Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- Directional derivative of functions of three variables.
- The gradient vector and directional derivatives.
- Properties of the the gradient vector.

Directional derivative of functions of two variables.

Remark: The directional derivative generalizes the partial derivatives to any direction.

Definition

The *directional derivative* of the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ is given by

$$\left(D_{\mathbf{u}} f\right)_{P_0} = \lim_{t \to 0} \frac{1}{t} \big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0) \big],$$

if the limit exists.

Notation: The directional derivative is also denoted as

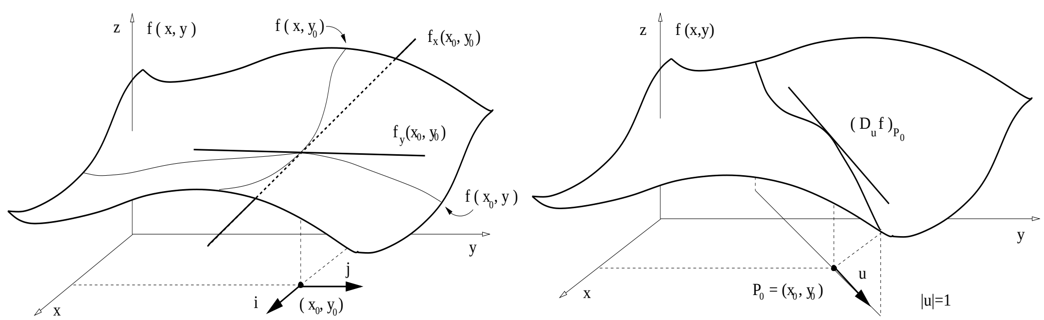
$$\left(\frac{df}{dt}\right)_{\mathbf{u}, P_0}.$$

Directional derivatives generalize partial derivatives.

Example

The partial derivatives f_x and f_y are particular cases of directional derivatives $\big(D_{\mathbf{u}} f\big)_{P_0} = \lim_{t\to 0} \frac{1}{t}\big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\big]$:

- $\mathbf{u} = \langle 1, 0\rangle = \mathbf{i}$, then $\big(D_{\mathbf{i}} f\big)_{P_0} = f_x(x_0, y_0)$.
- $\mathbf{u} = \langle 0, 1\rangle = \mathbf{j}$, then $\big(D_{\mathbf{j}} f\big)_{P_0} = f_y(x_0, y_0)$.



Directional derivative of functions of two variables.

Remark: The condition $|\mathbf{u}| = 1$ implies that the parameter t in the line $\mathbf{r}(t) = \langle x_0, y_0\rangle + \mathbf{u}\, t$ is the distance between the points $(x(t), y(t)) = (x_0 + u_x t, y_0 + u_y t)$ and (x_0, y_0).

Proof.

$$d = |\langle x - x_0, y - y_0\rangle|, = |\langle u_x t, u_y t\rangle|, = |t|\,|\mathbf{u}|,$$

that is, $d = |t|$. □

Remark: The directional derivative of $f(x, y)$ at $P_0 = (x_0, y_0)$ along $\mathbf{u}$, denoted as $\big(D_{\mathbf{u}} f\big)_{P_0}$, is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through (x_0, y_0).

Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
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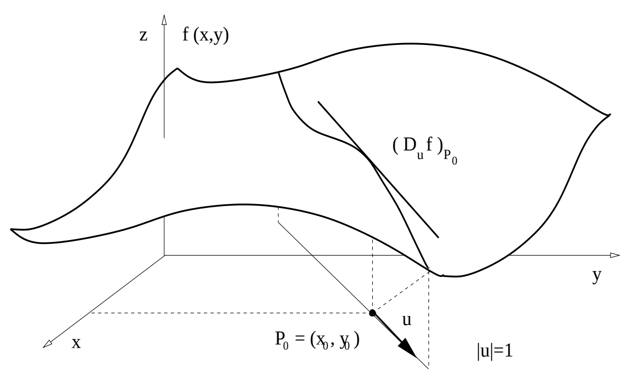
Directional derivative and partial derivatives.

Remark: The directional derivative $\big(D_{\mathbf{u}} f\big)_{P_0}$ is the derivative of f along the line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \mathbf{u}\, t$.



Theorem

If the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0)$ and $\mathbf{u} = \langle u_x, u_y \rangle$ is a unit vector, then

$$\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$$

Directional derivative and partial derivatives.

Proof.

The line $\mathbf{r}(t) = \langle x_0, y_0 \rangle + \langle u_x, u_y \rangle\, t$ has parametric equations:
$x(t) = x_0 + u_x t$ and $y(t) = y_0 + u_y t$;
Denote f evaluated along the line as $\hat{f}(t) = f(x(t), y(t))$.
Now, on the one hand, $\hat{f}'(0) = \big(D_{\mathbf{u}} f\big)_{P_0}$, since

$$\begin{aligned}
\hat{f}'(0) &= \lim_{t\to 0} \frac{1}{t}\big[\hat{f}(t) - \hat{f}(0)\big],\\
&= \lim_{t\to 0} \frac{1}{t}\big[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)\big],\\
&= D_{\mathbf{u}} f(x_0, y_0).
\end{aligned}$$

On the other hand, the chain rule implies:

$$\hat{f}'(0) = f_x(x_0, y_0)\, x'(0) + f_y(x_0, y_0)\, y'(0).$$

Therefore, $\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$ □

Directional derivative and partial derivatives.

Example

Compute the directional derivative of $f(x, y) = \sin(x + 3y)$ at the point $P_0 = (4, 3)$ in the direction of vector $\mathbf{v} = \langle 1, 2 \rangle$.

Solution: We need to find a unit vector in the direction of $\mathbf{v}$.

