Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- Directional derivative of functions of three variables.
- The gradient vector and directional derivatives.
- Properties of the the gradient vector.

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The *directional derivative* of the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ is given by

$$\left(D_{\mathbf{u}} f\right)_{P_0} = \lim_{t \to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right],$$

if the limit exists.

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if the limit exists.

Notation: The directional derivative is also denoted as

$$\Big(\frac{df}{dt}\Big)_{\mathbf{u}, P_0}.$$

Directional derivatives generalize partial derivatives.

Example

The partial derivatives f_x and f_y are particular cases of directional derivatives $\left(D_{\mathbf{u}}f\right)_{P_0} = \lim_{t\to 0} \frac{1}{t}\left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\right]$:

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- $\mathbf{u} = \langle 1, 0\rangle = \mathbf{i}$, then $\left(D_{\mathbf{i}}f\right)_{P_0} = f_x(x_0, y_0)$.

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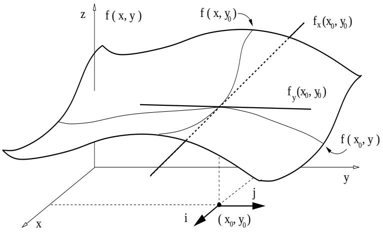

Directional derivative of functions of two variables.

Remark: The condition $|\mathbf{u}| = 1$ implies that the parameter t in the line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \mathbf{u}\, t$ is the distance between the points $(x(t), y(t)) = (x_0 + u_x t, y_0 + u_y t)$ and (x_0, y_0).

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Proof.

$$d = |\langle x - x_0, y - y_0 \rangle|, = |\langle u_x t, u_y t \rangle|, = |t|\,|\mathbf{u}|,$$

that is, $d = |t|$. □

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Remark: The directional derivative of $f(x, y)$ at $P_0 = (x_0, y_0)$ along $\mathbf{u}$, denoted as $\big(D_{\mathbf{u}} f\big)_{P_0}$, is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through (x_0, y_0).

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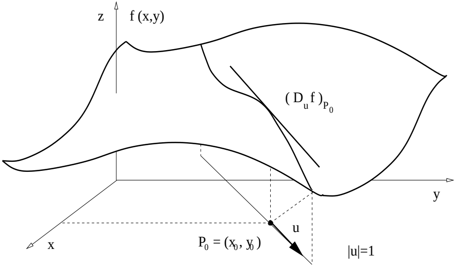
Remark: The directional derivative $\left(D_{\mathbf{u}} f\right)_{P_0}$ is the derivative of f along the line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \mathbf{u}\, t$.



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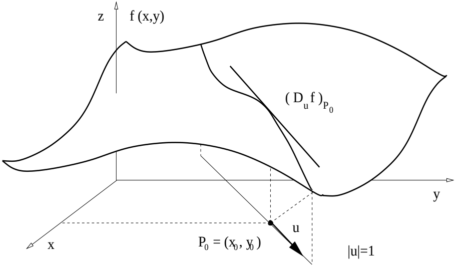

Theorem

If the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0)$ and $\mathbf{u} = \langle u_x, u_y\rangle$ is a unit vector, then

$$\big(D_{\mathbf{u}}f\big)_{P_0} = f_x(x_0, y_0)\,u_x + f_y(x_0, y_0)\,u_y.$$

Directional derivative and partial derivatives.

Proof.

The line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \langle u_x, u_y \rangle\, t$ has parametric equations:
$x(t) = x_0 + u_x t$ and $y(t) = y_0 + u_y t$;

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Denote f evaluated along the line as $\hat{f}(t) = f(x(t), y(t))$.

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Now, on the one hand, $\hat{f}'(0) = \big(D_{\mathbf{u}} f\big)_{P_0}$, since

$$\begin{aligned}
\hat{f}'(0) &= \lim_{t\to 0} \frac{1}{t}\big[\hat{f}(t) - \hat{f}(0)\big],\\
&= \lim_{t\to 0} \frac{1}{t}\big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\big],\\
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On the other hand, the chain rule implies:

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Therefore, $\left(D_{\mathbf{u}} f\right)_{P_0} = f_x(x_0, y_0) \, u_x + f_y(x_0, y_0) \, u_y$. □

Directional derivative and partial derivatives.

Example

Compute the directional derivative of $f(x, y) = \sin(x + 3y)$ at the point $P_0 = (4, 3)$ in the direction of vector $\mathbf{v} = \langle 1, 2 \rangle$.

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Such vector is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \dfrac{1}{\sqrt{5}} \langle 1, 2 \rangle.$

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Then , $\big(D_{\mathbf{u}} f\big)_{P_0} = (7/\sqrt{5}) \cos(10)$. $\lhd$

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Theorem

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$$\left(D_{\mathbf{u}} f\right)_{P_0} = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0)\, u_z.$$

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Find $\left(D_{\mathbf{u}} f\right)_{P_0}$ for $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at the point $P_0 = (3, 2, 1)$ along the direction given by $\mathbf{v} = \langle 2, 1, 1 \rangle$.

