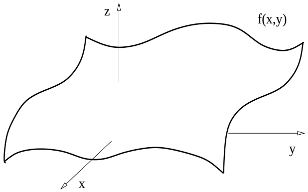
Scalar functions of several variables (Sect. 14.1).

- Functions of several variables.
- On open, closed sets.
- Functions of two variables:
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Level surfaces.



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Definition

A *scalar function of n variables* is a function $f : D \subset \mathbb{R}^n \to R \subset \mathbb{R}$, where $n \in \mathbb{N}$, the set D is called the *domain* of the function, and the set R is called the *range* of the function.

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$$f : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \to f(x, y).$$

- A vector function on the plane is a function

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \qquad t \to \mathbf{r}(t) = \langle x(t), y(t) \rangle.$$

Functions of several variables.

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.

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- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of this room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.

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$\triangleleft$

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Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is defined.

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The values of the function f are non-negative, that is, $f(x, y) = x^2 + y^2 \geqslant 0$ for all $(x, y) \in D$. Therefore, the range space is $R = [0, \infty)$. $\triangleleft$

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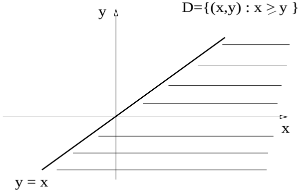

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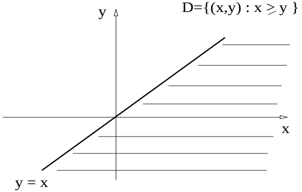

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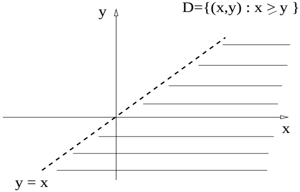

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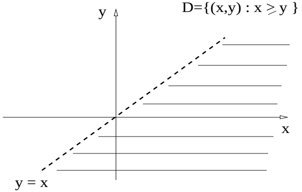

The values of the function f are positive, that is, $f(x, y) = 1/\sqrt{x - y} > 0$ for all $(x, y) \in D$. Therefore, the range space is $R = (0, \infty)$. $\triangleleft$

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On open and closed sets in $\mathbb{R}^n$.

We first generalize from $\mathbb{R}^3$ to $\mathbb{R}^n$ the definition of a ball of radius r centered at $\hat{P}$.

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Definition

A set $B_r(\hat{P}) \subset \mathbb{R}^n$, with $n \in \mathbb{N}$ and $r > 0$, is a *ball of radius r centered at $\hat{P} = (\hat{x}_1, \cdots, \hat{x}_n)$* iff

$$B_r(\hat{P}) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \ : \ (x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2 < r^2\}.$$

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Remark: A ball $B_r(\hat{P})$ contains the points inside a sphere of radius r centered at $\hat{P}$ without the points of the sphere.

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Definition

A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called an *interior point* iff there is a ball $B_r(P) \subset S$. A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called a *boundary point* iff every ball $B_r(P)$ contains points in S and points outside S. The *boundary* of a set S is the set of all boundary points of S.

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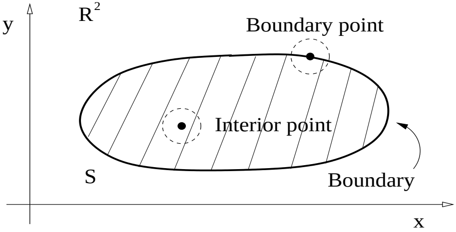

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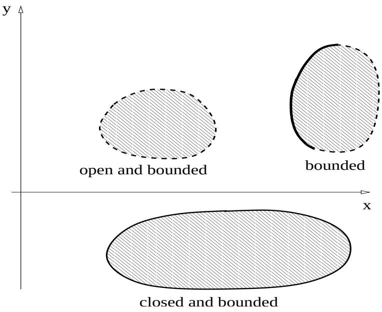

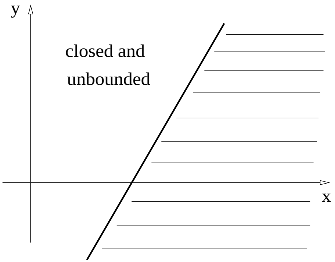

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Find and describe the maximum domain of the function $f(x, y) = \ln(x - y^2)$.

