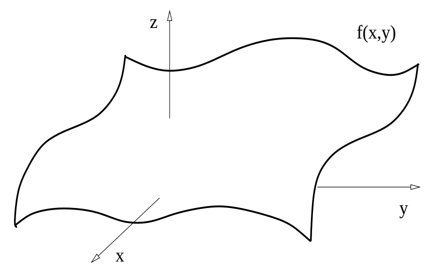
Scalar functions of several variables (Sect. 14.1).

- Functions of several variables.
- On open, closed sets.
- Functions of two variables:
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Level surfaces.



Scalar functions of several variables.

Definition
A *scalar function of n variables* is a function $f : D \subset \mathbb{R}^n \to R \subset \mathbb{R}$, where $n \in \mathbb{N}$, the set D is called the *domain* of the function, and the set R is called the *range* of the function.

Remark:
Comparison between $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$.

- A scalar function of two variables is a function

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \to f(x, y).$$

- A vector function on the plane is a function

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \qquad t \to \mathbf{r}(t) = \langle x(t), y(t) \rangle.$$

Functions of several variables.

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.
- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of this room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.
- Another example of a scalar function of three variables is the speed of the air in the room.
- An example of a vector-valued function of three variables, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$, is the velocity of the air in the room.

$\triangleleft$

Scalar functions of several variables.

Example

Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is defined.

Solution: The function $f(x, y) = x^2 + y^2$ is defined for all points $(x, y) \in \mathbb{R}^2$, therefore, $D = \mathbb{R}^2$.

The values of the function f are non-negative, that is, $f(x, y) = x^2 + y^2 \geqslant 0$ for all $(x, y) \in D$. Therefore, the range space is $R = [0, \infty)$. $\triangleleft$

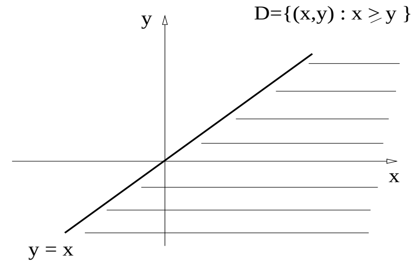
Scalar functions of several variables.

Example

Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x,y) = \sqrt{x-y}$ is defined.

Solution: The function $f(x,y) = \sqrt{x-y}$ is defined for points $(x,y) \in \mathbb{R}^2$ such that $x - y \geqslant 0$. Therefore,

$$D = \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant y\}.$$



The values of the function f are non-negative, that is, $f(x,y) = \sqrt{x-y} \geqslant 0$ for all $(x,y) \in D$. Therefore, the range space is $R = [0, \infty)$. ◁

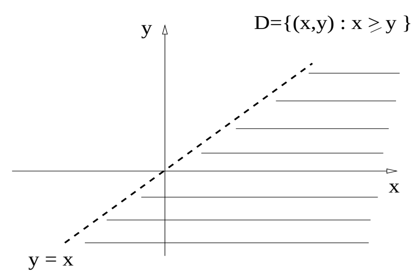
Scalar functions of several variables.

Example

Find the maximum domain D and range R sets where the function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ given by $f(x,y) = 1/\sqrt{x-y}$ is defined.

Solution: The function $f(x,y) = 1/\sqrt{x-y}$ is defined for points $(x,y) \in \mathbb{R}^2$ such that $x - y > 0$. Therefore,

$$D = \{(x,y) \in \mathbb{R}^2 \ : \ x > y\}.$$



The values of the function f are positive, that is, $f(x,y) = 1/\sqrt{x-y} > 0$ for all $(x,y) \in D$. Therefore, the range space is $R = (0, \infty)$. ◁

Scalar functions of several variables (Sect. 14.1).

- Functions of several variables.
- **On open, closed sets.**
- Functions of two variables:
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Level surfaces.

On open and closed sets in $\mathbb{R}^n$.

We first generalize from $\mathbb{R}^3$ to $\mathbb{R}^n$ the definition of a ball of radius r centered at $\hat{P}$.

