domain, with lower boundary

$$y = g_1(x) = x^2,$$

and upper boundary

$$y = g_2(x) = x.$$



A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain
$D = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

Solution: Recall: $\displaystyle\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
with $g_1(x) = x^2$ and $g_2(x) = x$, we obtain

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x^2+y^2)dy\,dx,$$
$$= \int_0^1 \Big[x^2 \Big(y\Big|_{x^2}^{x} \Big) + \Big(\frac{y^3}{3}\Big|_{x^2}^{x} \Big) \Big]\,dx.$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[x^2(x-x^2) + \frac{1}{3}(x^3-x^6) \Big]\,dx.$$

A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain
$D = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1, \quad x^2 \leqslant y \leqslant x\}$.

Solution: $\displaystyle\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[x^2(x-x^2) + \frac{1}{3}(x^3-x^6) \Big]\,dx.$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6 \Big]\,dx,$$
$$= \Big[\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^4}{12} - \frac{x^7}{21} \Big]\Big|_0^1,$$
$$= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{(3)(5)(7)}.$$

We conclude: $\displaystyle\iint_D f(x,y)\,dx\,dy = \frac{3}{35}.$ ◁

Summary: Fubini's Theorem on non-rectangular domains.

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then hold:

(a) *(Type I) If $D = \big\{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ y \in [g_1(x), g_2(x)]\big\}$, with g_1, g_2 continuous functions on $[a, b]$, then*

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

(b) *(Type II) If $D = \big\{(x, y) \in \mathbb{R}^2 : x \in [h_1(y), h_2(y)],\ y \in [c, d]\big\}$, with h_1, h_2 continuous functions on $[c, d]$, then*

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

A double integral on a Type II domain.

Example

Find the integral of $f(x, y) = x^2 + y^2$ on the domain $D = \{(x, y) \in \mathbb{R}^2 \ :\ y \leqslant x \leqslant \sqrt{y},\quad 0 \leqslant y \leqslant 1\}$.

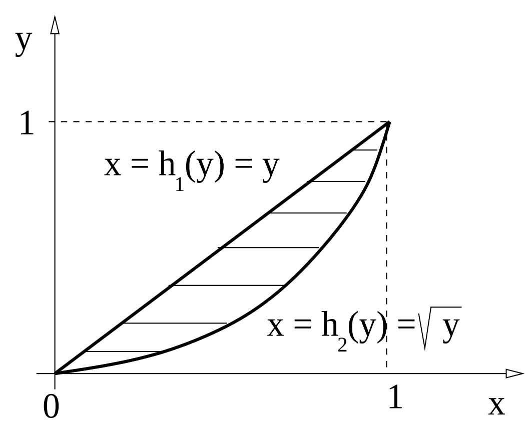
Solution:

This is a Type II domain, with left boundary

$$x = h_1(y) = y,$$

and right boundary

$$x = g_2(y) = \sqrt{y}.$$



Remark:

This domain is both Type I and Type II: $y = x^2 \Leftrightarrow x = \sqrt{y}$.

A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \;:\; y \leqslant x \leqslant \sqrt{y}, \quad 0 \leqslant y \leqslant 1\}$.

Solution: Recall: $\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$ with $h_1(y) = y$ and $h_2(y) = \sqrt{y}$, we obtain

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\,dx\,dy,$$

$$= \int_0^1 \Big[\Big(\frac{x^3}{3}\Big|_y^{\sqrt{y}}\Big) + y^2\Big(x\Big|_y^{\sqrt{y}}\Big)\Big]\,dy.$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[\frac{1}{3}\big(y^{3/2} - y^3\big) + y^2\big(y^{1/2} - y\big)\Big]\,dy.$$

A double integral on a Type I domain.

Example

Find the integral of $f(x,y) = x^2 + y^2$, on the domain $D = \{(x,y) \in \mathbb{R}^2 \;:\; y \leqslant x \leqslant \sqrt{y}, \quad 0 \leqslant y \leqslant 1\}$.

