Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives and continuity.

Recall: The following result holds for single variable functions.

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.

Proof.

$$\lim_{h\to 0}[f(x+h) - f(x)] = \lim_{h\to 0}\Big[\frac{f(x+h) - f(x)}{h}\Big]\, h,$$

$$= f'(x) \lim_{h\to 0} h$$

$$= 0.$$

That is, $\lim_{h\to 0} f(x+h) = f(x)$, so f is continuous. □

However, the claim "If $f_x(x,y)$ and $f_y(x,y)$ exist, then $f(x,y)$ is continuous" is false.

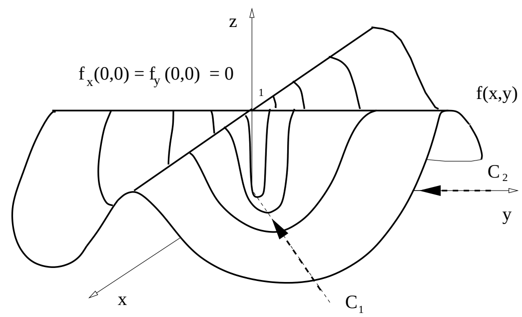
Partial derivatives and continuity.

Theorem

If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.

Remark:

- This Theorem is not true for the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$.
- There exist functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist but f is not continuous at (x_0, y_0).



Remark: This is a bad property for a differentiable function.

Partial derivatives and continuity.

Remark: Here is a discontinuous function at $(0,0)$ having partial derivatives at $(0,0)$.

Example

(a) Show that f is not continuous at $(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution:

(a) Choosing the path $x = 0$ we see that $f(0,y) = 0$, so $\lim_{y\to 0} f(0,y) = 0$. Choosing the path $x = y$ we see that $f(x,x) = 2x^2/2x^2 = 1$, so $\lim_{x\to 0} f(x,x) = 1$. The Two-Path Theorem implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example

(a) Show that f is not continuous at $(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution:
(b) The partial derivatives are defined at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{1}{h}\big[f(0+h,0) - f(0,0)\big] = \lim_{h\to 0} \frac{1}{h}\,[0-0] = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{1}{h}\big[f(0,0+h) - f(0,0)\big] = \lim_{h\to 0} \frac{1}{h}\,[0-0] = 0.$$

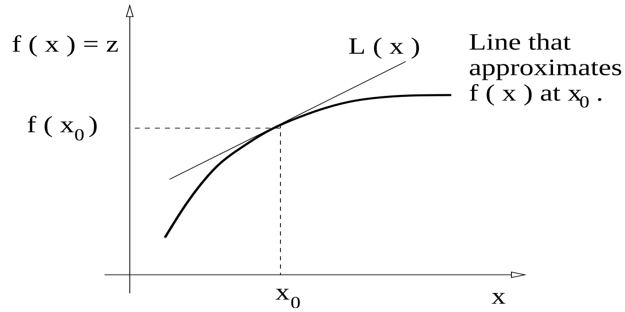
Therefore, $f_x(0,0) = f_y(0,0) = 0$. $\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- **Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.**
- Differentiability and continuity.
- A primer on differential equations.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Recall: A differentiable function $f : \mathbb{R} \to \mathbb{R}$ at x_0 must be approximated by a line $L(x)$ containing x_0 with slope $f'(x_0)$.



The equation of the tangent line is

$$L(x) = f'(x_0)\,(x - x_0) + f(x_0).$$

The function f is approximated by the line L near x_0 means

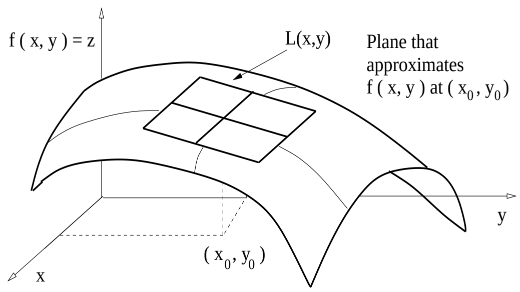
$$f(x) = L(x) + \epsilon_1\,(x - x_0)$$

with $\epsilon_1(x) \to 0$ as $x \to x_0$.

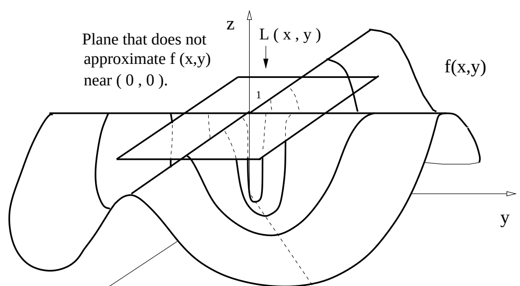
The graph of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is approximated by a line at every point in D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Remark: The idea to define differentiable functions:
The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



Function f is differentiable at (x_0 , y_0).