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Solution: We first find a unit vector along $\mathbf{v}$,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle.$$

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Then, $\big(D_{\mathbf{u}} f\big)$ is given by $\big(D_{\mathbf{u}} f\big) = (2x)u_x + (4y)u_y + (6z)u_z$.

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We conclude, $\left(D_{\mathbf{u}} f\right)_{P_0} = (6)\dfrac{2}{\sqrt{6}} + (8)\dfrac{1}{\sqrt{6}} + (6)\dfrac{1}{\sqrt{6}}$,

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that is, $\big(D_{\mathbf{u}} f\big)_{P_0} = \dfrac{26}{\sqrt{6}}$. $\triangleleft$

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- In the case of 2 variable functions: $D_{\mathbf{u}} f = f_x u_x + f_y u_y$

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y \rangle.$$

The gradient vector and directional derivatives.

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- In the case of 3 variable functions: $D_{\mathbf{u}} f = f_x u_x + f_y u_y + f_z u_z$,

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Definition

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at any point $(x, y) \in D$ is the vector $\nabla f = \langle f_x, f_y \rangle$.

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at any point $(x, y, z) \in D$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$.

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Notation:

- For two variable functions: $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j}$.
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Theorem

If $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, is a differentiable function and $\mathbf{u}$ is a unit vector, then,

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}.$$

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Example

Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

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Remark:
$\nabla f = 2\mathbf{r}$,
with
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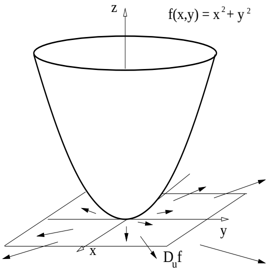

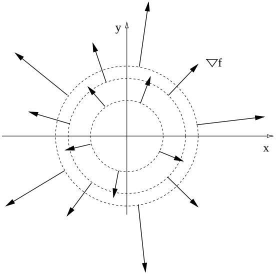

Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- Directional derivative of functions of three variables.
- The gradient vector and directional derivatives.
- **Properties of the the gradient vector.**

Properties of the the gradient vector.

Remark: If θ is the angle between ∇f and $\mathbf{u}$, then holds

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos(\theta).$$

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- The function f does not change along level curve or surfaces, that is, $D_{\mathbf{u}} f = 0$. Therefore, ∇f is perpendicular to the level curves or level surfaces.

Properties of the the gradient vector.

Example

Find the direction of maximum increase of the function $f(x, y) = x^2/4 + y^2/9$ at an arbitrary point (x, y), and also at the points $(1, 0)$ and $(0, 1)$.

Properties of the the gradient vector.

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Solution: The direction of maximum increase of f is the direction of its gradient vector:

$$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle.$$

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At the points $(1, 0)$ and $(0, 1)$ we obtain, respectively,

$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle. \qquad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle.$$

$\lhd$

Properties of the the gradient vector.

Example

Given the function $f(x, y) = x^2/4 + y^2/9$, find the equation of a line tangent to a level curve $f(x, y) = 1$ at the point $P_0 = (1, -3\sqrt{3}/2)$.

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$$\frac{1}{4} + \frac{(9)(3)}{4}\,\frac{1}{9} = 1.$$

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The equation of the line we look for is

$$\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t\,\langle v_x, v_y \rangle,$$

where $\mathbf{v} = \langle v_x, v_y \rangle$ is tangent to the level curve $f(x,y) = 1$ at P_0.

Properties of the the gradient vector.

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Solution: Therefore, $\mathbf{v} \perp \nabla f$ at P_0.

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Therefore,

$$0 = \mathbf{v} \cdot \big(\nabla f\big)_{P_0}$$

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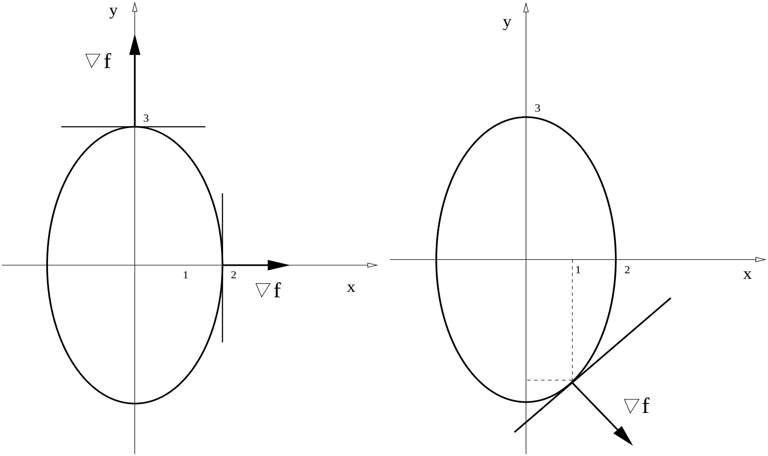
The line is $\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \,\langle 2, \sqrt{3} \rangle$. $\triangleleft$

Properties of the the gradient vector.