Definition
A set $B_r(\hat{P}) \subset \mathbb{R}^n$, with $n \in \mathbb{N}$ and $r > 0$, is a *ball of radius r centered at* $\hat{P} = (\hat{x}_1, \cdots, \hat{x}_n)$ iff

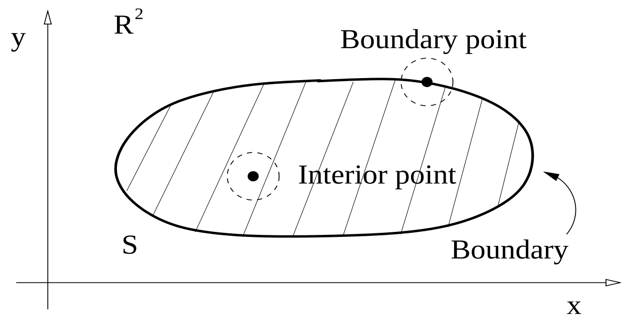
$$B_r(\hat{P}) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \ : \ (x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2 < r^2\}.$$

Remark: A ball $B_r(\hat{P})$ contains the points inside a sphere of radius r centered at $\hat{P}$ without the points of the sphere.

On open and closed sets in $\mathbb{R}^n$.

Definition

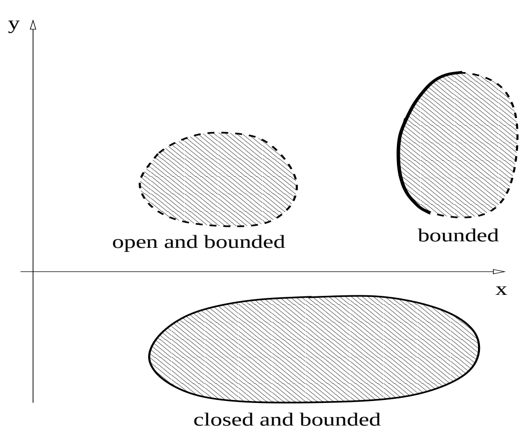
A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called an *interior point* iff there is a ball $B_r(P) \subset S$. A point $P \in S \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is called a *boundary point* iff every ball $B_r(P)$ contains points in S and points outside S. The *boundary* of a set S is the set of all boundary points of S.

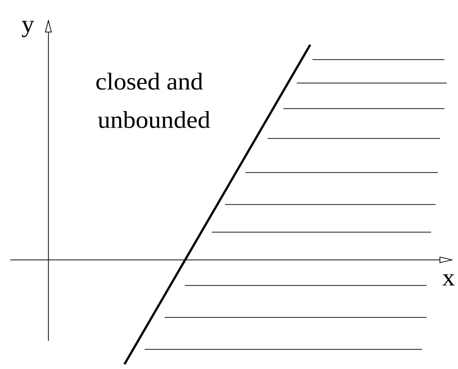


On open and closed sets in $\mathbb{R}^n$.

Definition

A set $S \in \mathbb{R}^n$, with $n \in \mathbb{N}$, is called *open* iff every point in S is an interior point. The set S is called *closed* iff S contains its boundary. A set S is called *bounded* iff S is contained in ball, otherwise S is called *unbounded*.





On open and closed sets in $\mathbb{R}^n$.

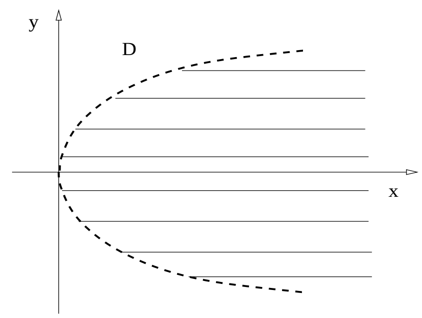
Example

Find and describe the maximum domain of the function $f(x, y) = \ln(x - y^2)$.

Solution: The maximum domain of f is the set

$$D = \{(x, y) \in \mathbb{R}^2 \; : \; x > y^2\}.$$

D is an open, unbounded set. $\triangleleft$



Scalar functions of several variables (Sect. 14.1).

- Functions of several variables.
- On open, closed sets.
- **Functions of two variables:**
  - Graph of the function.
  - Level curves, contour curves.
- Functions of three variables.
  - Level surfaces.

The graph of a function of two variables is a surface in $\mathbb{R}^3$.

Definition

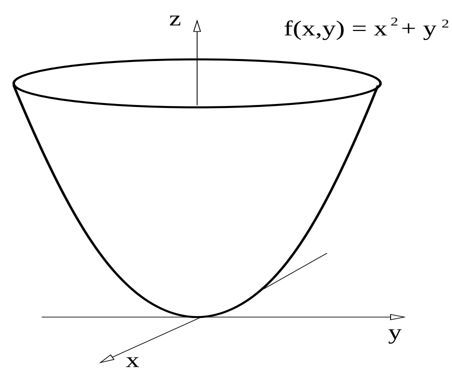
The *graph* of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points (x, y, z) in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$. The graph of a function f is also called the surface $z = f(x, y)$.