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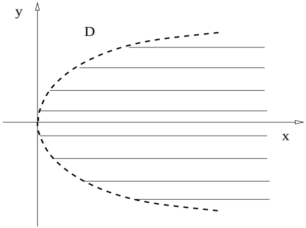

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The *graph* of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points (x, y, z) in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$. The graph of a function f is also called the surface $z = f(x, y)$.

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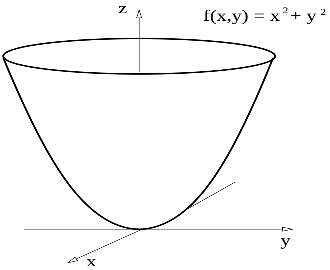

Level curves, contour curves.

Definition

The *level curves* of a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ are the curves in the domain $D \subset \mathbb{R}^2$ of f solutions of the equation $f(x, y) = k$, where $k \in R$ is a constant in the range of f.
The *contour curves* of function f are the curves in $\mathbb{R}^3$ given by the intersection of the graph of f with horizontal planes $z = k$, where $k \in R$ is a constant in the range of f.

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Find and draw few level curves and contour curves for the function $f(x, y) = x^2 + y^2$.

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The level curves are solutions of the equation $x^2 + y^2 = k$ with $k \geqslant 0$. These curves are circles of radius $\sqrt{k}$ in $D = \mathbb{R}^2$.

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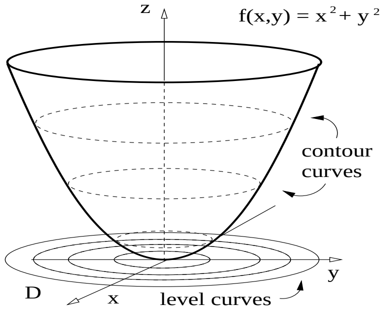

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$f(x, y) = \dfrac{1}{1 + x^2 + y^2}$.

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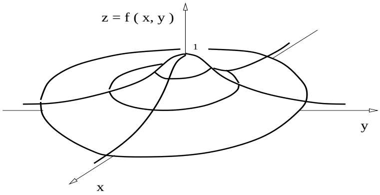
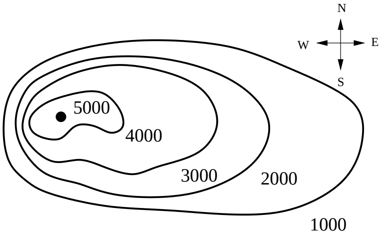


Level curves, contour curves.

Example

Given the topographic map in the figure, which way do you choose to the summit?



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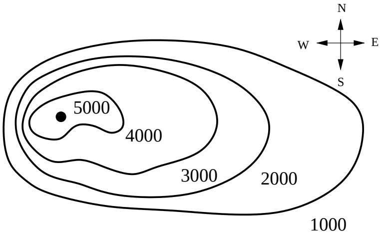

Solution:

From the east.

Level curves, contour curves.

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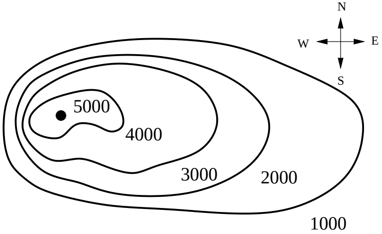

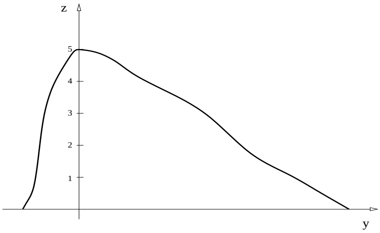

Solution:

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Scalar functions of three variables.

Definition

The *graph* of a scalar function of three variables, $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$, is the set of points in $\mathbb{R}^4$ of the form $(x, y, z, f(x, y, z))$ for every $(x, y, z) \in D$.

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The graph a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ requires four space dimensions. We cannot picture such graph.

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Definition

The *level surfaces* of a function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$ are the surfaces in the domain $D \subset \mathbb{R}^3$ of f solutions of the equation $f(x, y, z) = k$, where $k \in R$ is a constant in the range of f.

Scalar functions of three variables.

Example

Draw one level surface of the function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$
$f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

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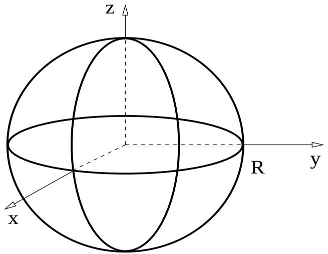
$f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

Solution: The domain of f is $D = \mathbb{R}^3$ and its range is $R = (0, \infty)$. Writing $k = 1/R^2$, the level surfaces $f(x, y, z) = k$ are spheres $x^2 + y^2 + z^2 = R^2$. $\lhd$



Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
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The limit of functions of several variables.

