Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives and continuity.

Recall: The following result holds for single variable functions.

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ *is differentiable, then* f *is continuous.*

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Proof.

$$\begin{aligned}
\lim_{h\to 0}[f(x+h) - f(x)] &= \lim_{h\to 0}\Big[\frac{f(x+h) - f(x)}{h}\Big]\, h, \\
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That is, $\lim_{h\to 0} f(x+h) = f(x)$, so f is continuous. $\square$

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However, the claim "If $f_x(x,y)$ and $f_y(x,y)$ exist, then $f(x,y)$ is continuous" is false.

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- This Theorem is not true for the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$.
- There exist functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist but f is not continuous at (x_0, y_0).

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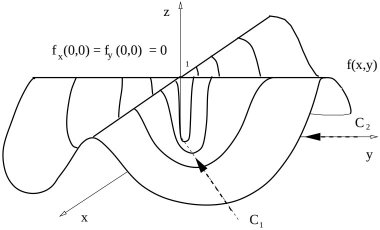

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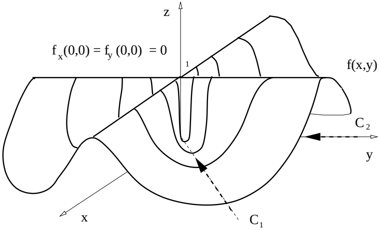

Remark: This is a bad property for a differentiable function.

Partial derivatives and continuity.

Remark: Here is a discontinuous function at $(0,0)$ having partial derivatives at $(0,0)$.

Example

(a) Show that f is not continuous at $(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

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Solution:
(a) Choosing the path $x = 0$ we see that $f(0,y) = 0$, so $\lim_{y\to 0} f(0,y) = 0$. Choosing the path $x = y$ we see that $f(x,x) = 2x^2/2x^2 = 1$, so $\lim_{x\to 0} f(x,x) = 1$.

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$$f_x(0,0) = \lim_{h\to 0} \frac{1}{h}\left[f(0+h,0) - f(0,0)\right] = \lim_{h\to 0} \frac{1}{h}\left[0-0\right] = 0.$$

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Therefore, $f_x(0,0) = f_y(0,0) = 0$. $\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
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Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Recall: A differentiable function $f : \mathbb{R} \to \mathbb{R}$ at x_0 must be approximated by a line $L(x)$ containing x_0 with slope $f'(x_0)$.

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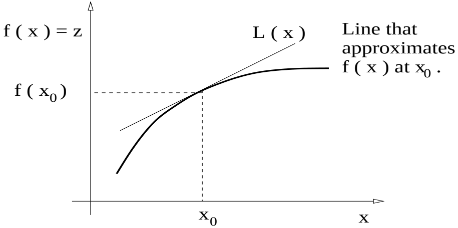

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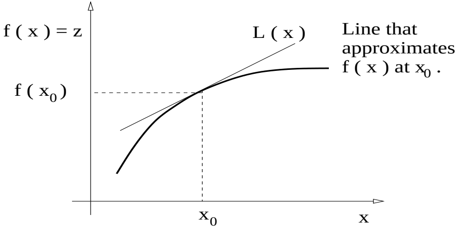

The equation of the tangent line is

$$L(x) = f'(x_0)\,(x - x_0) + f(x_0).$$

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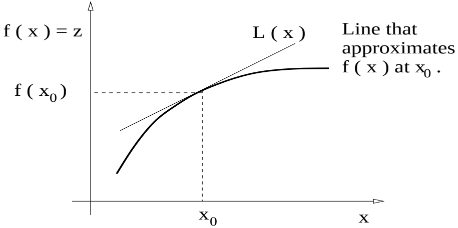

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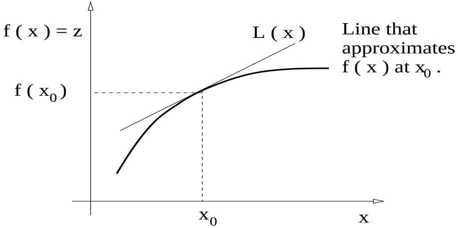

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The graph of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is approximated by a line at every point in D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Remark: The idea to define differentiable functions:
The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.