$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle, \quad \mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t \left\langle 2, \sqrt{3} \right\rangle.$$

Further properties of the the gradient vector.

Theorem

If f, g are differentiable scalar valued vector functions, $g \neq 0$, and $k \in R$ any constant, then holds,

1. $\nabla(kf) = k\,(\nabla f)$;
2. $\nabla(f \pm g) = \nabla f \pm \nabla g$;
3. $\nabla(fg) = (\nabla f)\,g + f\,(\nabla g)$;
4. $\nabla\Big(\dfrac{f}{g}\Big) = \dfrac{(\nabla f)\,g - f\,(\nabla g)}{g^2}$.

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.

Review: Differentiable functions of two variables.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point $(x_0, y_0) \in D$, let L be the linear function

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at* (x_0, y_0) iff the function f is approximated by the linear function L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.

Review: Differentiable functions of two variables.

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Theorem

If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

Review: Differentiable functions of two variables.

Example

Show that the function $f(x, y) = x^2 + y^2$ is differentiable for all $(x, y) \in \mathbb{R}^2$. Furthermore, find the linear function L, mentioned in the definition of a differentiable function, at the point $(1, 2)$.

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Solution: The partial derivatives of f are given by $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, which are continuous functions. Therefore, the function f is differentiable.

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Therefore, $L(x, y) = 2(x - 1) + 4(y - 2) + 5$. $\triangleleft$

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- **The tangent plane to the graph of a function.**
- The linear approximation of a differentiable function.
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  - Review: Scalar functions of one variable.
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The tangent plane to the graph of a function.

Remark:
The function $L(x, y) = 2(x - 1) + 4(y - 2) + 5$ is a plane in $\mathbb{R}^3$.

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The function $L(x, y) = 2(x - 1) + 4(y - 2) + 5$ is a plane in $\mathbb{R}^3$.
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$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

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$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

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This is a plane passing through $\tilde{P}_0 = (x_0, y_0, f(x_0,y_0))$ with normal vector $\mathbf{n} = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$.

The tangent plane to the graph of a function.

Theorem

The plane tangent to the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ *at the point* (x_0, y_0) *is given by*

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

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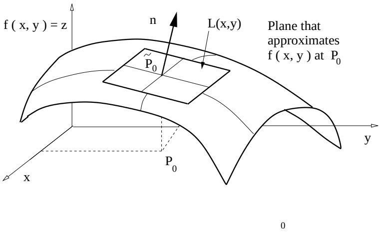
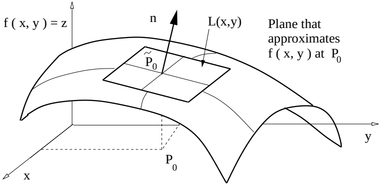
Proof

The plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0, y_0))$. We only need to find its normal vector **n**.



The tangent plane to the graph of a function.

The vector **n** normal to the plane $L(x, y)$ is a vector perpendicular to the surface $z = f(x, y)$ at $P_0 = (x_0, y_0)$.



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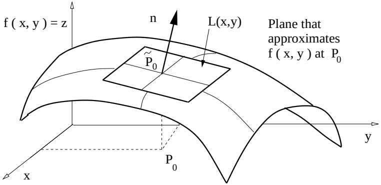

This surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$.

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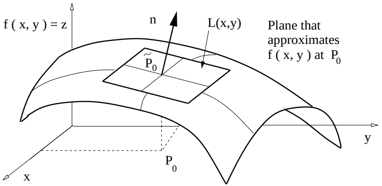

This surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. A vector normal to this level surface is its gradient ∇F.

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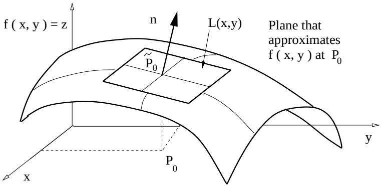

This surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. A vector normal to this level surface is its gradient ∇F. That is, $\nabla F = \langle F_x, F_y, F_z \rangle$

The tangent plane to the graph of a function.

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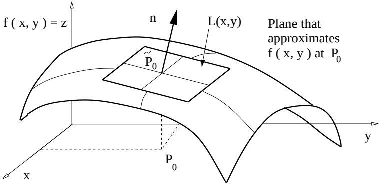

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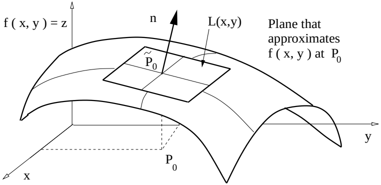

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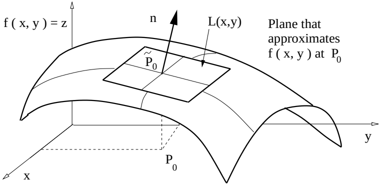

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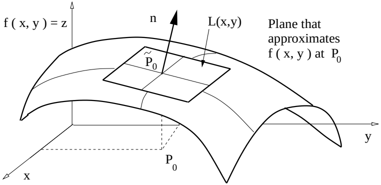

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$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

The tangent plane to the graph of a function.