Definition

The function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has the number $L \in \mathbb{R}$ as *limit at the point* $\hat{P} \in \mathbb{R}^n$, denoted as $\lim_{P \to \hat{P}} f(P) = L$, iff the following holds: For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $|P - \hat{P}| < \delta$ then $|f(P) - L| < \epsilon$.

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Remarks:

- In Cartesian coordinates $P = (x_1, \cdots, x_n)$, $\hat{P} = (\hat{x}_1, \cdots, \hat{x}_n)$. Then, $|P - \hat{P}|$ is the distance between points in $\mathbb{R}^n$,

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- $|f(P) - L| \in \mathbb{R}$ is the absolute value of real numbers.

The limit of functions $f : \mathbb{R}^2 \to \mathbb{R}$.

The function with values $f(x, y)$ has the number L as limit at the point $P_0 = (x_0, y_0)$ iff holds: For all points $P = (x, y)$ near $P_0 = (x_0, y_0)$ the value of $f(x, y)$ differs little from L.

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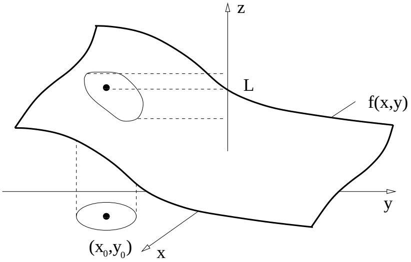
We denote it as follows:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$



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Computing limits by definition usually is not easy.

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Use the definition of limit to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^2}{x^2+y^2}$.

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So, if L exists, it must be $L = 0$.
Fix any number $\epsilon > 0$. Given that ϵ, find a number $\delta > 0$ such that

$$\sqrt{(x-0)^2 + (y-0)^2} < \delta \quad \Rightarrow \quad \Big| \frac{2yx^2}{x^2+y^2} - 0 \Big| < \epsilon.$$

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We conclude that $L = 0$. $\lhd$

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Properties of limits of functions.

Theorem

If $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$, *with* $n \in \mathbb{N}$, *and* $\lim_{P \to \hat{P}} f(P) = L$, $\lim_{P \to \hat{P}} g(P) = M$, *then the following statements hold:*

1. $\lim_{P \to \hat{P}} f(P) \pm g(P) = L \pm M$;
2. *If* $k \in \mathbb{R}$, *then* $\lim_{P \to \hat{P}} kf(P) = kL$;
3. $\lim_{P \to \hat{P}} f(P)\, g(P) = LM$;
4. *If* $M \neq 0$, *then* $\displaystyle \lim_{P \to \hat{P}} \left(\frac{f(P)}{g(P)} \right) = \frac{L}{M}$.
5. *If* $k \in \mathbb{Z}$ *and* $s \in \mathbb{N}$, *then* $\displaystyle \lim_{P \to \hat{P}} \big[f(P) \big]^{r/s} = L^{r/s}$.

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Remark:

The Theorem above implies that: If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a rational function $f = R/S$, (quotient of two polynomials), with $S(\hat{P}) \neq 0$, then $\lim_{P\to\hat{P}} f(P) = f(\hat{P})$.

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Limits of R/S at $\hat{P}$ where $S(\hat{P}) \neq 0$ are simple to find.

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Compute $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 + 2y - x}{\sqrt{x - y}}$.

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Compute $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 + 2y - x}{\sqrt{x - y}}$.

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that is,

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◁

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

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Continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

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A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, is called *continuous at* $\hat{P} \in D$ iff holds $\lim_{P \to \hat{P}} f(P) = f(\hat{P})$.

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- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is *continuous* iff f is continuous at every point in D.
- Continuous functions have graphs without holes or jumps.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

- Polynomial functions are continuous in $\mathbb{R}^n$.

  For example, $P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2$.

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- Composition of continuous functions are continuous.

  For example, $f(x, y) = \cos(x^2 + y^2)$.

Continuous functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Example

Compute $\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2)$.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

Compute $\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2)$.

Solution:

The function $f(x, y) = \cos(x^2 + y^2)$ is continuous for all $(x, y) \in \mathbb{R}^2$.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

Compute $\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2)$.