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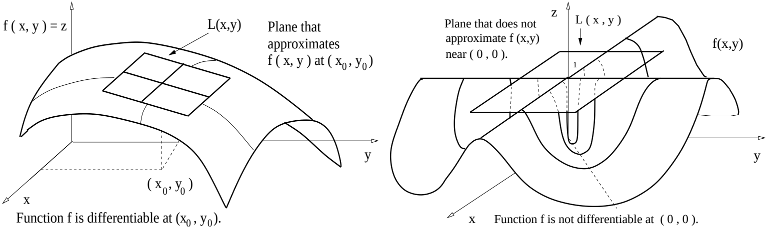

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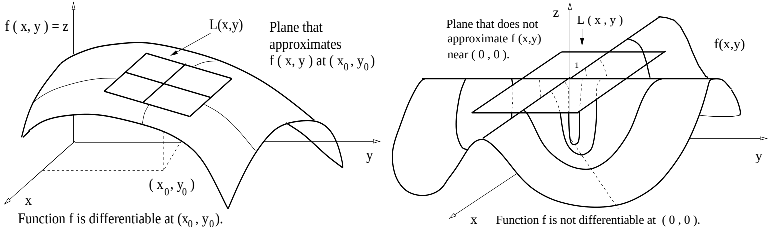

We will show next week that the equation of the plane L is

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point (x_0, y_0) in D, let L be the plane given by

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at (x_0, y_0)* iff the function f is approximated by the plane L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.
The function f is *differentiable* iff f is differentiable at every interior point of D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Remark: Recalling that the equation for the plane L is

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$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

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(Equation used in the textbook to define a differentiable function.)

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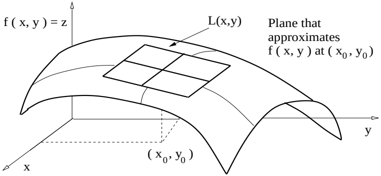

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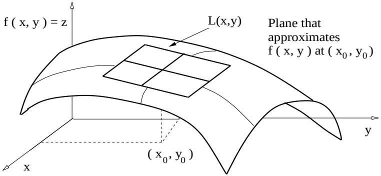

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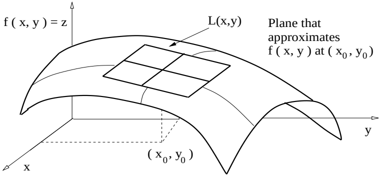

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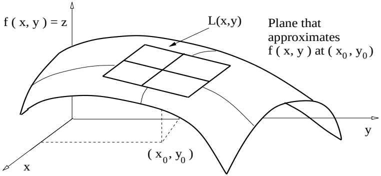

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A primer on differential equations.

Remark: A differential equation is an equation where the unknown is a function and the function together with its derivatives appear in the equation.

Example

Given a constant $k \in \mathbb{R}$, find all solutions $f : \mathbb{R} \to \mathbb{R}$ to the differential equation

$$f'(x) = k\, f(x).$$

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Therefore, $f(x) = c\, e^{kx}$. $\lhd$

A primer on differential equations.

There are three differential equations for functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4$, that appear in several physical applications.

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- The Wave equation: (Light, sound, gravitation.)

$$\partial_t^2 f = v\big(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f\big).$$

A primer on differential equations.

Example

Verify that $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation : $f_{xx} + f_{yy} + f_{zz} = 0$.

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Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $f_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$.
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$$f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5}$$

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We conclude that $f_{xx} + f_{yy} + f_{zz} = 0$. $\lhd$

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Example

Verify that the function $T(t,x) = e^{-4t}\sin(2x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.

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Therefore $T_t = T_{xx}$. $\triangleleft$

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Verify that the function $f(t, x) = (vt - x)^3$, with $v \in \mathbb{R}$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

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Therefore $f_{tt} = v^2 f_{xx}$. $\triangleleft$

A primer on differential equations.

Example

Verify that every function $f(t,x) = u(vt - x)$, with $v \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

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Therefore $f_{tt} = v^2 f_{xx}$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

The chain rule for change of coordinates in a line.

Theorem

If the functions $f : [x_0, x_1] \to \mathbb{R}$ *and* $x : [t_0, t_1] \to [x_0, x_1]$ *are differentiable, then the function* $\hat{f} : [t_0, t_1] \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(x(t)\big)$ *is differentiable and*

$$\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}\big(x(t)\big)\,\frac{dx}{dt}(t).$$

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Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{df}{dx}\,\dfrac{dx}{dt}$.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

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Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{df}{dx}\,\dfrac{dx}{dt}$.

Alternative notations are $\hat{f}'(t) = f'\big(x(t)\big)\,x'(t)$ and $\hat{f}' = f'\,x'$.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

Example

The volume V of a gas in a balloon depends on the temperature F in Fahrenheit as follows: $V(F) = k\,F^2$. Let $F(C) = (9/5)C + 32$ be the temperature in Fahrenheit corresponding to C in Celsius. Find $\hat{V}(C) = V\big(F(C)\big)$ and $\hat{V}'(C)$.