Function f is not differentiable at (0 , 0).

We will show next week that the equation of the plane L is

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point (x_0, y_0) in D, let L be the plane given by

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at* (x_0, y_0) iff the function f is approximated by the plane L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.
The function f is *differentiable* iff f is differentiable at every interior point of D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Remark: Recalling that the equation for the plane L is

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0),$$

an equivalent expression for f being differentiable,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0),$$

is the following: Denote $z = f(x, y)$ and $z_0 = f(x_0, y_0)$, and introduce the increments

$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

then, the equation above is

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

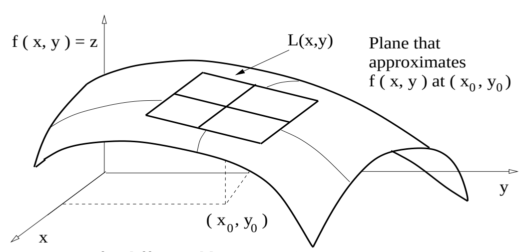
(Equation used in the textbook to define a differentiable function.)

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- **Differentiability and continuity.**
- A primer on differential equations.

Differentiability and continuity.

Recall: The graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



Function f is differentiable at (x_0, y_0).

Remark: A simple sufficient condition on a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ guarantees that f is differentiable:

Theorem
If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

Theorem
If a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then f is continuous.

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives and continuity.
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- **A primer on differential equations.**

A primer on differential equations.

Remark: A differential equation is an equation where the unknown is a function and the function together with its derivatives appear in the equation.

Example

Given a constant $k \in \mathbb{R}$, find all solutions $f : \mathbb{R} \to \mathbb{R}$ to the differential equation

$$f'(x) = k\, f(x).$$

Solution: Multiply the equation above $f'(x) - kf(x) = 0$ by e^{-kx}, that is, $f'(x)\, e^{-kx} - f(x)\, ke^{-kx} = 0$.

The left-hand side is a total derivative, $\big[f(x)\, e^{-kx}\big]' = 0$.

The solution of the equation above is $f(x)e^{-kx} = c$, with $c \in \mathbb{R}$.

Therefore, $f(x) = c\, e^{kx}$. $\triangleleft$

A primer on differential equations.

There are three differential equations for functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4$, that appear in several physical applications.

- The Laplace equation: (Gravitation, electrostatics.)

$$\partial_x^2 f + \partial_y^2 f + \partial_z^2 f = 0.$$

- The Heat equation: (Heat propagation, diffusion.)

$$\partial_t f = k\big(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f\big).$$

- The Wave equation: (Light, sound, gravitation.)

$$\partial_t^2 f = v\big(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f\big).$$

A primer on differential equations.

Example

Verify that $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation : $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution: Recall: $f_x = -x/\big(x^2 + y^2 + z^2\big)^{3/2}$. Then,

$$f_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2}\,\frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $f_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$.
Analogously, $f_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$, and $f_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$. Then,

$$f_{xx} + f_{yy} + f_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0.$$

We conclude that $f_{xx} + f_{yy} + f_{zz} = 0$. $\triangleleft$

A primer on differential equations.

Example

Verify that the function $T(t,x) = e^{-4t}\sin(2x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.

Solution: We first compute T_t,

$$T_t = -4e^{-t}\sin(2x).$$

Now compute T_{xx},

$$T_x = 2e^{-t}\cos(2x) \quad \Rightarrow \quad T_{xx} = -4e^{-t}\sin(2x)$$

Therefore $T_t = T_{xx}$. $\triangleleft$

A primer on differential equations.

Example

Verify that the function $f(t,x) = (vt - x)^3$, with $v \in \mathbb{R}$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

Solution: We first compute f_{tt},

$$f_t = 3v(vt - x)^2 \quad \Rightarrow \quad f_{tt} = 6v^2(vt - x).$$

Now compute f_{xx},

$$f_x = -3(vt - x)^2 \quad \Rightarrow \quad f_{xx} = 6(vt - x).$$

Therefore $f_{tt} = v^2 f_{xx}$. $\triangleleft$

A primer on differential equations.

Example

Verify that every function $f(t,x) = u(vt - x)$, with $v \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

Solution: We first compute f_{tt},

$$f_t = v\, u'(vt - x) \quad \Rightarrow \quad f_{tt} = v^2\, u''(vt - x).$$

Now compute f_{xx},

$$f_x = -u'(vt - x)^2 \quad \Rightarrow \quad f_{xx} = u''(vt - x).$$

Therefore $f_{tt} = v^2 f_{xx}$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

The chain rule for change of coordinates in a line.