Example

Draw the graph of $f(x, y) = x^2 + y^2$.

Solution: The graph of f is the surface $z = x^2 + y^2$. This is a paraboloid along the z axis.

◁



Level curves, contour curves.

Definition

The *level curves* of a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$ are the curves in the domain $D \subset \mathbb{R}^2$ of f solutions of the equation $f(x, y) = k$, where $k \in R$ is a constant in the range of f.

The *contour curves* of function f are the curves in $\mathbb{R}^3$ given by the intersection of the graph of f with horizontal planes $z = k$, where $k \in R$ is a constant in the range of f.

Curves of constant f in $D \subset \mathbb{R}^2$ are called level curves.

Curves of constant f in $\mathbb{R}^3$ are called contour curves.

Level curves, contour curves.

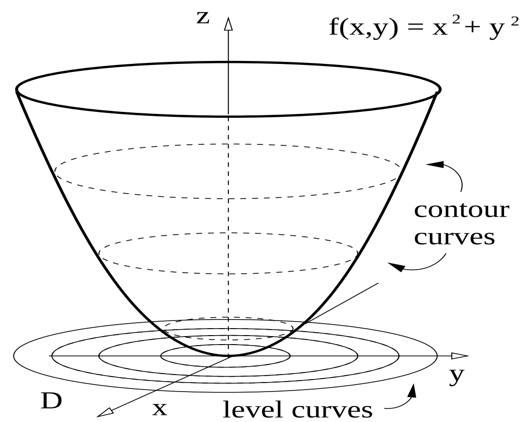
Example

Find and draw few level curves and contour curves for the function $f(x, y) = x^2 + y^2$.

Solution:

The level curves are solutions of the equation $x^2 + y^2 = k$ with $k \geqslant 0$. These curves are circles of radius $\sqrt{k}$ in $D = \mathbb{R}^2$.

The contour curves are the circles $\{(x, y, z) \; : \; x^2+y^2 = k, \; z = k\}$.



Level curves, contour curves.

Example

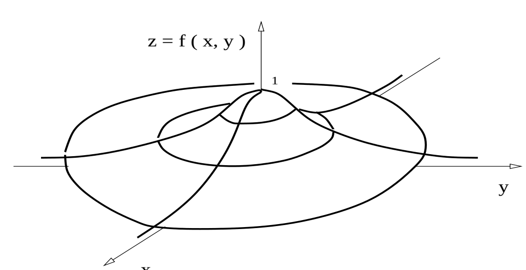
Find the maximum domain, range of, and graph the function $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$.

Solution:

Since the denominator never vanishes, hence $D = \mathbb{R}^2$.

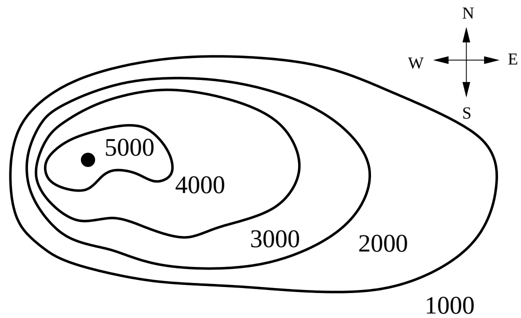
Since $0 < \dfrac{1}{1 + x^2 + y^2} \leqslant 1$, the range of f is $R = (0, 1]$.

The contour curves are circles on horizontal planes in $(0, 1]$. $\triangleleft$



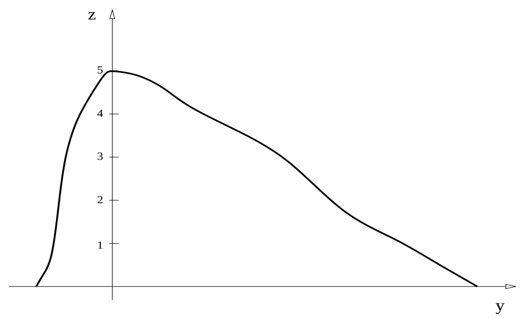
Level curves, contour curves.