Summary: We have shown that the linear L given by

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$$

is the plane tangent to the graph of f at (x_0, y_0).

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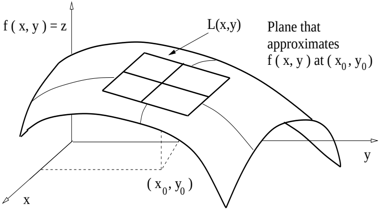

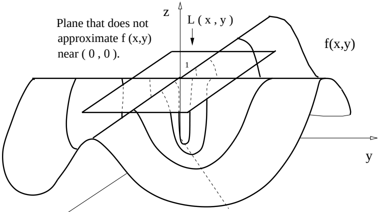

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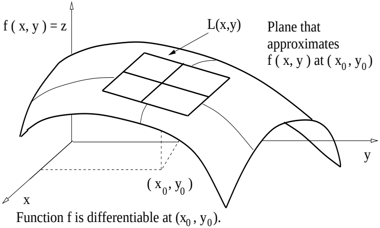

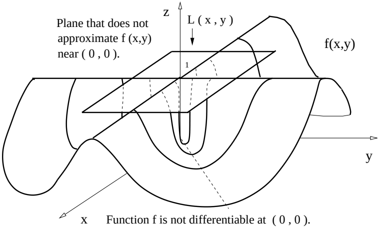

Remark: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by the tangent plane L at every point in D.

The tangent plane to the graph of a function.

Example

Show that $f(x, y) = \arctan(x + 2y)$ is differentiable and find the plane tangent to $f(x, y)$ at $(1, 0)$.

The tangent plane to the graph of a function.

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Solution: The partial derivatives of f are given by

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Then, $L(x, y) = \frac{1}{2}(x - 1) + y + \frac{\pi}{4}$. $\lhd$

Tangent planes and linear approximations (Sect. 14.6).

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- Bounds for the error of a linear approximation.
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Definition

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Find the linear approximation of $f = \sqrt{17 - x^2 - 4y^2}$ at $(2, 1)$.

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We need three numbers: $f(2, 1)$, $f_x(2, 1)$, and $f_y(2, 1)$.

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These are: $f(2, 1) = 3$, $f_x(2, 1) = -2/3$, and $f_y(2, 1) = -4/3$.

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Then the plane is given by $L(x, y) = -\dfrac{2}{3}(x - 2) - \dfrac{4}{3}(y - 1) + 3. \triangleleft$

Tangent planes and linear approximations (Sect. 14.6).

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Bounds for the error of a linear approximation.

Theorem

Assume that the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has first and second partial derivatives continuous on an open set containing a rectangular region $R \subset D$ centered at the point (x_0, y_0).
If $M \in \mathbb{R}$ is the upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in R, then the error $E(x, y) = f(x, y) - L(x, y)$ satisfies the inequality

$$|E(x,y)| \leqslant \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2,$$

where $L(x, y)$ is the linearization of f at (x_0, y_0), that is,

$$L(x,y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Bounds for the error of a linear approximation.

Example

Find an upper bound for the error in the linear approximation of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$ over the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \ : \ |x - 1| < 0.1, \quad |y - 2| < 0.1\}$$

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Solution: The second derivatives of f are $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$.

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$$|E(x,y)| \leqslant \big(|x-1| + |y-2|\big)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x,y)| < 0.04$.

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$$|E(x,y)| \leqslant \left(|x-1| + |y-2|\right)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x,y)| < 0.04$. Since $f(1,2) = 5$, the percentage relative error $100\,E(x,y)/f(1,2)$ is bounded by 0.8% $\triangleleft$

Tangent planes and linear approximations (Sect. 14.6).

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Review: Differential of functions of one variable.

Definition

The *differential at* $x_0 \in D$ of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is the linear function

$$df(x) = L(x) - f(x_0).$$

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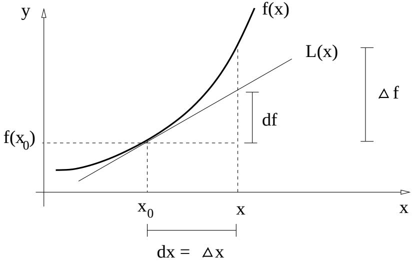

Differential of functions of more than one variable.

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Remark: The linear approximation of $f(x, y)$ at (x_0, y_0) is the plane $L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$.