Such vector is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \dfrac{1}{\sqrt{5}} \langle 1, 2 \rangle.$

We now use the formula $\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$

That is, $\big(D_{\mathbf{u}} f\big)_{P_0} = \cos(x_0 + 3y_0)(1/\sqrt{5}) + 3\cos(x_0 + 3y_0)(2/\sqrt{5}).$

Equivalently, $\big(D_{\mathbf{u}} f\big)_{P_0} = (7/\sqrt{5})\cos(x_0 + 3y_0).$

Then , $\big(D_{\mathbf{u}} f\big)_{P_0} = (7/\sqrt{5})\cos(10).$ ◁

Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
- Partial derivatives and directional derivatives.
- **Directional derivative of functions of three variables.**
- The gradient vector and directional derivatives.
- Properties of the the gradient vector.

Directional derivative of functions of three variables.

Definition

The *directional derivative* of the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at the point $P_0 = (x_0, y_0, z_0) \in D$ in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is given by

$$\big(D_{\mathbf{u}} f\big)_{P_0} = \lim_{t \to 0} \frac{1}{t} \big[f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0) \big],$$

if the limit exists.

Theorem

If the function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $P_0 = (x_0, y_0, z_0)$ and $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is a unit vector, then

$$\big(D_{\mathbf{u}} f\big)_{P_0} = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0)\, u_z.$$

Directional derivative of functions of three variables.

Example

Find $\left(D_{\mathbf{u}} f\right)_{P_0}$ for $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at the point $P_0 = (3,2,1)$ along the direction given by $\mathbf{v} = \langle 2,1,1 \rangle$.

Solution: We first find a unit vector along $\mathbf{v}$,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{6}} \langle 2,1,1 \rangle.$$

Then, $\left(D_{\mathbf{u}} f\right)$ is given by $\left(D_{\mathbf{u}} f\right) = (2x)u_x + (4y)u_y + (6z)u_z$.

We conclude, $\left(D_{\mathbf{u}} f\right)_{P_0} = (6)\frac{2}{\sqrt{6}} + (8)\frac{1}{\sqrt{6}} + (6)\frac{1}{\sqrt{6}}$,

that is, $\left(D_{\mathbf{u}} f\right)_{P_0} = \frac{26}{\sqrt{6}}$. $\triangleleft$

Directional derivatives and gradient vectors (Sect. 14.5).

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- **The gradient vector and directional derivatives.**
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The gradient vector and directional derivatives.

Remark: The directional derivative of a function can be written in terms of a dot product.

- In the case of 2 variable functions: $D_{\mathbf{u}}f = f_x u_x + f_y u_y$

$$D_{\mathbf{u}}f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y \rangle.$$

- In the case of 3 variable functions: $D_{\mathbf{u}}f = f_x u_x + f_y u_y + f_z u_z$,

$$D_{\mathbf{u}}f = (\nabla f) \cdot \mathbf{u}, \quad \text{with} \quad \nabla f = \langle f_x, f_y, f_z \rangle.$$

The gradient vector and directional derivatives.

Definition

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at any point $(x, y) \in D$ is the vector $\nabla f = \langle f_x, f_y \rangle$.

The *gradient vector* of a differentiable function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ at any point $(x, y, z) \in D$ is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$.

Notation:

- For two variable functions: $\nabla f = f_x\, \mathbf{i} + f_y\, \mathbf{j}$.
- For two variable functions: $\nabla f = f_x\, \mathbf{i} + f_y\, \mathbf{j} + f_z\, \mathbf{k}$.

Theorem

If $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, is a differentiable function and $\mathbf{u}$ *is a unit vector, then,*

$$D_{\mathbf{u}}f = (\nabla f) \cdot \mathbf{u}.$$

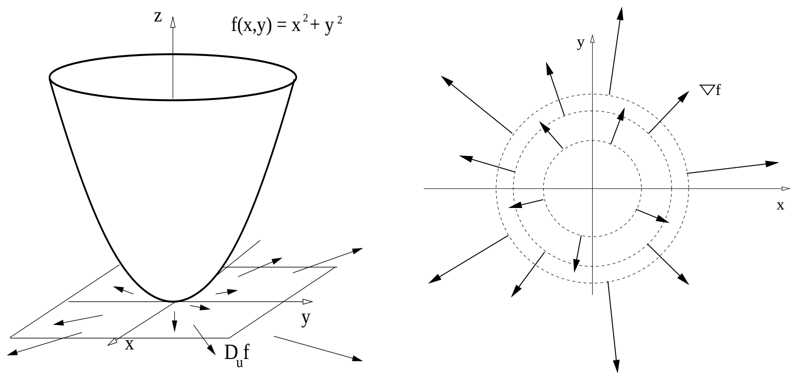
The gradient vector and directional derivatives.

Example

Find the gradient vector at any point in the domain of the function $f(x, y) = x^2 + y^2$.

Solution: The gradient is $\nabla f = \langle f_x, f_y \rangle$, that is, $\nabla f = \langle 2x, 2y \rangle$. $\triangleleft$

Remark:
$\nabla f = 2\mathbf{r}$,
with
$\mathbf{r} = \langle x, y \rangle$.



Directional derivatives and gradient vectors (Sect. 14.5).

- Directional derivative of functions of two variables.
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- **Properties of the the gradient vector.**

Properties of the the gradient vector.

Remark: If θ is the angle between ∇f and $\mathbf{u}$, then holds

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos(\theta).$$

The formula above implies:

- The function f increases the most rapidly when $\mathbf{u}$ is in the direction of ∇f, that is, $\theta = 0$. The maximum increase rate of f is $|\nabla f|$.
- The function f decreases the most rapidly when $\mathbf{u}$ is in the direction of $-\nabla f$, that is, $\theta = \pi$. The maximum decrease rate of f is $-|\nabla f|$.
- The function f does not change along level curve or surfaces, that is, $D_{\mathbf{u}} f = 0$. Therefore, ∇f is perpendicular to the level curves or level surfaces.

Properties of the the gradient vector.

Example

Find the direction of maximum increase of the function $f(x, y) = x^2/4 + y^2/9$ at an arbitrary point (x, y), and also at the points $(1, 0)$ and $(0, 1)$.