Example
Given the topographic map in the figure, which way do you choose to the summit?



Solution:

From the east. ◁



Scalar functions of several variables (Sect. 14.1).

- Functions of several variables.
- On open, closed sets.
- Functions of two variables:
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  - Level curves, contour curves.
- **Functions of three variables.**
  - Level surfaces.

Scalar functions of three variables.

Definition

The *graph* of a scalar function of three variables, $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$, is the set of points in $\mathbb{R}^4$ of the form $(x, y, z, f(x, y, z))$ for every $(x, y, z) \in D$.

Remark:

The graph a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ requires four space dimensions. We cannot picture such graph.

Definition

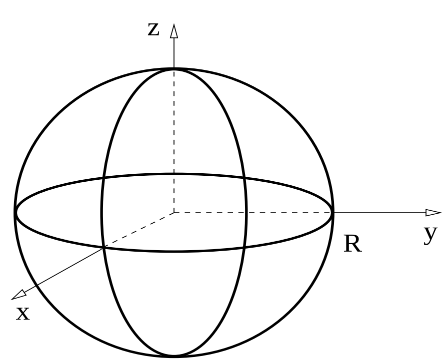
The *level surfaces* of a function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$ are the surfaces in the domain $D \subset \mathbb{R}^3$ of f solutions of the equation $f(x, y, z) = k$, where $k \in R$ is a constant in the range of f.

Scalar functions of three variables.

Example

Draw one level surface of the function $f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$
$f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

Solution: The domain of f is $D = \mathbb{R}^3$ and its range is $R = (0, \infty)$. Writing $k = 1/R^2$, the level surfaces $f(x, y, z) = k$ are spheres $x^2 + y^2 + z^2 = R^2$. $\triangleleft$



Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

The limit of functions of several variables.

Definition
The function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has the number $L \in \mathbb{R}$ as *limit at the point* $\hat{P} \in \mathbb{R}^n$, denoted as $\lim_{P \to \hat{P}} f(P) = L$, iff the following holds: For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $|P - \hat{P}| < \delta$ then $|f(P) - L| < \epsilon$.

Remarks:

- In Cartesian coordinates $P = (x_1, \cdots, x_n)$, $\hat{P} = (\hat{x}_1, \cdots, \hat{x}_n)$. Then, $|P - \hat{P}|$ is the distance between points in $\mathbb{R}^n$,

$$|P - \hat{P}| = |\overrightarrow{P\hat{P}}| = \sqrt{(x_1 - \hat{x}_1)^2 + \cdots + (x_n - \hat{x}_n)^2}.$$

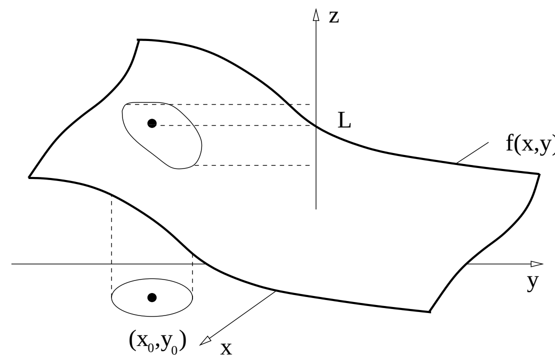
- $|f(P) - L| \in \mathbb{R}$ is the absolute value of real numbers.

The limit of functions $f : \mathbb{R}^2 \to \mathbb{R}$.

The function with values $f(x, y)$ has the number L as limit at the point $P_0 = (x_0, y_0)$ iff holds: For all points $P = (x, y)$ near $P_0 = (x_0, y_0)$ the value of $f(x, y)$ differs little from L.

We denote it as follows:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$



Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- **Example: Computing a limit by the definition.**
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Computing limits by definition usually is not easy.

Example

Use the definition of limit to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^2}{x^2+y^2}$.

Solution: The function above is not defined at $(0,0)$.
First: Guess what the limit L is.
Along the line $x = 0$ the function above vanishes for all $y \neq 0$.
So, if L exists, it must be $L = 0$.
Fix any number $\epsilon > 0$. Given that ϵ, find a number $\delta > 0$ such that

$$\sqrt{(x-0)^2 + (y-0)^2} < \delta \quad \Rightarrow \quad \Big| \frac{2yx^2}{x^2+y^2} - 0 \Big| < \epsilon.$$

Example

Use the definition of limit to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^2}{x^2+y^2}$.