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Therefore $df(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

Denoting $dx = x - x_0$ and $dy = (y - y_0)$ we obtain the usual expression

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.$$

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The *differential at* $(x_0, y_0) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the linear function

$$df(x, y) = L(x, y) - f(x_0, y_0).$$

Remark: The linear approximation of $f(x, y)$ at (x_0, y_0) is the plane $L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$.

Therefore $df(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

Denoting $dx = x - x_0$ and $dy = (y - y_0)$ we obtain the usual expression

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.$$

Therefore, df and L are similar concepts: The linear approximation of a differentiable function f.

Differential of functions of more than one variable.

Example

Compute the df of the function $f(x, y) = \ln(1 + x^2 + y^2)$ at the point $(1, 1)$. Evaluate this df for $dx = 0.1$, $dy = 0.2$.

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Solution: The differential of f at (x_0, y_0) is given by

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$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

The partial derivatives f_x and f_y are given by

$$f_x(x, y) = \frac{2x}{1 + x^2 + y^2}, \qquad f_y(x, y) = \frac{2y}{1 + x^2 + y^2}.$$

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Therefore, $f_x(1,1) = 2/3 = f_y(1,1)$. Then $df = \frac{2}{3}\,dx + \frac{2}{3}\,dy$.
Evaluating this differential at $dx = 0.1$ and $dy = 0.2$ we obtain

$$df = \frac{2}{3}\,\frac{1}{10} + \frac{2}{3}\,\frac{2}{10} = \frac{2}{3}\,\frac{3}{10} \quad \Rightarrow \quad df = \frac{1}{5}.$$

Differential of functions of more than one variable.

Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of $8cm$ and height of $12cm$ if the aluminum is $0.04cm$ thick.

Differential of functions of more than one variable.

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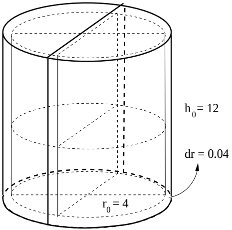
Solution:

The data of the problem is: $h_0 = 12cm$, $r_0 = 4cm$, $dr = 0.04cm$ and $dh = 0.08cm$. The function to consider is the mass of the cylinder, $M = \rho V$, where $\rho = 2.7gr/cm^3$ is the aluminum density and V is the volume of the cylinder,

$$V(r, h) = \pi r^2 h.$$



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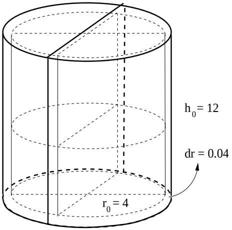

The metal to build the can is given by

$$\Delta M = \rho \left[V(r + dr, h + dh) - V(r, h) \right], \quad \big(\text{recall } dh = 2dr.\big)$$

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Since $V(r, h) = \pi r^2 h$, we obtain $dV = 2\pi r_0 h_0 dr + \pi r_0^2 dh$.

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Therefore, $dV = 16.1 \, cm^3$.

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Since $V(r, h) = \pi r^2 h$, we obtain $dV = 2\pi r_0 h_0 dr + \pi r_0^2 dh$.

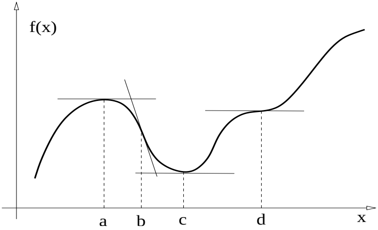
Therefore, $dV = 16.1 \, cm^3$. Since $dM = \rho \, dV$, a linear estimate for the aluminum needed to build the can is $dM = 43.47 \, gr$. $\triangleleft$

Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

Review: Local extrema for functions of one variable.

Recall: Main results on local extrema for $f(x)$:



at	f	f'	f''
a	max.	0	< 0
b	infl.	$\neq 0$	$\pm\, 0\, \mp$
c	min.	0	> 0
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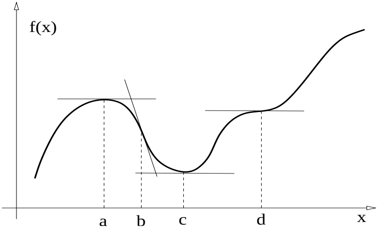

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Remarks: Assume that f is twice continuously differentiable.

- If x_0 is local maximum or minimum of f, then $f'(x_0) = 0$.

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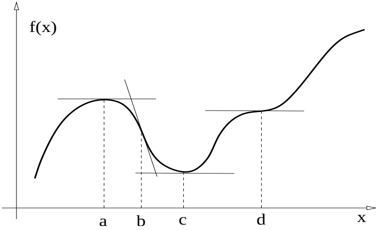

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- If $f'(x_0) = 0$ then x_0 is a critical point of f, that is, x_0 is a maximum or a minimum or an inflection point.