Solution: The direction of maximum increase of f is the direction of its gradient vector:

$$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle.$$

At the points $(1, 0)$ and $(0, 1)$ we obtain, respectively,

$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle. \qquad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle.$$

$\triangleleft$

Properties of the the gradient vector.

Example

Given the function $f(x,y) = x^2/4 + y^2/9$, find the equation of a line tangent to a level curve $f(x,y) = 1$ at the point $P_0 = (1, -3\sqrt{3}/2)$.

Solution: We first verify that P_0 belongs to the level curve $f(x,y) = 1$. This is the case, since

$$\frac{1}{4} + \frac{(9)(3)}{4}\,\frac{1}{9} = 1.$$

The equation of the line we look for is

$$\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t\,\langle v_x, v_y \rangle,$$

where $\mathbf{v} = \langle v_x, v_y \rangle$ is tangent to the level curve $f(x,y) = 1$ at P_0.

Properties of the the gradient vector.

Example

Given the function $f(x,y) = x^2/4 + y^2/9$, find the equation of a line tangent to a level curve $f(x,y) = 1$ at the point $P_0 = (1, -3\sqrt{3}/2)$.

Solution: Therefore, $\mathbf{v} \perp \nabla f$ at P_0. Since,

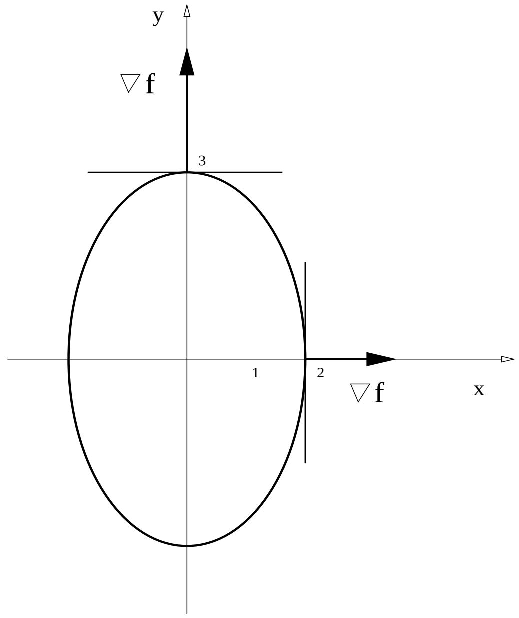
$$\nabla f = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle \quad \Rightarrow \quad \left(\nabla f\right)_{P_0} = \left\langle \frac{1}{2}, -\frac{2}{9}\,\frac{3\sqrt{3}}{2} \right\rangle = \left\langle \frac{1}{2}, -\frac{1}{\sqrt{3}} \right\rangle.$$

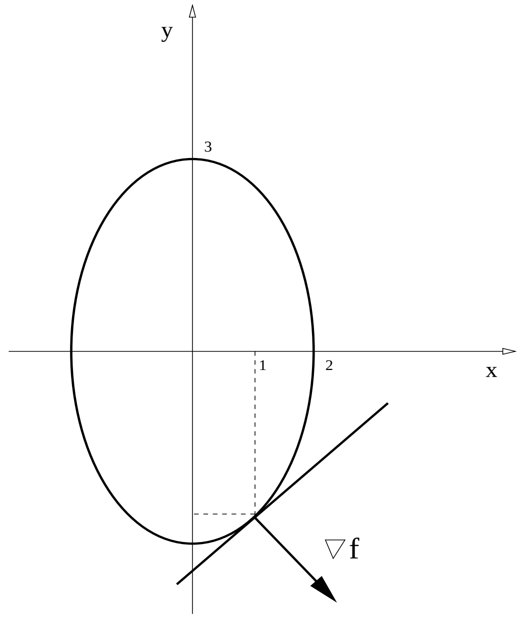
Therefore,

$$0 = \mathbf{v} \cdot \left(\nabla f\right)_{P_0} \quad \Rightarrow \quad \frac{1}{2} v_x = \frac{1}{\sqrt{3}} v_y \quad \Rightarrow \quad \mathbf{v} = \langle 2, \sqrt{3} \rangle.$$

The line is $\mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t\,\langle 2, \sqrt{3} \rangle$. ◁

Properties of the the gradient vector.





$$\nabla f = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \nabla f = \left\langle 0, \frac{2}{9} \right\rangle, \quad \mathbf{r}(t) = \left\langle 1, -\frac{3\sqrt{3}}{2} \right\rangle + t\,\langle 2, \sqrt{3} \rangle.$$

Further properties of the the gradient vector.

Theorem

If f, g are differentiable scalar valued vector functions, $g \neq 0$, and $k \in R$ any constant, then holds,

1. $\nabla(kf) = k\,(\nabla f)$;
2. $\nabla(f \pm g) = \nabla f \pm \nabla g$;
3. $\nabla(fg) = (\nabla f)\,g + f\,(\nabla g)$;
4. $\nabla\left(\frac{f}{g}\right) = \frac{(\nabla f)\,g - f\,(\nabla g)}{g^2}$.

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- The differential of a function.
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.

Review: Differentiable functions of two variables.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point $(x_0, y_0) \in D$, let L be the linear function

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at* (x_0, y_0) iff the function f is approximated by the linear function L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.

Theorem

If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

Review: Differentiable functions of two variables.

Example

Show that the function $f(x, y) = x^2 + y^2$ is differentiable for all $(x, y) \in \mathbb{R}^2$. Furthermore, find the linear function L, mentioned in the definition of a differentiable function, at the point $(1, 2)$.

Solution: The partial derivatives of f are given by $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, which are continuous functions. Therefore, the function f is differentiable. The linear function L at $(1, 2)$ is

$$L(x, y) = f_x(1, 2)\,(x - 1) + f_y(1, 2)\,(y - 2) + f(1, 2).$$

That is, we need three numbers to find the linear function L: $f_x(1, 2)$, $f_y(1, 2)$, and $f(1, 2)$. These numbers are:

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4, \quad f(1, 2) = 5.$$

Therefore, $L(x, y) = 2(x - 1) + 4(y - 2) + 5$. ◁

Tangent planes and linear approximations (Sect. 14.6).