Solution: Given any $\epsilon > 0$, find a number $\delta > 0$ such that

$$\sqrt{x^2+y^2} < \delta \quad \Rightarrow \quad \Big| \frac{2yx^2}{x^2+y^2} \Big| < \epsilon.$$

Recall: $x^2 \leqslant x^2 + y^2$, that is, $\dfrac{x^2}{x^2+y^2} \leqslant 1$. Then

$$\Big| \frac{2yx^2}{x^2+y^2} \Big| = \frac{2|y|x^2}{x^2+y^2} \leqslant 2|y| = 2\sqrt{y^2} \leqslant 2\sqrt{x^2+y^2}.$$

Choose $\delta = \epsilon/2$. If $\sqrt{x^2+y^2} < \delta$, then $\Big| \dfrac{2yx^2}{x^2+y^2} \Big| < 2\delta = \epsilon$.

We conclude that $L = 0$. $\triangleleft$

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- **Properties of limits of functions.**
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Properties of limits of functions.

Theorem

If $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, and $\lim_{P\to\hat{P}} f(P) = L$, $\lim_{P\to\hat{P}} g(P) = M$, then the following statements hold:

1. $\lim_{P\to\hat{P}} f(P) \pm g(P) = L \pm M$;
2. *If* $k \in \mathbb{R}$, *then* $\lim_{P\to\hat{P}} kf(P) = kL$;
3. $\lim_{P\to\hat{P}} f(P)\, g(P) = LM$;
4. *If* $M \neq 0$, *then* $\displaystyle\lim_{P\to\hat{P}} \Big(\frac{f(P)}{g(P)}\Big) = \frac{L}{M}$.
5. *If* $k \in \mathbb{Z}$ *and* $s \in \mathbb{N}$, *then* $\displaystyle\lim_{P\to\hat{P}} \big[f(P)\big]^{r/s} = L^{r/s}$.

Remark:

The Theorem above implies that: If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a rational function $f = R/S$, (quotient of two polynomials), with $S(\hat{P}) \neq 0$, then $\lim_{P\to\hat{P}} f(P) = f(\hat{P})$.

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- **Examples: Computing limits of simple functions.**
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Limits of R/S at $\hat{P}$ where $S(\hat{P}) \neq 0$ are simple to find.

Example

Compute $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 + 2y - x}{\sqrt{x-y}}$.

Solution: The function above is a rational function in x and y and its denominator does not vanish at $(1, 2)$. Therefore

$$\lim_{(x,y)\to(1,2)} \frac{x^2 + 2y - x}{\sqrt{x-y}} = \frac{1 + 2(2) - 1}{\sqrt{1-2}},$$

that is,

$$\lim_{(x,y)\to(1,2)} \frac{x^2 + 2y - x}{\sqrt{x-y}} = 4.$$

◁

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- **Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.**
- Computing limits of non-continuous functions:
  - Two-path test for the non-existence of limits.
  - The sandwich test for the existence of limits.

Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition
A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, is called *continuous at* $\hat{P} \in D$ iff holds $\lim_{P \to \hat{P}} f(P) = f(\hat{P})$.

Remarks:

- The definition above says:
  (a) $f(\hat{P})$ is defined;
  (b) $\lim_{P \to \hat{P}} f(P) = L$ exists;
  (c) $L = f(\hat{P})$.
- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is *continuous* iff f is continuous at every point in D.
- Continuous functions have graphs without holes or jumps.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

- Polynomial functions are continuous in $\mathbb{R}^n$.

  For example, $P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2$.

- Rational functions $f = R/S$ are continuous on their domain.

  For example, $f(x, y) = \dfrac{x^2 + 3y - x^2 y^2 + y^4}{x^2 - y^2}$, with $x \neq \pm y$.

- Composition of continuous functions are continuous.

  For example, $f(x, y) = \cos(x^2 + y^2)$.

Continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

Compute $\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2)$.

Solution:

The function $f(x, y) = \cos(x^2 + y^2)$ is continuous for all $(x, y) \in \mathbb{R}^2$. Therefore,

$$\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2) = \cos(\pi + 0),$$

that is,

$$\lim_{(x,y)\to(\sqrt{\pi},0)} \cos(x^2 + y^2) = -1.$$

$\triangleleft$

Limits and continuity for $f : \mathbb{R}^n \to \mathbb{R}$ (Sect. 14.2).