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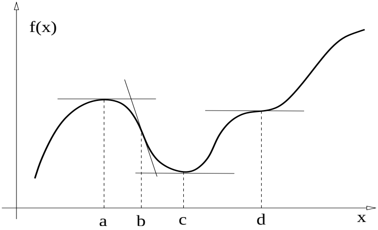

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- The second derivative test determines whether a critical point is a maximum, minimum of or an inflection point.

Local and absolute extrema, saddle points (Sect. 14.7).

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Definition of local extrema for functions of two variables.

Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local maximum* at the point $(a, b) \in D$ iff holds that $f(x, y) \leqslant f(a, b)$ for every point (x, y) in a neighborhood of (a, b).

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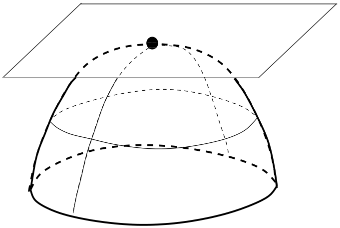
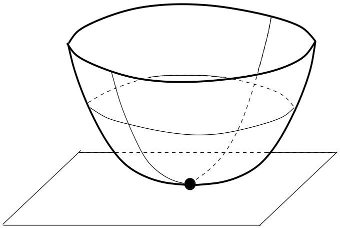
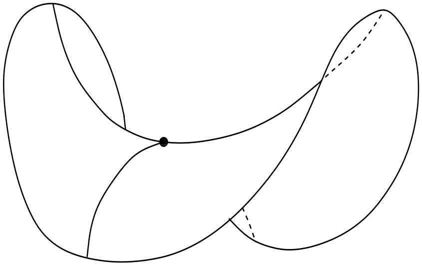




Definition of local extrema for functions of two variables.

Definition

A differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *saddle point* at an interior point $(a, b) \in D$ iff in every open disk in D centered at (a, b) there always exist points (x, y) where $f(x, y) > f(a, b)$ and other points (x, y) where $f(x, y) < f(a, b)$.



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Characterization of local extrema.

First derivative test.

Theorem

If a differentiable function f *has a local maximum or minimum at* (a, b) *then holds* $\left(\nabla f\right)\Big|_{(a,b)} = \langle 0, 0\rangle$.

Characterization of local extrema.
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Theorem

If a differentiable function f has a local maximum or minimum at (a, b) then holds $\left(\nabla f\right)\Big|_{(a,b)} = \langle 0, 0 \rangle$.

Remark: The tangent plane at a local extremum is horizontal, since its normal vector is $\mathbf{n} = \langle f_x, f_y, -1 \rangle = \langle 0, 0, -1 \rangle$.

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Definition

The interior point $(a, b) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a *critical point* of f iff $\left(\nabla f\right)\Big|_{(a,b)} = \langle 0, 0\rangle$.

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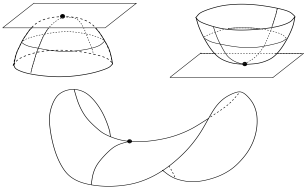
Remark:
Critical points include local maxima, local minima, and saddle points.



Characterization of local extrema.

First derivative test.

Example

Find the critical points of the function $f(x, y) = -x^2 - y^2$

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Solution: The critical points are the points where ∇f vanishes. Since $\nabla f = \langle -2x, -2y \rangle$, the only solution to $\nabla f = \langle 0, 0 \rangle$ is $x = 0$, $y = 0$. That is, $(a, b) = (0, 0)$. $\lhd$

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Remark: Since $f(x, y) \leqslant 0$ for all $(x, y) \in \mathbb{R}^2$ and $f(0, 0) = 0$, then the point $(0, 0)$ must be a local maximum of f.

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Characterization of local extrema.

Second derivative test.

Theorem

Let (a, b) be a critical point of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, that is, $(\nabla f)\big|_{(a,b)} = \langle 0, 0 \rangle$. Assume that f has continuous second derivatives in an open disk in D with center in (a, b) and denote

$$D = f_{xx}(a, b)\, f_{yy}(a, b) - \big[f_{xy}(a, b)\big]^2.$$

Then, the following statements hold:

- *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
- *If $D < 0$, then $f(a, b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

Notation: The number D is called the discriminant of f at (a, b).

Characterization of local extrema.

Second derivative test.

Example

Find the local extrema of $f(x, y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

Characterization of local extrema.
Second derivative test.

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Characterization of local extrema.

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Since $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = 2$, and $f_{xy}(0, 0) = 0$,

Characterization of local extrema.

Second derivative test.

Example

Find the local extrema of $f(x, y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

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$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0,0).$$

$\triangleleft$

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Second derivative test.

Example

Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2 x^2$?

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Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2 x^2$?

Solution: We first verify that $(0, 0)$ is a critical point of f:

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therefore, $(0, 0)$ is a critical point.