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  - Scalar functions of more than one variable.

The tangent plane to the graph of a function.

Remark:
The function $L(x, y) = 2(x - 1) + 4(y - 2) + 5$ is a plane in $\mathbb{R}^3$.
We usually write down the equation of a plane using the notation $z = L(x, y)$, that is, $z = 2(x - 1) + 4(y - 2) + 5$, or equivalently

$$2(x - 1) + 4(y - 2) - (z - 5) = 0.$$

This is a plane passing through $\tilde{P}_0 = (1, 2, 5)$ with normal vector $\mathbf{n} = \langle 2, 4, -1 \rangle$. Analogously, the function

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

is a plane in $\mathbb{R}^3$. Using the notation $z = L(x, y)$ we obtain

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

This is a plane passing through $\tilde{P}_0 = (x_0, y_0, f(x_0, y_0))$ with normal vector $\mathbf{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$.

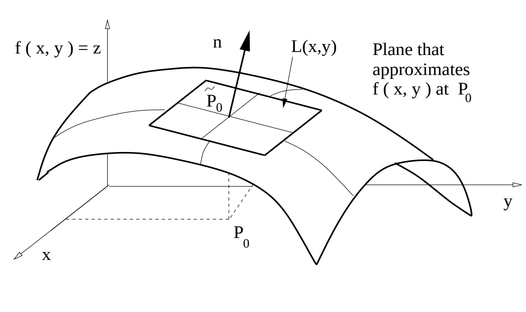
The tangent plane to the graph of a function.

Theorem
The plane tangent to the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point (x_0, y_0) is given by

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

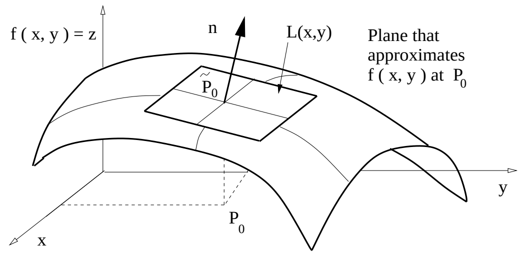
Proof

The plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0, y_0))$. We only need to find its normal vector $\mathbf{n}$.



The tangent plane to the graph of a function.

The vector **n** normal to the plane $L(x, y)$ is a vector perpendicular to the surface $z = f(x, y)$ at $P_0 = (x_0, y_0)$.



This surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. A vector normal to this level surface is its gradient ∇F. That is, $\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$.

Therefore, the normal to the tangent plane $L(x, y)$ at the point P_0 is $\mathbf{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$. Recall that the plane contains the point $\tilde{P}_0 = (x_0, y_0, f(x_0, y_0))$. The equation for the plane is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

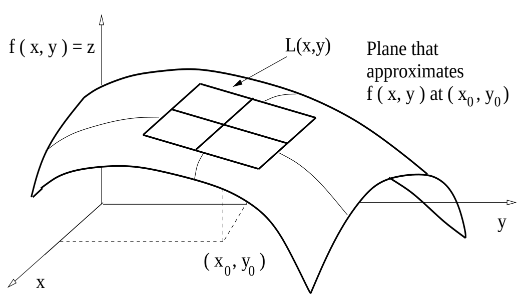
$\square$

The tangent plane to the graph of a function.

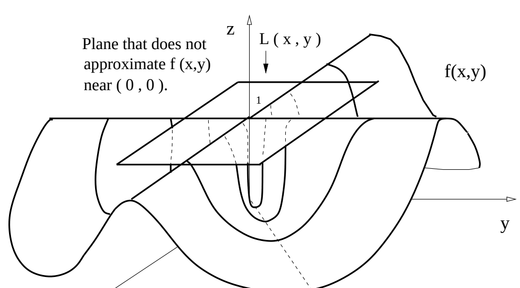
Summary: We have shown that the linear L given by

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0)$$

is the plane tangent to the graph of f at (x_0, y_0).



Function f is differentiable at (x_0, y_0).



Function f is not differentiable at (0 , 0).

Remark: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by the tangent plane L at every point in D.

The tangent plane to the graph of a function.

Example

Show that $f(x,y) = \arctan(x+2y)$ is differentiable and find the plane tangent to $f(x,y)$ at $(1,0)$.

Solution: The partial derivatives of f are given by

$$f_x(x,y) = \frac{1}{1+(x+2y)^2}, \quad f_y(x,y) = \frac{2}{1+(x+2y)^2}.$$

These functions are continuous in $\mathbb{R}^2$, so $f(x,y)$ is differentiable at every point in $\mathbb{R}^2$. The plane $L(x,y)$ at $(1,0)$ is given by

$$L(x,y) = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0),$$

where $f(1,0) = \arctan(1) = \pi/4$, $f_x(1,0) = 1/2$, $f_y(1,0) = 1$.

Then, $L(x,y) = \dfrac{1}{2}(x-1) + y + \dfrac{\pi}{4}$. $\triangleleft$

Tangent planes and linear approximations (Sect. 14.6).

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- Bounds for the error of a linear approximation.
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The linear approximation of a differentiable function.

Definition

The *linear approximation* of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ at the point $(x_0, y_0) \in D$ is the plane

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Example

Find the linear approximation of $f = \sqrt{17 - x^2 - 4y^2}$ at $(2, 1)$.

Solution: $L(x, y) = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1)$.

We need three numbers: $f(2, 1)$, $f_x(2, 1)$, and $f_y(2, 1)$.

These are: $f(2, 1) = 3$, $f_x(2, 1) = -2/3$, and $f_y(2, 1) = -4/3$.