Solution:

The function $f(x, y) = \cos(x^2 + y^2)$ is continuous for all $(x, y) \in \mathbb{R}^2$. Therefore,

$$\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2) = \cos(\pi + 0),$$

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$\lhd$

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Two-path test for the non-existence of limits.

Theorem

If a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has two different limits along to different paths as P approaches $\hat{P}$, then $\lim_{P \to \hat{P}} f(P)$ does not exist.

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Remark: Consider the case $f : D \subset \mathbb{R}^2 \to \mathbb{R}$:

If

- $f(x, y) \to L_1$ along a path C_1 as $(x, y) \to (x_0, y_0)$,
- $f(x, y) \to L_2$ along a path C_2 as $(x, y) \to (x_0, y_0)$,
- $L_1 \neq L_2$,

then $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ does not exist.

Two-path test for the non-existence of limits.

Theorem

If a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has two different limits along to different paths as P approaches $\hat{P}$, then $\lim_{P \to \hat{P}} f(P)$ does not exist.

Remark: Consider the case $f : D \subset \mathbb{R}^2 \to \mathbb{R}$:

If

- $f(x, y) \to L_1$ along a path C_1 as $(x, y) \to (x_0, y_0)$,
- $f(x, y) \to L_2$ along a path C_2 as $(x, y) \to (x_0, y_0)$,
- $L_1 \neq L_2$,

then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

When side limits do not agree, the limit does not exist.

Two-path test for the non-existence of limits.

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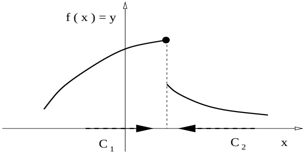

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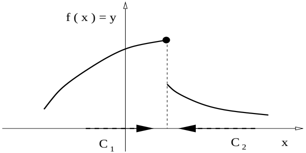

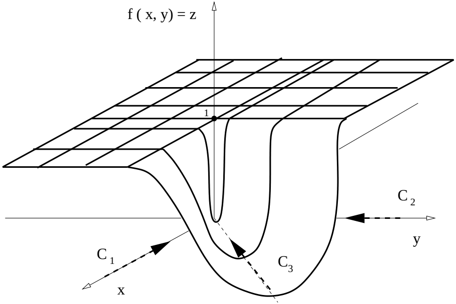

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Example

Compute $\lim_{(x,y)\to(0,0)} \dfrac{3x^2}{x^2+2y^2}$.

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We try to show that the limit above does not exist.
If path C_1 is the x-axis, $(y=0)$, then,

$$f(x,0) = \frac{3x^2}{x^2} = 3, \quad \Rightarrow \quad \lim_{(x,0)\to(0,0)} f(x,0) = 3.$$

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If path C_2 is the y-axis, $(x=0)$, then,

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Therefore, $\lim_{(x,y)\to(0,0)} \dfrac{3x^2}{x^2+2y^2}$ does not exist. $\triangleleft$

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Remark:

In the example above one could compute the limit for arbitrary lines, that is, C_m given by $y = mx$, with m a constant.

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That is,

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The limits along these paths are:

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This agrees what we concluded: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2 + 2y^2}$ does not exist.

The sandwich test for the existence of limits.

Theorem

If functions $f, g, h : D \subset \mathbb{R}^n \to \mathbb{R}$, *with* $n \in \mathbb{N}$, *satisfy:*

(a) $g(P) \leqslant f(P) \leqslant h(P)$ *for all* P *near* $\hat{P} \in D$;

(b) $\lim_{P \to \hat{P}} g(P) = L = \lim_{P \to \hat{P}} h(P)$;

then $\lim_{P \to \hat{P}} f(P) = L$.

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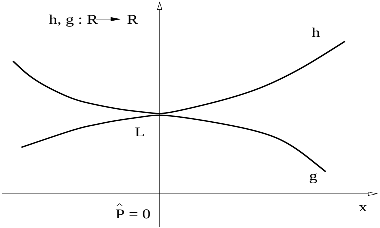

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Solution: Notice: $\dfrac{x^2}{x^2 + y^2} \leqslant 1$, for all $(x, y) \neq (0, 0)$.

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Solution: Notice: $\dfrac{x^2}{x^2 + y^2} \leqslant 1$, for all $(x, y) \neq (0, 0)$.