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Solution:

The function $\hat{V}$ is the composition $\hat{V}(C) = k\left(\frac{9}{5}\,C + 32\right)^2$.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

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Solution:

The function $\hat{V}$ is the composition $\hat{V}(C) = k\left(\frac{9}{5}\, C + 32\right)^2$.
Which could also be written as

$$\hat{V}(C) = k\frac{81}{25}\, C^2 + 64k\frac{9}{5}\, C + k(32)^2.$$

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

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Which could also be written as

$$\hat{V}(C) = k\frac{81}{25}\,C^2 + 64k\frac{9}{5}\,C + k(32)^2.$$

The formula $\dfrac{d\hat{V}}{dC} = \dfrac{dV}{dF}\,\dfrac{dF}{dC}$ implies $\hat{V}'(C) = 2k\left(\frac{9}{5}\,C + 32\right)\frac{9}{5}$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
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- **Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.**
  - The chain rule for functions defined on a curve in a plane.
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- A formula for implicit differentiation.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ *and* $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ *are differentiable, with* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$*, then the function* $\hat{f} : \mathbb{R} \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ *is differentiable and holds*

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}\big(\mathbf{r}(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(\mathbf{r}(t)\big)\,\frac{dy}{dt}(t).$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

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If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}\big(\mathbf{r}(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(\mathbf{r}(t)\big)\,\frac{dy}{dt}(t).$$

Notation:

The equation above is usually written as $\displaystyle \frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

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Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$.

An alternative notation is $\hat{f}' = \big(\partial_x f\big)\, x' + \big(\partial_y f\big)\, y'$.

The chain rule for functions defined on a curve in a plane.

Example

Evaluate the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(2t)\rangle$. Furthermore, compute the derivative of f along that curve.

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Evaluate the function $f(x,y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(2t)\rangle$. Furthermore, compute the derivative of f along that curve.

Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

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Evaluate the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$. Furthermore, compute the derivative of f along that curve.

Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

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Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

The derivative of f along the curve $\mathbf{r}$ is $\hat{f}'$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

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Example

Evaluate the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$. Furthermore, compute the derivative of f along that curve.

Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

The derivative of f along the curve $\mathbf{r}$ is $\hat{f}'$. The result is

$$\begin{aligned} \hat{f}'(t) &= 2x(t)\, x'(t) + 6\big(y(t)\big)^2\, y'(t), \\ &= 2x(t)\cos(t) - 12\big(y(t)\big)^2 \sin(2t) \end{aligned}$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Example

Evaluate the function $f(x,y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(2t)\rangle$. Furthermore, compute the derivative of f along that curve.

Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

The derivative of f along the curve $\mathbf{r}$ is $\hat{f}'$. The result is

$$\begin{aligned} \hat{f}'(t) &= 2x(t)\,x'(t) + 6\big(y(t)\big)^2\,y'(t), \\ &= 2x(t)\cos(t) - 12\big(y(t)\big)^2\sin(2t) \end{aligned}$$

We conclude: $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. $\triangleleft$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ *and the change of coordinate functions* $x, y : \mathbb{R}^2 \to \mathbb{R}$ *are differentiable, with* $x(t,s)$ *and* $y(t,s)$*, then the function* $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ *given by the composition* $\hat{f}(t,s) = f\big(x(t,s), y(t,s)\big)$ *is differentiable and holds*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ *and the change of coordinate functions* $x, y : \mathbb{R}^2 \to \mathbb{R}$ *are differentiable, with* $x(t,s)$ *and* $y(t,s)$*, then the function* $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ *given by the composition* $\hat{f}(t,s) = f\big(x(t,s), y(t,s)\big)$ *is differentiable and holds*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

Remark:

We denote by $f(x,y)$ are the function values in the coordinates (x,y), while we denote by $\hat{f}(t,s)$ are the function values in the coordinates (t,s).