Then the plane is given by $L(x, y) = -\frac{2}{3}(x - 2) - \frac{4}{3}(y - 1) + 3.\lhd$

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
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Bounds for the error of a linear approximation.

Theorem

Assume that the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has first and second partial derivatives continuous on an open set containing a rectangular region $R \subset D$ centered at the point (x_0, y_0).
If $M \in \mathbb{R}$ is the upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in R, then the error $E(x,y) = f(x,y) - L(x,y)$ satisfies the inequality

$$|E(x,y)| \leqslant \frac{1}{2} M \big(|x - x_0| + |y - y_0|\big)^2,$$

where $L(x,y)$ is the linearization of f at (x_0, y_0), that is,

$$L(x,y) = f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + f(x_0,y_0).$$

Bounds for the error of a linear approximation.

Example

Find an upper bound for the error in the linear approximation of $f(x,y) = x^2 + y^2$ at the point $(1,2)$ over the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \;:\; |x - 1| < 0.1, \quad |y - 2| < 0.1\}$$

Solution: The second derivatives of f are $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$.

Therefore, we can take $M = 2$.

Then the formula $|E(x,y)| \leqslant \frac{1}{2} M \big(|x - x_0| + |y - y_0|\big)^2$, implies

$$|E(x,y)| \leqslant \big(|x - 1| + |y - 2|\big)^2 < (0.1 + 0.1)^2 = 0.04,$$

that is $|E(x,y)| < 0.04$. Since $f(1,2) = 5$, the percentage relative error $100\,E(x,y)/f(1,2)$ is bounded by 0.8% ◁

Tangent planes and linear approximations (Sect. 14.6).

- Review: Differentiable functions of two variables.
- The tangent plane to the graph of a function.
- The linear approximation of a differentiable function.
- Bounds for the error of a linear approximation.
- **The differential of a function.**
  - Review: Scalar functions of one variable.
  - Scalar functions of more than one variable.

Review: Differential of functions of one variable.

Definition
The *differential at $x_0 \in D$* of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is the linear function

$$df(x) = L(x) - f(x_0).$$

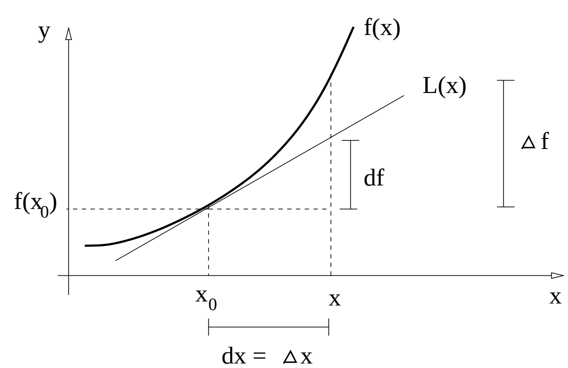
Remark: The linear approximation of $f(x)$ at x_0 is the line given by $L(x) = f'(x_0)\,(x - x_0) + f(x_0)$.

Therefore

$$df(x) = f'(x_0)\,(x - x_0).$$

Denoting $dx = x - x_0$,

$$df = f'(x_0)\,dx.$$



Differential of functions of more than one variable.

Definition

The *differential at* $(x_0, y_0) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the linear function

$$df(x,y) = L(x,y) - f(x_0,y_0).$$

Remark: The linear approximation of $f(x,y)$ at (x_0,y_0) is the plane $L(x,y) = f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) + f(x_0,y_0)$.

Therefore $df(x,y) = f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)$.

Denoting $dx = x - x_0$ and $dy = (y - y_0)$ we obtain the usual expression

$$df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.$$

Therefore, df and L are similar concepts: The linear approximation of a differentiable function f.

Differential of functions of more than one variable.

Example

Compute the df of the function $f(x,y) = \ln(1 + x^2 + y^2)$ at the point $(1,1)$. Evaluate this df for $dx = 0.1$, $dy = 0.2$.

Solution: The differential of f at (x_0,y_0) is given by

$$df = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy.$$

The partial derivatives f_x and f_y are given by

$$f_x(x,y) = \frac{2x}{1+x^2+y^2}, \qquad f_y(x,y) = \frac{2y}{1+x^2+y^2}.$$

Therefore, $f_x(1,1) = 2/3 = f_y(1,1)$. Then $df = \frac{2}{3}\,dx + \frac{2}{3}\,dy$. Evaluating this differential at $dx = 0.1$ and $dy = 0.2$ we obtain

$$df = \frac{2}{3}\,\frac{1}{10} + \frac{2}{3}\,\frac{2}{10} = \frac{2}{3}\,\frac{3}{10} \quad \Rightarrow \quad df = \frac{1}{5}.$$

Differential of functions of more than one variable.

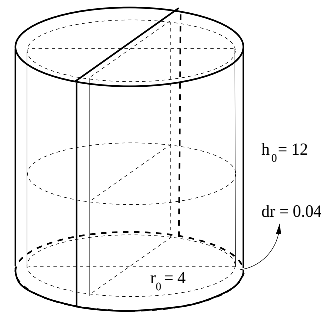
Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of $8cm$ and height of $12cm$ if the aluminum is $0.04cm$ thick.

Solution:

The data of the problem is: $h_0 = 12cm$, $r_0 = 4cm$, $dr = 0.04cm$ and $dh = 0.08cm$. The function to consider is the mass of the cylinder, $M = \rho V$, where $\rho = 2.7gr/cm^3$ is the aluminum density and V is the volume of the cylinder,

$$V(r,h) = \pi r^2 h.$$



The metal to build the can is given by

$$\Delta M = \rho\left[V(r+dr, h+dh) - V(r,h)\right], \quad \left(\text{recall } dh = 2dr.\right)$$

Differential of functions of more than one variable.

Example

Use differentials to estimate the amount of aluminum needed to build a closed cylindrical can with internal diameter of $8cm$ and height of $12cm$ if the aluminum is $0.04cm$ thick.