So, $\left|\dfrac{x^2 y}{x^2 + y^2}\right| \leqslant |y|$, for all $(x, y) \neq (0, 0)$. Hence,

$$-|y| \leqslant \frac{x^2 y}{x^2 + y^2} \leqslant |y|.$$

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Since $\lim_{y\to 0} |y| = 0$, the Sandwich Theorem with $g = -|y|$, $h = |y|$, implies

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$

$\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Next class:

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, the *partial derivative of $f(x, y)$ with respect to x* at a point $(x, y) \in D$ is given by

$$f_x(x, y) = \lim_{h \to 0} \frac{1}{h}\big[f(x + h, y) - f(x, y)\big].$$

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- To compute $f_y(x, y)$ derivate $f(x, y)$ keeping x constant.

Computing $f_x(x, y)$ at (x_0, y_0).

- Evaluate the function f at $y = y_0$. The result is a single variable function $f(x, y_0)$.
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Geometrical meaning of partial derivatives.

$f_x(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x, y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zx-plane.

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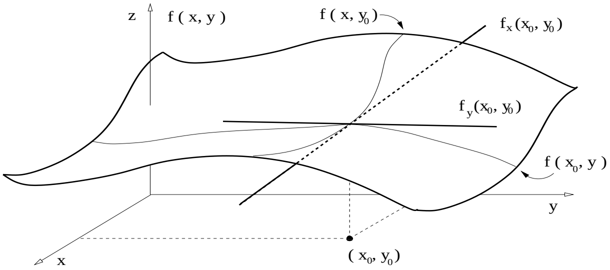

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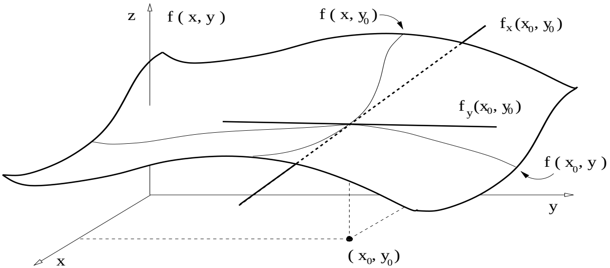

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Partial derivatives can be computed on any point in D.

Example

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Solution:

$$f_x(x, y) = \frac{2(x + 2y) - (2x - y)}{(x + 2y)^2} \quad \Rightarrow \quad f_x(x, y) = \frac{5y}{(x + 2y)^2}.$$

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$\triangleleft$

The derivative of a function is a new function.

Recall: The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is itself a function.

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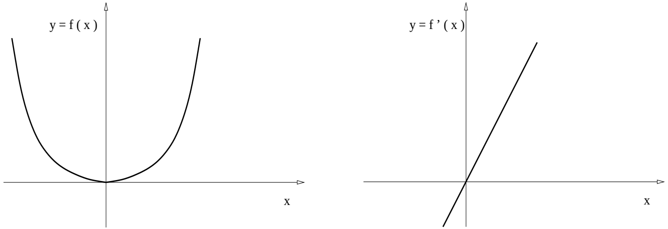

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Example

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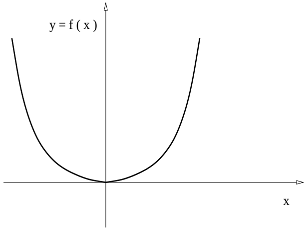

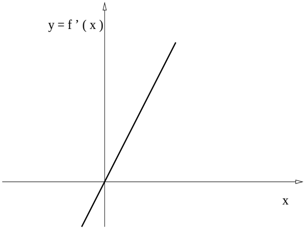

The same statement is true for partial derivatives.

The partial derivatives of a function are new functions.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$, the *functions partial derivatives of* $f(x, y)$ are denoted by $f_x(x, y)$ and $f_y(x, y)$, and they are given by the expressions

$$f_x(x, y) = \lim_{h \to 0} \frac{1}{h} \left[f(x + h, y) - f(x, y) \right],$$
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Notation:

Partial derivatives of $f(x,y)$ are denoted in several ways:

$$f_x(x,y), \qquad \frac{\partial f}{\partial x}(x,y), \qquad \partial_x f(x,y).$$

$$f_y(x,y), \qquad \frac{\partial f}{\partial y}(x,y), \qquad \partial_y f(x,y)$$

The partial derivatives of a paraboloid are planes

Example

Find the functions partial derivatives of $f(x, y) = x^2 + y^2$.