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: The polar coordinates (r, θ) are related to Cartesian coordinates (x, y) by the formula

$$x(r, \theta) = r\cos(\theta), \quad y(r, \theta) = r\sin(\theta).$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x,y) = x^2 + 3y^2$, in Cartesian coordinates (x,y), find $\hat{f}(r,\theta)$ in polar coordinates (r,θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: The polar coordinates (r,θ) are related to Cartesian coordinates (x,y) by the formula

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta).$$

The function $\hat{f}(r,\theta) = f(x(r,\theta), y(r,\theta))$ is simple to compute,

$$\hat{f}(r,\theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta).$$

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

$$\hat{f}_r = f_x\, x_r + f_y\, y_r = 2x\cos(\theta) + 6y\sin(\theta).$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

$$\hat{f}_r = f_x\, x_r + f_y\, y_r = 2x\cos(\theta) + 6y\sin(\theta).$$

we obtain $\hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta)$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

$$\hat{f}_r = f_x\, x_r + f_y\, y_r = 2x\cos(\theta) + 6y\sin(\theta).$$

we obtain $\hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta)$. Analogously,

$$\hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta = -2xr\sin(\theta) + 6yr\cos(\theta).$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

$$\hat{f}_r = f_x\, x_r + f_y\, y_r = 2x\cos(\theta) + 6y\sin(\theta).$$

we obtain $\hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta)$. Analogously,

$$\hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta = -2xr\sin(\theta) + 6yr\cos(\theta).$$

we obtain $\hat{f}_\theta = -2r^2\cos(\theta)\sin(\theta) + 6r^2\cos(\theta)\sin(\theta)$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- **Functions of three variables,** $f : D \subset \mathbb{R}^3 \to \mathbb{R}$**.**
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ *are differentiable, with* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*, then the function* $\hat{f} : \mathbb{R} \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ *is differentiable and holds*

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt} + \frac{\partial f}{\partial z}\,\frac{dz}{dt}.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: We first compute $\hat{f}(t) = f\big(x(t), y(t), z(t)\big)$, that is,

$$\hat{f}(t) = \cos^2(t) + \sin^3(t) + 81\, t^4.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: We first compute $\hat{f}(t) = f\big(x(t), y(t), z(t)\big)$, that is,

$$\hat{f}(t) = \cos^2(t) + \sin^3(t) + 81\, t^4.$$

The derivative of f along the curve $\mathbf{r}$ is the derivative of $\hat{f}$, that is,

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: We first compute $\hat{f}(t) = f\big(x(t), y(t), z(t)\big)$, that is,

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The derivative of f along the curve $\mathbf{r}$ is the derivative of $\hat{f}$, that is,

$$\hat{f}' = f_x\,x' + f_y\,y' + f_z\,z' = -2x\sin(t) + 3y^2\cos(t) + 4z^3(3).$$

We obtain $\hat{f}' = -2\cos(t)\sin(t) + 3\sin^2(t)\cos(t) + 4(3)(3^3)t^3$. $\triangleleft$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the surface given by functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, and $z(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f\big(x(t,s), y(t,s), z(t,s)\big)$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$

$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ *and the surface given by functions* $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ *are differentiable, with* $x(t, s)$ *and* $y(t, s)$*, and* $z(t, s)$*, then the function* $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ *given by the composition* $\hat{f}(t, s) = f\big(x(t, s), y(t, s), z(t, s)\big)$ *is differentiable and holds*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x, y, z)$ the function values in the coordinates (x, y, z), while we denote by $\hat{f}(t, s)$ the function values at the surface point with coordinates (t, s).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute:

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.
The derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$ are given by $\hat{f}_t$ and $\hat{f}_s$;

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.
The derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$ are given by $\hat{f}_t$ and $\hat{f}_s$; which are given by

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \quad \hat{f}_S = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.
The derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$ are given by $\hat{f}_t$ and $\hat{f}_s$; which are given by

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \quad \hat{f}_S = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

We obtain $\hat{f}_t = 2(t + s) + 6(t^2 - s^2)(2t) + 4(t - s)$,

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.
The derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$ are given by $\hat{f}_t$ and $\hat{f}_s$; which are given by

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \quad \hat{f}_S = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

We obtain $\hat{f}_t = 2(t + s) + 6(t^2 - s^2)(2t) + 4(t - s)$,
and $\hat{f}_s = 2(t + s) + 6(t^2 - s^2)(-2s) - 4(t - s)$. $\triangleleft$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the change of coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ are differentiable, with $x(t,s,r)$, $y(t,s,r)$, and $z(t,s,r)$, then the function $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ given by the composition $\hat{f}(t,s,r) = f\big(x(t,s,r), y(t,s,r), z(t,s,r)\big)$ is differentiable and