Solution: The metal to build the can is given by

$$\Delta M = \rho\left[V(r+dr, h+dh) - V(r,h)\right],$$

A linear approximation to $\Delta V = V(r+dr, h+dh) - V(r,h)$ is $dV = V_r\, dr + V_h\, dh$, that is,

$$dV = V_r(r_0,h_0)dr + V_h(r_0,h_0)dh.$$

Since $V(r,h) = \pi r^2 h$, we obtain $dV = 2\pi r_0 h_0 dr + \pi r_0^2 dh$.

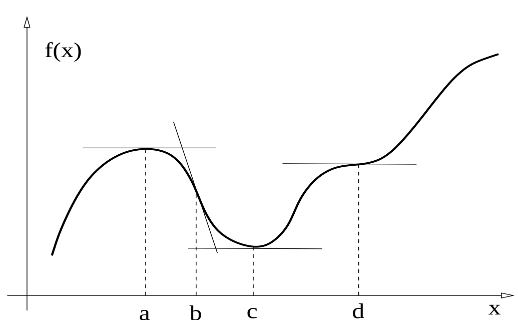
Therefore, $dV = 16.1\, cm^3$. Since $dM = \rho\, dV$, a linear estimate for the aluminum needed to build the can is $dM = 43.47\, gr$. $\triangleleft$

Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

Review: Local extrema for functions of one variable.

Recall: Main results on local extrema for $f(x)$:



at	f	f'	f''
a	max.	0	< 0
b	infl.	$\neq 0$	$\pm\, 0 \mp$
c	min.	0	> 0
d	infl.	$= 0$	$\pm\, 0 \mp$

Remarks: Assume that f is twice continuously differentiable.

- If x_0 is local maximum or minimum of f, then $f'(x_0) = 0$.
- If $f'(x_0) = 0$ then x_0 is a critical point of f, that is, x_0 is a maximum or a minimum or an inflection point.
- The second derivative test determines whether a critical point is a maximum, minimum or an inflection point.

Local and absolute extrema, saddle points (Sect. 14.7).

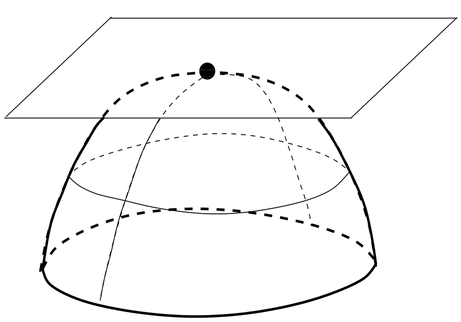
- Review: Local extrema for functions of one variable.
- **Definition of local extrema.**
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

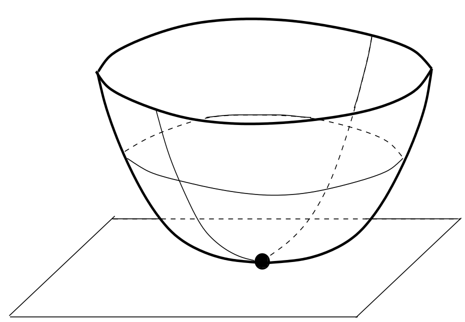
Definition of local extrema for functions of two variables.

Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local maximum* at the point $(a, b) \in D$ iff holds that $f(x, y) \leqslant f(a, b)$ for every point (x, y) in a neighborhood of (a, b).

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *local minimum* at the point $(a, b) \in D$ iff holds that $f(x, y) \geqslant f(a, b)$ for every point (x, y) in a neighborhood of (a, b).

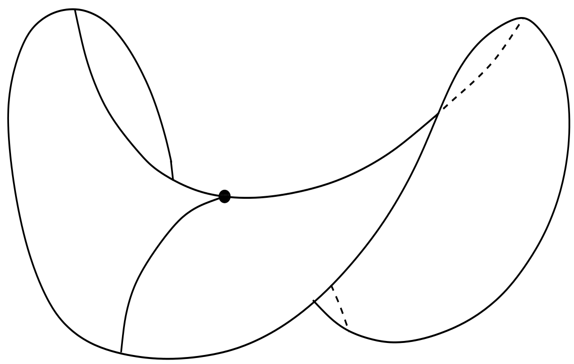




Definition of local extrema for functions of two variables.

Definition

A differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has a *saddle point* at an interior point $(a, b) \in D$ iff in every open disk in D centered at (a, b) there always exist points (x, y) where $f(x, y) > f(a, b)$ and other points (x, y) where $f(x, y) < f(a, b)$.



Local and absolute extrema, saddle points (Sect. 14.7).

- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- **Characterization of local extrema.**
  - First derivative test.
  - Second derivative test.
- Absolute extrema of a function in a domain.

Characterization of local extrema.

First derivative test.

Theorem

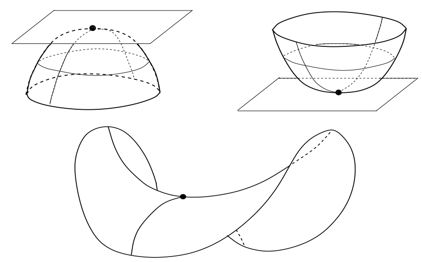
If a differentiable function f has a local maximum or minimum at (a, b) then holds $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$.

Remark: The tangent plane at a local extremum is horizontal, since its normal vector is $\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 0, 0, -1\rangle$.

Definition

The interior point $(a, b) \in D$ of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a *critical point* of f iff $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$.

Remark:
Critical points include local maxima, local minima, and saddle points.



Characterization of local extrema.

First derivative test.

Example

Find the critical points of the function $f(x, y) = -x^2 - y^2$

Solution: The critical points are the points where ∇f vanishes. Since $\nabla f = \langle -2x, -2y\rangle$, the only solution to $\nabla f = \langle 0, 0\rangle$ is $x = 0$, $y = 0$. That is, $(a, b) = (0, 0)$. $\triangleleft$

Remark: Since $f(x, y) \leqslant 0$ for all $(x, y) \in \mathbb{R}^2$ and $f(0, 0) = 0$, then the point $(0, 0)$ must be a local maximum of f.