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hence $D = 0$ and the test is inconclusive. $\triangleleft$

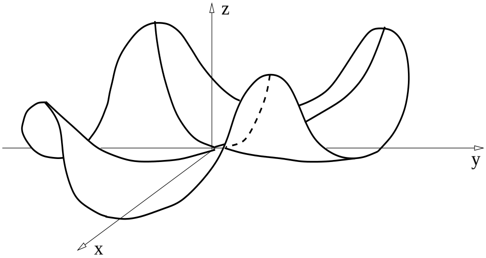
Characterization of local extrema.

Second derivative test.

Example

Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2x^2$?

Solution: From the graph of $f = x^2y^2$ is simple to see that $(0, 0)$ is a local minimum: (also a global minimum.) $\triangleleft$



Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- **Absolute extrema of a function in a domain.**

Absolute extrema of a function in a domain.

Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute maximum at the point $(a, b) \in D$ iff $f(x, y) \leqslant f(a, b)$ for all $(x, y) \in D$.
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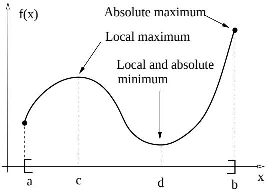

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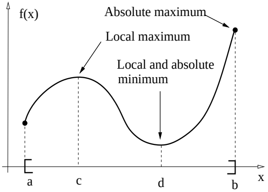
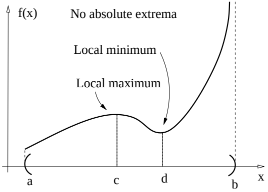


Remark: Absolute extrema may not be defined on open intervals.



Review: Functions of one variable.

Theorem

Every continuous functions $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, *with* $a < b \in \mathbb{R}$ *always has absolute extrema.*

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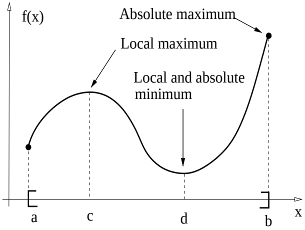


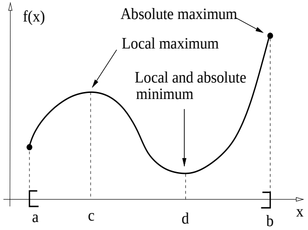
Recall:

- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.

Review: Functions of one variable.

Theorem

Every continuous functions $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, *with* $a < b \in \mathbb{R}$ *always has absolute extrema.*



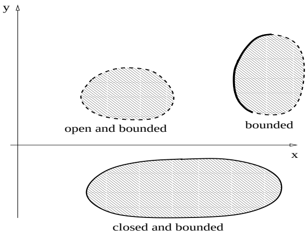
Recall:

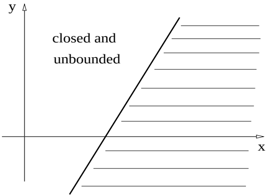
- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.
- The set $[a, b]$ is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.

Recall: On open and closed sets in $\mathbb{R}^n$.

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





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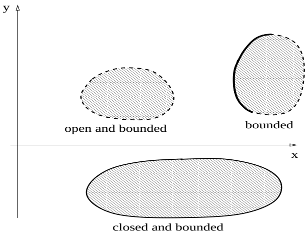

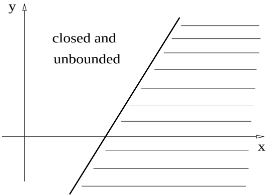

Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in a closed and bounded set D, then f has an absolute maximum and an absolute minimum in D.

Absolute extrema on closed and bounded sets.

Problem:
Find the absolute extrema of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ in a closed and bounded set D.

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Solution:

(1) Find every critical point of f in the interior of D and evaluate f at these points.

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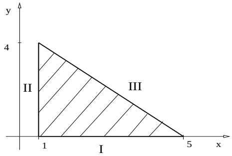
(3) Look at the list of values for f found in the previous two steps.

If $f(x_0, y_0)$ is the biggest (smallest) value of f in the list above, then (x_0, y_0) is the absolute maximum (minimum) of f in D.

Absolute extrema on closed and bounded sets.

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



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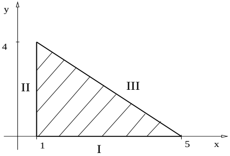

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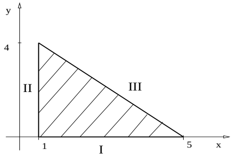

Solution:

(1) We find all critical points in the interior of the domain:

$$\nabla f = \langle (y - 1), (x + 2) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad (x_0, y_0) = (-2, 1).$$

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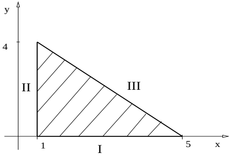

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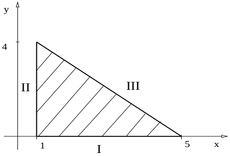

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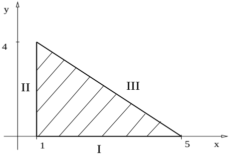

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Boundary I: The segment $y = 0$, $x \in [1, 5]$.