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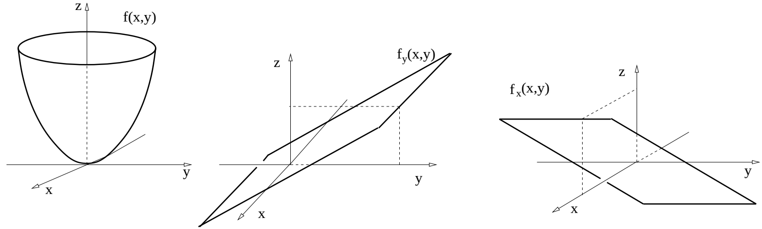

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Example

Find the partial derivatives of $f(x, y) = x^2 + \frac{y^2}{4}$.

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Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- **Higher-order partial derivatives.**
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Higher-order partial derivatives.

Remark:

Higher derivatives of a function are partial derivatives of its partial derivatives.

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The second partial derivatives of $f(x, y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

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Notation: f_{xx}, $\dfrac{\partial^2 f}{\partial x^2}$, $\partial_{xx} f$, and also f_{xy}, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\partial_{xy} f$.

Higher-order partial derivatives.

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Find all second order derivatives of the function
$f(x, y) = x^3 e^{2y} + 3y$.

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Find all second order derivatives of the function $f(x, y) = x^3 e^{2y} + 3y$.

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$$f_x(x, y) = 3x^2 e^{2y}, \qquad f_y(x, y) = 2x^3 e^{2y} + 3.$$

Higher-order partial derivatives.

Example

Find all second order derivatives of the function $f(x, y) = x^3 e^{2y} + 3y$.

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$$f_x(x, y) = 3x^2 e^{2y}, \qquad f_y(x, y) = 2x^3 e^{2y} + 3.$$

$$f_{xx}(x, y) = 6xe^{2y}, \qquad f_{yy}(x, y) = 4x^3 e^{2y}.$$

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$$f_{xy} = 6x^2 e^{2y}, \qquad f_{yx} = 6x^2 e^{2y}.$$

◁

Partial derivatives and differentiability (Sect. 14.3).

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Higher-order partial derivatives sometimes commute.

Theorem

If the partial derivatives f_x, f_y, f_{xy} and f_{yx} of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ exist and all are continuous functions, then holds

$$f_{xy} = f_{yx}.$$

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Example

Find f_{xy} and f_{yx} for $f(x, y) = \cos(xy)$.

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Find f_{xy} and f_{yx} for $f(x, y) = \cos(xy)$.

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Examples of implicit partial differentiation.

Remark: Implicit differentiation rules for partial derivatives are similar to those for functions of one variable.

Example

Find $\partial_x z(x,y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

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Find $\partial_x z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

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Compute $\partial_x z$ as a function of x, y and $z(x, y)$, as follows,

$$(\partial_x z)\Big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\Big] = -yz,$$

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that is, $(\partial_x z) = -\dfrac{yz}{\Big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\Big]}.$ ◁

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Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
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- **Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.**

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, the *partial derivative of* $f(x_1, \cdots, x_n)$ *with respect to* x_i, with $i = 1, \cdots, n$, at a point $(x_1, \cdots, x_n) \in D$ is given by

$$f_{x_i} = \lim_{h \to 0} \frac{1}{h} \big[f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n) \big].$$

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Notation: $f_{x_i}, \qquad f_i, \qquad \dfrac{\partial f}{\partial x_i}, \qquad \partial_{x_i} f, \qquad \partial_i f.$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Example

Compute all first partial derivatives of the function $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Example

Compute all first partial derivatives of the function $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

Solution:

$$\phi_x = -\frac{1}{2}\frac{2x}{(x^2 + y^2 + z^2)^{3/2}} \quad \Rightarrow \quad \phi_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}.$$

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Analogously, the other partial derivatives are given by

$$\phi_y = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \phi_z = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}.$$

$\triangleleft$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Example

Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation : $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$.

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Denote $r = \sqrt{x^2 + y^2 + z^2}$,

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Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $\phi_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$.

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

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$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5}$$

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Verify that $\phi(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies the Laplace equation : $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$.

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Analogously, $\phi_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$, and $\phi_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$. Then,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5}.$$

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Solution: Recall: $\phi_x = -x/\big(x^2 + y^2 + z^2\big)^{3/2}$. Then,

$$\phi_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2}\,\frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $\phi_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$.
Analogously, $\phi_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$, and $\phi_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$. Then,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5}.$$

We conclude that $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$. $\triangleleft$