Example

Find the critical points of the function $f(x, y) = x^2 - y^2$

Solution: Since $\nabla f = \langle 2x, -2y\rangle$, the only solution to $\nabla f = \langle 0, 0\rangle$ is $x = 0$, $y = 0$. That is, we again obtain $(a, b) = (0, 0)$. $\triangleleft$

Characterization of local extrema.

Second derivative test.

Theorem

Let (a,b) be a critical point of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, that is, $(\nabla f)\big|_{(a,b)} = \langle 0, 0\rangle$. Assume that f has continuous second derivatives in an open disk in D with center in (a,b) and denote

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2.$$

Then, the following statements hold:

- *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
- *If $D < 0$, then $f(a,b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

Notation: The number D is called the discriminant of f at (a,b).

Characterization of local extrema.

Second derivative test.

Example

Find the local extrema of $f(x,y) = y^2 - x^2$ and determine whether they are local maximum, minimum, or saddle points.

Solution: We first find the critical points:

$$\nabla f = \langle -2x, 2y\rangle \quad \Rightarrow \quad (\nabla f)\big|_{(a,b)} = \langle 0,0\rangle \ \text{ iff } \ (a,b) = (0,0).$$

The only critical point is $(a,b) = (0,0)$.

We need to compute $D = f_{xx}(a,b)\, f_{yy}(a,b) - \big[f_{xy}(a,b)\big]^2$.

Since $f_{xx}(0,0) = -2$, $f_{yy}(0,0) = 2$, and $f_{xy}(0,0) = 0$, we get

$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0,0).$$

◁

Characterization of local extrema.
Second derivative test.

Example
Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2x^2$?

Solution: We first verify that $(0, 0)$ is a critical point of f:

$$\nabla f(x, y) = \langle 2xy^2, 2yx^2 \rangle, \quad \Rightarrow \quad \big(\nabla f\big)\big|_{(0,0)} = \langle 0, 0 \rangle,$$

therefore, $(0, 0)$ is a critical point.

Remark: The whole axes $x = 0$ and $y = 0$ are critical points of f.

We need to compute $D = f_{xx}(a, b)\, f_{yy}(a, b) - \big[f_{xy}(a, b)\big]^2$.

Since $f_{xx}(x, y) = 2y^2$, $f_{yy}(x, y) = 2x^2$, and $f_{xy}(x, y) = 4xy$,

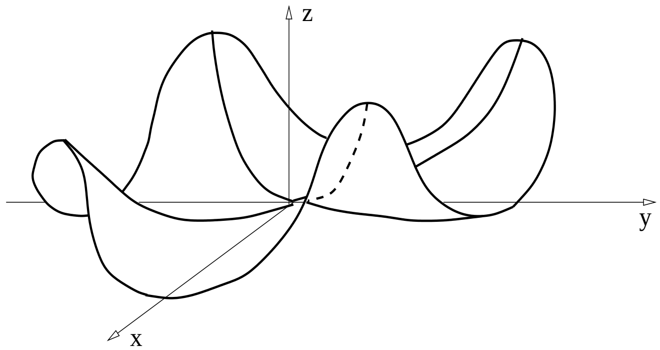
we obtain $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, and $f_{xy}(0, 0) = 0$,

hence $D = 0$ and the test is inconclusive. ◁

Characterization of local extrema.
Second derivative test.

Example
Is the point $(a, b) = (0, 0)$ a local extrema of $f(x, y) = y^2x^2$?

Solution: From the graph of $f = x^2y^2$ is simple to see that $(0, 0)$ is a local minimum: (also a global minimum.) ◁



Local and absolute extrema, saddle points (Sect. 14.7).

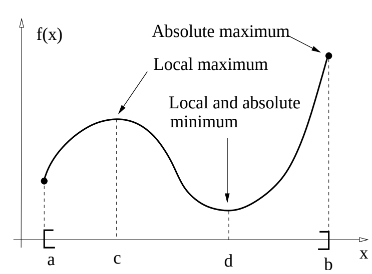
- Review: Local extrema for functions of one variable.
- Definition of local extrema.
- Characterization of local extrema.
  - First derivative test.
  - Second derivative test.
- **Absolute extrema of a function in a domain.**

Absolute extrema of a function in a domain.

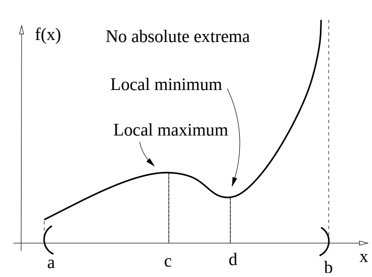
Definition

A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute maximum at the point $(a,b) \in D$ iff $f(x,y) \leqslant f(a,b)$ for all $(x,y) \in D$.
A function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has an absolute minimum at the point $(a,b) \in D$ iff $f(x,y) \geqslant f(a,b)$ for all $(x,y) \in D$.

Remark: Local extrema need not be the absolute extrema.



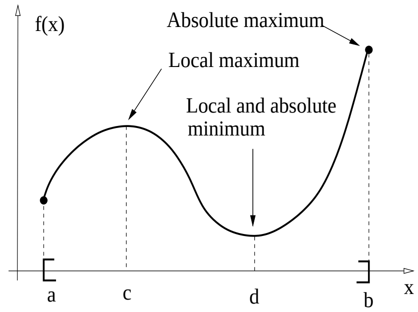
Remark: Absolute extrema may not be defined on open intervals.



Review: Functions of one variable.

Theorem

Every continuous functions $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, with $a < b \in \mathbb{R}$ always has absolute extrema.