$$
\begin{aligned}
\hat{f}_t &= f_x\, x_t + f_y\, y_t + f_z\, z_t \\
\hat{f}_s &= f_x\, x_s + f_y\, y_s + f_z\, z_s \\
\hat{f}_r &= f_x\, x_r + f_y\, y_r + f_z\, z_r .
\end{aligned}
$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the change of coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ are differentiable, with $x(t,s,r)$, $y(t,s,r)$, and $z(t,s,r)$, then the function $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ given by the composition $\hat{f}(t,s,r) = f\big(x(t,s,r), y(t,s,r), z(t,s,r)\big)$ is differentiable and

$$\begin{aligned}
\hat{f}_t &= f_x\, x_t + f_y\, y_t + f_z\, z_t \\
\hat{f}_s &= f_x\, x_s + f_y\, y_s + f_z\, z_s \\
\hat{f}_r &= f_x\, x_r + f_y\, y_r + f_z\, z_r .
\end{aligned}$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s,r)$ the function values in the coordinates (t,s,r).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its derivatives in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

The chain rule for change of coordinates in space.

Example

Given the function $f(x,y,z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x,y,z), find $\hat{f}(r,\theta,\phi)$ and its derivatives in spherical coordinates (r,θ,ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: We first compute the function
$\hat{f}(r,\theta,\phi) = f\big(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\big)$,

$$\begin{aligned}\hat{f} &= r^2\cos^2(\phi)\sin^2(\theta) + 3r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\\ &= r^2\sin^2(\theta) + 2r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\end{aligned}$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its derivatives in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: We first compute the function $\hat{f}(r, \theta, \phi) = f\big(x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi)\big)$,

$$\begin{aligned}\hat{f} &= r^2\cos^2(\phi)\sin^2(\theta) + 3r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\\ &= r^2\sin^2(\theta) + 2r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\end{aligned}$$

so we obtain

$$\hat{f} = r^2 + 2r^2\sin^2(\phi)\sin^2(\theta).$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its r-derivative in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

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$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: The r-derivative of $\hat{f}$ is given by

$$\begin{aligned}
\hat{f}_r &= 2x\,x_r + 6y\,y_r + 2z\,z_r \\
&= 2r\cos^2(\phi)\sin^2(\theta) + 6r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta) \\
&= 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).
\end{aligned}$$

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The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its r-derivative in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

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\hat{f}_r &= 2x\, x_r + 6y\, y_r + 2z\, z_r \\
&= 2r\cos^2(\phi)\sin^2(\theta) + 6r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta) \\
&= 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).
\end{aligned}$$

We conclude that $\hat{f}_r = 2r + 4r\sin^2(\phi)\sin^2(\theta)$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- **A formula for implicit differentiation.**

A formula for implicit differentiation.

Theorem

Assume that the differentiable function with values $F(x, y)$ *defines implicitly a function with values* $y(x)$ *by the equation* $F(x, y) = 0$*. If the function* $F_y \neq 0$*, then* y *is differentiable and*

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

A formula for implicit differentiation.

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Proof.

Since $\hat{F}(x) = F(x, y(x)) = 0$, then $0 = \dfrac{d\hat{F}}{dx} = F_x + F_y\, y'$.

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Since $\hat{F}(x) = F(x, y(x)) = 0$, then $0 = \dfrac{d\hat{F}}{dx} = F_x + F_y\, y'$.

We conclude that $y' = -\dfrac{F_x}{F_y}$. $\square$

A formula for implicit differentiation.

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x, y) = 0$, where $F(x, y) = x\, e^y + \cos(x - y)$.

A formula for implicit differentiation.

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The partial derivatives of function F are

$$F_x = e^y - \sin(x - y), \qquad F_y = x\, e^y + \sin(x - y).$$

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$$F_x = e^y - \sin(x - y), \qquad F_y = x\,e^y + \sin(x - y).$$

Therefore,

$$y'(x) = \frac{\big[\sin(x - y) - e^y\big]}{\big[x\,e^y + \sin(x - y)\big]}.$$

$\triangleleft$

A formula for implicit differentiation.

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Solution: (Old method.)
Since $F(x, y(x)) = x\, e^y + \cos(x - y) = 0$, then

$$e^y + x\, y'\, e^y - \sin(x - y) - \sin(x - y)(-y') = 0.$$

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Reordering terms,

$$y' \left[x\, e^y + \sin(x - y) \right] = \sin(x - y) - e^y.$$

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Reordering terms,

$$y'\,\big[x\,e^y + \sin(x-y)\big] = \sin(x-y) - e^y.$$

We conclude that: $y'(x) = \dfrac{\big[\sin(x-y) - e^y\big]}{\big[x\,e^y + \sin(x-y)\big]}$. $\triangleleft$