- The limit of functions $f : \mathbb{R}^n \to \mathbb{R}$.
- Example: Computing a limit by the definition.
- Properties of limits of functions.
- Examples: Computing limits of simple functions.
- Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.
- **Computing limits of non-continuous functions:**
  - Two-path test for the **non-existence** of limits.
  - The sandwich test for the **existence** of limits.

Two-path test for the non-existence of limits.

Theorem

If a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, has two different limits along to different paths as P approaches $\hat{P}$, then $\lim_{P \to \hat{P}} f(P)$ does not exist.

Remark: Consider the case $f : D \subset \mathbb{R}^2 \to \mathbb{R}$:

If

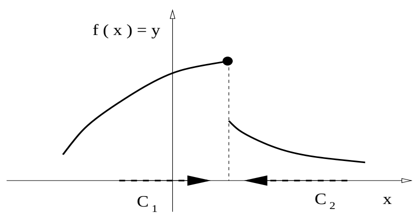
- $f(x, y) \to L_1$ along a path C_1 as $(x, y) \to (x_0, y_0)$,
- $f(x, y) \to L_2$ along a path C_2 as $(x, y) \to (x_0, y_0)$,
- $L_1 \neq L_2$,

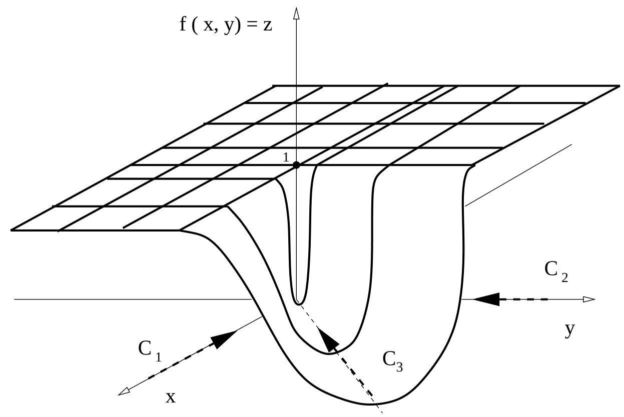
then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

When side limits do not agree, the limit does not exist.

Two-path test for the non-existence of limits.

When side limits do not agree, the limit does not exist.





Two-path test for the non-existence of limits.

Example

Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}$.

Solution: $f(x,y) = (3x^2)/(x^2+2y^2)$ is not continuous at $(0,0)$.
We try to show that the limit above does not exist.
If path C_1 is the x-axis, $(y=0)$, then,

$$f(x,0) = \frac{3x^2}{x^2} = 3, \quad \Rightarrow \quad \lim_{(x,0)\to(0,0)} f(x,0) = 3.$$

If path C_2 is the y-axis, $(x=0)$, then,

$$f(0,y) = 0, \quad \Rightarrow \quad \lim_{(0,y)\to(0,0)} f(0,y) = 0.$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}$ does not exist. ◁

Two-path test for the non-existence of limits.

Remark:

In the example above one could compute the limit for arbitrary lines, that is, C_m given by $y = mx$, with m a constant.

That is,

$$f(x, mx) = \frac{3x^2}{x^2 + 2m^2x^2} = \frac{3}{1 + 2m^2}.$$

The limits along these paths are:

$$\lim_{(x,mx)\to(0,0)} f(x, mx) = \frac{3}{1 + 2m^2}$$

which are different for each value of m.

This agrees what we concluded: $\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2 + 2y^2}$ does not exist.

The sandwich test for the existence of limits.

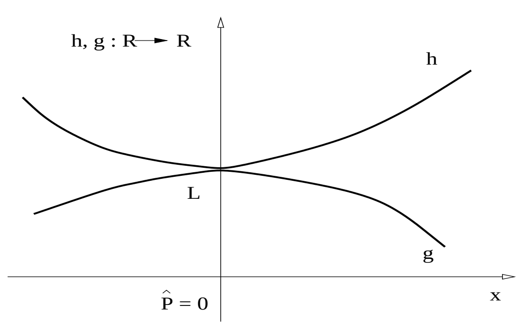
Theorem

If functions $f, g, h : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, satisfy:

(a) $g(P) \leqslant f(P) \leqslant h(P)$ *for all* P *near* $\hat{P} \in D$;

(b) $\lim_{P\to\hat{P}} g(P) = L = \lim_{P\to\hat{P}} h(P)$;

then $\lim_{P\to\hat{P}} f(P) = L$.