Solution:

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[\frac{1}{3}\big(y^{3/2} - y^3\big) + y^2\big(y^{1/2} - y\big)\Big]\,dy.$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\Big]\,dy,$$

$$= \Big[\frac{1}{3}\,\frac{2}{5}\,y^{5/2} - \frac{1}{3}\,\frac{y^4}{4} + \frac{2}{7}\,y^{7/2} - \frac{y^4}{4}\Big]\Big|_0^1,$$

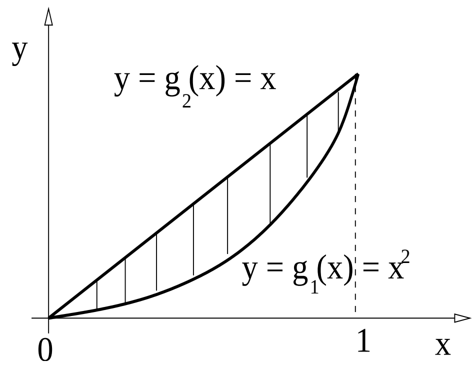
$$= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{(3)(5)(7)}.$$

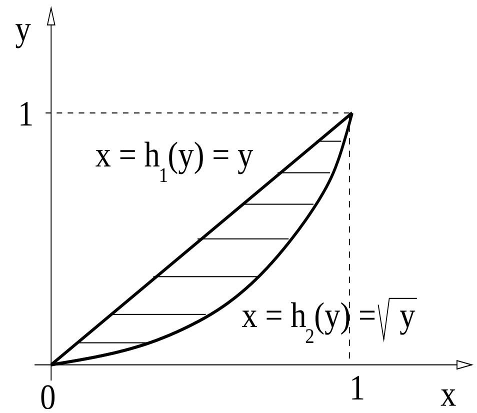
We conclude $\iint_D f(x,y)\,dx\,dy = \frac{3}{35}$. ◁

Domains Type I and Type II.

Summary: We have shown that a double integral of a function f on the domain D given in the pictures below holds,

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx = \int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy.$$





Double integrals on regions (Sect. 15.1)

- Review: Fubini's Theorem on rectangular domains.
- Fubini's Theorem on non-rectangular domains.
  - Type I: Domain functions $y(x)$.
  - Type II: Domain functions $x(y)$.
- **Finding the limits of integration.**

Domains Type I and Type II.

Example

Find the limits of integration of $\int\int_D f(x,y)\,dxdy$ in the domain $D = \left\{(x,y) \in \mathbb{R}^2 \ : \ \frac{x^2}{9} + \frac{y^2}{4} \leqslant 1\right\}$ when D is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1.$
So, the boundary as Type I is given by

$$y = -2\sqrt{1 - \frac{x^2}{9}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).$$

The boundary as Type II is given by

$$x = -3\sqrt{1 - \frac{y^2}{4}} = h_1(y), \quad x = 3\sqrt{1 - \frac{y^2}{4}} = h_2(y).$$

$\triangleleft$

Domains Type I and Type II.

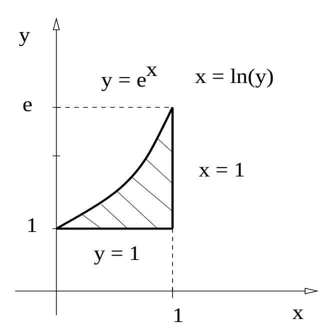
Example

Reverse the order of integration in $\displaystyle\int_0^1 \int_1^{e^x} dy\,dx.$

Solution:
This integral is written as Type I, since we first integrate on vertical intervals $[1, e^x]$, with boundaries $y = e^x$, $y = 1$, while $x \in [0,1]$.

By inverting the first equation and looking at the figure we get the left and right boundaries:

$$x = \ln(y), \quad x = 1, \quad \text{with} \quad y \in [1, e].$$



Therefore, we conclude that $\displaystyle\int_0^1 \int_1^{e^x} dy\,dx = \int_1^e \int_{\ln(y)}^1 dx\,dy.$ $\triangleleft$