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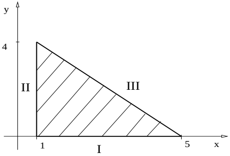

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Boundary I: The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.

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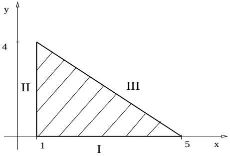

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We look for critical point on the interior of Boundary I:

Absolute extrema on closed and bounded sets.

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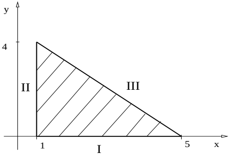

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(2) Three segments form the boundary of D:

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We look for critical point on the interior of Boundary I: Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$.

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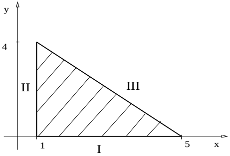

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Since $(-2, 1)$ does not belong to the domain, we discard it.

(2) Three segments form the boundary of D:

Boundary I: The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.

We look for critical point on the interior of Boundary I: Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$. No critical points in the interior of Boundary I.

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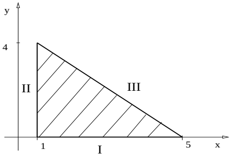

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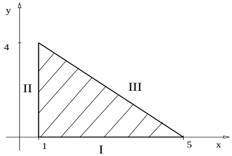

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$.

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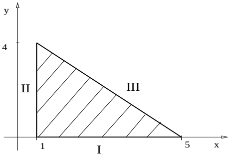

Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$.

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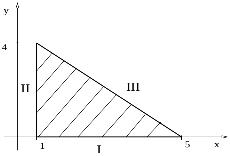

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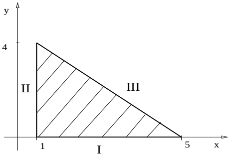

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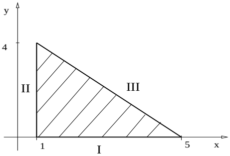

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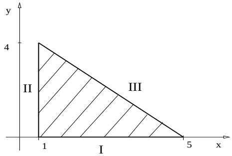

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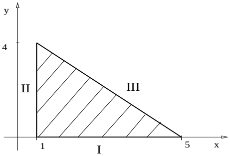

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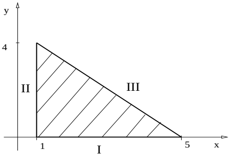

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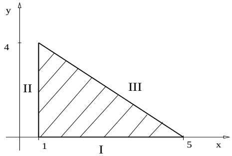

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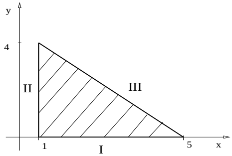

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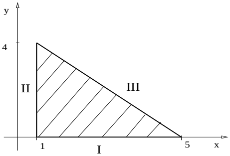

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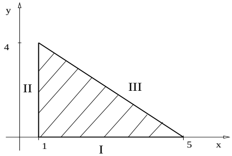

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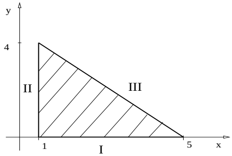

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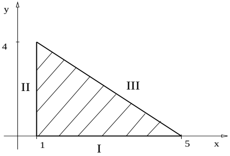

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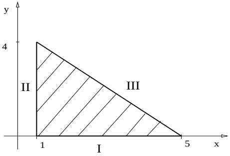

Solution:

(3) Our list of values is:

$$f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2.$$

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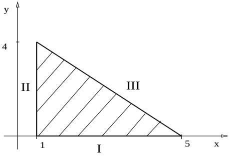

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(3) Our list of values is:

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We conclude:

- Absolute maximum at $(1, 4)$,
- Absolute minimum at $(5, 0)$.

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A maximization problem with a constraint.

Example

Find the maximum volume of a closed rectangular box with a given surface area A_0.

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Solution: This problem can be solved by finding the local maximum of an appropriate function f.
The function f is obtained as follows: Recall the functions volume and area of a rectangular box with vertex at $(0,0,0)$ and sides x, y and z:

$$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

Since $A(x,y,z) = A_0$, we obtain $z = \dfrac{A_0 - 2xy}{2(x+y)}$, that is

$$f(x,y) = \frac{A_0 xy - 2x^2y^2}{2(x+y)}.$$

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$$f_x = \frac{2A_0 y^2 - 4x^2y^2 - 8xy^3}{4(x+y)^2}, \quad f_y = \frac{2A_0 x^2 - 4x^2y^2 - 8yx^3}{4(x+y)^2}.$$

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$\left.\begin{aligned} A_0 &= 2x^2 + 4xy, \\ A_0 &= 2y^2 + 4xy, \end{aligned}\right\} \quad \Rightarrow \quad x = y.$$

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so, $z = \dfrac{A_0 - 2x^2}{4x} = y$. Therefore, $x_0 = y_0 = z_0 = \sqrt{A_0/6}$.