Theorem

If the functions $f : [x_0, x_1] \to \mathbb{R}$ and $x : [t_0, t_1] \to [x_0, x_1]$ are differentiable, then the function $\hat{f} : [t_0, t_1] \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(x(t)\big)$ is differentiable and

$$\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}\big(x(t)\big)\,\frac{dx}{dt}(t).$$

Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{df}{dx}\,\dfrac{dx}{dt}$.

Alternative notations are $\hat{f}'(t) = f'\big(x(t)\big)\,x'(t)$ and $\hat{f}' = f'\,x'$.

Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.

Example

The volume V of a gas in a balloon depends on the temperature F in Fahrenheit as follows: $V(F) = k\,F^2$. Let $F(C) = (9/5)C + 32$ be the temperature in Fahrenheit corresponding to C in Celsius. Find $\hat{V}(C) = V\big(F(C)\big)$ and $\hat{V}'(C)$.

Solution:

The function $\hat{V}$ is the composition $\hat{V}(C) = k\left(\frac{9}{5}\,C + 32\right)^2$.

Which could also be written as

$$\hat{V}(C) = k\frac{81}{25}\,C^2 + 64k\frac{9}{5}\,C + k(32)^2.$$

The formula $\dfrac{d\hat{V}}{dC} = \dfrac{dV}{dF}\,\dfrac{dF}{dC}$ implies $\hat{V}'(C) = 2k\Big(\frac{9}{5}\,C + 32\Big)\,\frac{9}{5}$. $\lhd$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- **Functions of two variables,** $f : D \subset \mathbb{R}^2 \to \mathbb{R}$**.**
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}\big(\mathbf{r}(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(\mathbf{r}(t)\big)\,\frac{dy}{dt}(t).$$

Notation:

The equation above is usually written as $\dfrac{d\hat{f}}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

An alternative notation is $\hat{f}' = \big(\partial_x f\big)\,x' + \big(\partial_y f\big)\,y'$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Example

Evaluate the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \sin(t), \cos(2t)\rangle$. Furthermore, compute the derivative of f along that curve.

Solution: The function f along the curve $\mathbf{r}(t)$ is denoted as $\hat{f}(t) = f(x(t), y(t))$. The result is $\hat{f}(t) = \sin^2(t) + 2\cos^3(2t)$.

The derivative of f along the curve $\mathbf{r}$ is $\hat{f}'$. The result is

$$\begin{aligned}\hat{f}'(t) &= 2x(t)\,x'(t) + 6\big(y(t)\big)^2\,y'(t),\\ &= 2x(t)\cos(t) - 12\big(y(t)\big)^2\sin(2t)\end{aligned}$$

We conclude: $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. $\triangleleft$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and the change of coordinate functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t, s)$ and $y(t, s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t, s) = f\big(x(t, s), y(t, s)\big)$ is differentiable and holds

$$\begin{aligned}\hat{f}_t &= f_x\,x_t + f_y\,y_t\\ \hat{f}_s &= f_x\,x_s + f_y\,y_s.\end{aligned}$$

Remark:

We denote by $f(x, y)$ are the function values in the coordinates (x, y), while we denote by $\hat{f}(t, s)$ are the function values in the coordinates (t, s).

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: The polar coordinates (r, θ) are related to Cartesian coordinates (x, y) by the formula

$$x(r, \theta) = r\cos(\theta), \quad y(r, \theta) = r\sin(\theta).$$

The function $\hat{f}(r, \theta) = f(x(r, \theta), y(r, \theta))$ is simple to compute,

$$\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta).$$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find $\hat{f}(r, \theta)$ in polar coordinates (r, θ). Furthermore, compute $\hat{f}_r$ and $\hat{f}_\theta$.

Solution: Recall: $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$.

Compute the derivatives of $\hat{f}(r, \theta) = r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$.

$$\hat{f}_r = f_x\, x_r + f_y\, y_r = 2x\cos(\theta) + 6y\sin(\theta).$$

we obtain $\hat{f}_r = 2r\cos^2(\theta) + 6r\sin^2(\theta)$. Analogously,

$$\hat{f}_\theta = f_x\, x_\theta + f_y\, y_\theta = -2xr\sin(\theta) + 6yr\cos(\theta).$$

we obtain $\hat{f}_\theta = -2r^2\cos(\theta)\sin(\theta) + 6r^2\cos(\theta)\sin(\theta)$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- **Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.**
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: We first compute $\hat{f}(t) = f\big(x(t), y(t), z(t)\big)$, that is,

$$\hat{f}(t) = \cos^2(t) + \sin^3(t) + 81\, t^4.$$

The derivative of f along the curve $\mathbf{r}$ is the derivative of $\hat{f}$, that is,