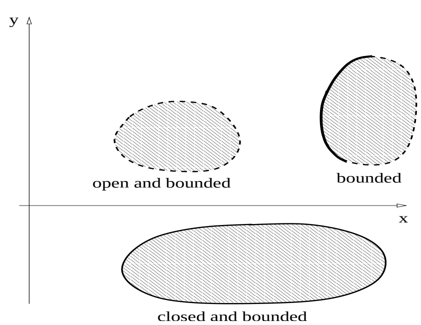
Recall:

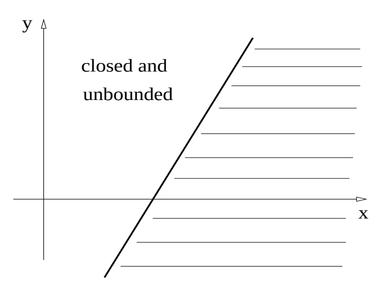
- Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$.
- The set $[a, b]$ is closed, since the boundary points belong to the set, and it is bounded, since it does not extend to infinity.

Recall: On open and closed sets in $\mathbb{R}^n$.

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





Theorem

If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in a closed and bounded set D, then f has an absolute maximum and an absolute minimum in D.

Absolute extrema on closed and bounded sets.

Problem:
Find the absolute extrema of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ in a closed and bounded set D.

Solution:

(1) Find every critical point of f in the interior of D and evaluate f at these points.

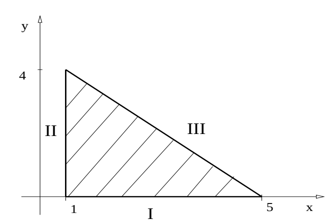
(2) Find the boundary points of D where f has local extrema, and evaluate f at these points.

(3) Look at the list of values for f found in the previous two steps.

If $f(x_0, y_0)$ is the biggest (smallest) value of f in the list above, then (x_0, y_0) is the absolute maximum (minimum) of f in D.

Absolute extrema on closed and bounded sets.

Example
Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:

(1) We find all critical points in the interior of the domain:

$$\nabla f = \langle (y-1), (x+2) \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad (x_0, y_0) = (-2, 1).$$

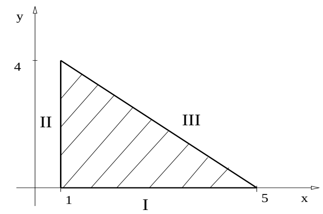
Since $(-2, 1)$ does not belong to the domain, we discard it.

(2) Three segments form the boundary of D:
Boundary I: The segment $y = 0$, $x \in [1, 5]$. We select the end points $(1, 0)$, $(5, 0)$, and we record: $f(1, 0) = 2$ and $f(5, 0) = -2$.
We look for critical point on the interior of Boundary I: Since $g(x) = f(x, 0) = 3 - x$, so $g' = -1 \neq 0$. No critical points in the interior of Boundary I.

Absolute extrema on closed and bounded sets.

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



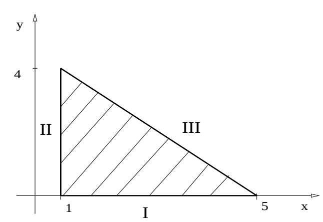
Solution: Boundary II: The segment $x = 1$, $y \in [0, 4]$. We select the end point $(1, 4)$ and we record: $f(1, 4) = 14$.
We look for critical point on the interior of Boundary II: Since $g(y) = f(1, y) = 3 + y - 1 + 2y = 2 + 3y$, so $g' = 3 \neq 0$. No critical points in the interior of Boundary II.

Boundary III: The segment $y = -x + 5$, $x \in [1, 5]$.
We look for critical point on the interior of Boundary III: Since $g(x) = f(x, -x + 5) = 3 + x(-x + 5) - x + 2(-x + 5)$. We obtain $g(x) = -x^2 + 2x + 13$, hence $g'(x) = -2x + 2 = 0$ implies $x = 1$. So, $y = 4$, and we selected the point $(1, 4)$, which was already in our list. No critical points in the interior of Boundary III.

Absolute extrema on closed and bounded sets.

Example

Find the absolute extrema of the function $f(x, y) = 3 + xy - x + 2y$ on the closed domain given in the Figure.



Solution:
(3) Our list of values is:

$$f(1, 0) = 2 \quad f(1, 4) = 14 \quad f(5, 0) = -2.$$

We conclude:

- Absolute maximum at $(1, 4)$,
- Absolute minimum at $(5, 0)$.

◁

A maximization problem with a constraint.

Example

Find the maximum volume of a closed rectangular box with a given surface area A_0.

Solution: This problem can be solved by finding the local maximum of an appropriate function f.
The function f is obtained as follows: Recall the functions volume and area of a rectangular box with vertex at $(0,0,0)$ and sides x, y and z:

$$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

Since $A(x,y,z) = A_0$, we obtain $z = \dfrac{A_0 - 2xy}{2(x+y)}$, that is

$$f(x,y) = \frac{A_0 xy - 2x^2y^2}{2(x+y)}.$$

A maximization problem with a constraint.

Example

Find the maximum volume of a closed rectangular box with a given surface area A_0.

Solution:

We must find the critical points of $f(x,y) = \dfrac{A_0 xy - 2x^2y^2}{2(x+y)}$.

$$f_x = \frac{2A_0y^2 - 4x^2y^2 - 8xy^3}{4(x+y)^2}, \quad f_y = \frac{2A_0x^2 - 4x^2y^2 - 8yx^3}{4(x+y)^2}.$$

The conditions $f_x = 0$ and $f_y = 0$ and $x \neq 0$, $y \neq 0$ imply

$$\left.\begin{aligned} A_0 &= 2x^2 + 4xy, \\ A_0 &= 2y^2 + 4xy, \end{aligned}\right\} \quad \Rightarrow \quad x = y. \quad \text{Recall} \quad z = \frac{A_0 - 2xy}{2(x+y)},$$

so, $z = \dfrac{A_0 - 2x^2}{4x} = y$. Therefore, $x_0 = y_0 = z_0 = \sqrt{A_0/6}$.