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).$$

We obtain $\hat{f}' = -2\cos(t)\sin(t) + 3\sin^2(t)\cos(t) + 4(3)(3^3)t^3$. $\lhd$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the surface given by functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, with $x(t,s)$ and $y(t,s)$, and $z(t,s)$, then the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ given by the composition $\hat{f}(t,s) = f\big(x(t,s), y(t,s), z(t,s)\big)$ is differentiable and holds

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s)$ the function values at the surface point with coordinates (t,s).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find $\hat{f}(t, s)$, the values of f and its derivatives on the surface given by $x(t, s) = t + s$, $y(t, s) = t^2 - s^2$, $z(t, s) = t - s$.

Solution: The function $\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$ is simple to compute: $\hat{f}(t, s) = (t + s)^2 + 3(t^2 - s^2)^2 + 2(t - s)^2$.
The derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$ are given by $\hat{f}_t$ and $\hat{f}_s$; which are given by

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t \qquad \hat{f}_S = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

We obtain $\hat{f}_t = 2(t + s) + 6(t^2 - s^2)(2t) + 4(t - s)$,
and $\hat{f}_s = 2(t + s) + 6(t^2 - s^2)(-2s) - 4(t - s)$. $\triangleleft$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the change of coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ are differentiable, with $x(t, s, r)$, $y(t, s, r)$, and $z(t, s, r)$, then the function $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ given by the composition $\hat{f}(t, s, r) = f\big(x(t, s, r), y(t, s, r), z(t, s, r)\big)$ is differentiable and

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s$$
$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r.$$

Remark:

We denote by $f(x, y, z)$ the function values in the coordinates (x, y, z), while we denote by $\hat{f}(t, s, r)$ the function values in the coordinates (t, s, r).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its derivatives in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: We first compute the function $\hat{f}(r, \theta, \phi) = f\big(x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi)\big)$,

$$\begin{aligned}\hat{f} &= r^2\cos^2(\phi)\sin^2(\theta) + 3r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\\ &= r^2\sin^2(\theta) + 2r^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\theta)\end{aligned}$$

so we obtain

$$\hat{f} = r^2 + 2r^2\sin^2(\phi)\sin^2(\theta).$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

The chain rule for change of coordinates in space.

Example

Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates (x, y, z), find $\hat{f}(r, \theta, \phi)$ and its r-derivative in spherical coordinates (r, θ, ϕ), where

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

Solution: The r-derivative of $\hat{f}$ is given by

$$\begin{aligned}\hat{f}_r &= 2x\,x_r + 6y\,y_r + 2z\,z_r\\ &= 2r\cos^2(\phi)\sin^2(\theta) + 6r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta)\\ &= 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).\end{aligned}$$

We conclude that $\hat{f}_r = 2r + 4r\sin^2(\phi)\sin^2(\theta)$. $\triangleleft$

The chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - The chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in a plane.
  - The chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - The chain rule for functions defined on a curve in space.
  - The chain rule for functions defined on surfaces in space.
  - The chain rule for change of coordinates in space.
- **A formula for implicit differentiation.**

A formula for implicit differentiation.

Theorem

Assume that the differentiable function with values $F(x, y)$ defines implicitly a function with values $y(x)$ by the equation $F(x, y) = 0$. If the function $F_y \neq 0$, then y is differentiable and

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Proof.

Since $\hat{F}(x) = F(x, y(x)) = 0$, then $0 = \dfrac{d\hat{F}}{dx} = F_x + F_y\, y'$.

We conclude that $y' = -\dfrac{F_x}{F_y}$. $\square$

A formula for implicit differentiation.

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x, y) = 0$, where $F(x, y) = x\,e^y + \cos(x - y)$.

Solution:

The partial derivatives of function F are

$$F_x = e^y - \sin(x - y), \qquad F_y = x\,e^y + \sin(x - y).$$

Therefore,

$$y'(x) = \frac{\big[\sin(x - y) - e^y\big]}{\big[x\,e^y + \sin(x - y)\big]}.$$

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A formula for implicit differentiation.

Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x, y) = 0$, where $F(x, y) = x\,e^y + \cos(x - y)$.

Solution: (Old method.)

Since $F(x, y(x)) = x\,e^y + \cos(x - y) = 0$, then

$$e^y + x\,y'\,e^y - \sin(x - y) - \sin(x - y)(-y') = 0.$$

Reordering terms,

$$y'\,\big[x\,e^y + \sin(x - y)\big] = \sin(x - y) - e^y.$$

We conclude that: $y'(x) = \dfrac{\big[\sin(x - y) - e^y\big]}{\big[x\,e^y + \sin(x - y)\big]}.$ ◁