The sandwich test for the existence of limits.

Example

Compute $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2}$.

Solution: $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ is not continuous at $(0,0)$.
The Two-Path Theorem does not prove non-existence of the limit.
Consider paths C_m given by $y = mx$, with $m \in \mathbb{R}$. Then

$$f(x,mx) = \frac{x^2 mx}{x^2 + m^2x^2} = \frac{mx}{1+m^2},$$

which implies $\lim_{(x,mx)\to(0,0)} f(x,mx) = 0, \quad \forall\, m \in \mathbb{R}$.
We cannot conclude that the limit does not exist.
We cannot conclude that the limit exists.

The sandwich test for the existence of limits.

Example

Compute $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2}$.

Solution: Notice: $\dfrac{x^2}{x^2+y^2} \leqslant 1$, for all $(x,y) \neq (0,0)$.

So, $\left|\dfrac{x^2 y}{x^2+y^2}\right| \leqslant |y|$, for all $(x,y) \neq (0,0)$. Hence,

$$-|y| \leqslant \frac{x^2 y}{x^2+y^2} \leqslant |y|.$$

Since $\lim_{y\to 0} |y| = 0$, the Sandwich Theorem with $g = -|y|$, $h = |y|$, implies

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$

$\lhd$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Next class:

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, the *partial derivative of $f(x, y)$ with respect to x* at a point $(x, y) \in D$ is given by

$$f_x(x, y) = \lim_{h \to 0} \frac{1}{h}\big[f(x + h, y) - f(x, y)\big].$$

The *partial derivative of $f(x, y)$ with respect to y* at a point $(x, y) \in D$ is given by

$$f_y(x, y) = \lim_{h \to 0} \frac{1}{h}\big[f(x, y + h) - f(x, y)\big].$$

Remark:

- To compute $f_x(x, y)$ derivate $f(x, y)$ keeping y constant.
- To compute $f_y(x, y)$ derivate $f(x, y)$ keeping x constant.

Computing $f_x(x, y)$ at (x_0, y_0).

- Evaluate the function f at $y = y_0$. The result is a single variable function $f(x, y_0)$.
- Compute the derivative of $f(x, y_0)$ and evaluate it at $x = x_0$.
- The result is $f_x(x_0, y_0)$.

Example

Find $f_x(1, 3)$ for $f(x, y) = x^2 + y^2/4$.

Solution:

- $f(x, 3) = x^2 + 9/4$;
- $f_x(x, 3) = 2x$;
- $f_x(1, 3) = 2$. $\lhd$

To compute $f_x(x, y)$ derivate $f(x, y)$ keeping y constant.

Computing $f_y(x, y)$ at (x_0, y_0).

- Evaluate the function f at $x = x_0$. The result is a single variable function $f(x_0, y)$.
- Compute the derivative of $f(x_0, y)$ and evaluate it at $y = y_0$.
- The result is $f_y(x_0, y_0)$.

Example

Find $f_y(1, 3)$ for $f(x, y) = x^2 + y^2/4$.

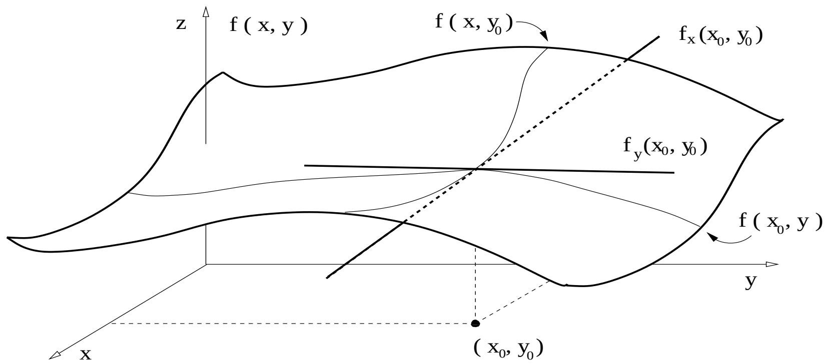
Solution:

- $f(1, y) = 1 + y^2/4$;
- $f_y(1, y) = y/2$;
- $f_y(1, 3) = 3/2$. $\lhd$

To compute $f_y(x, y)$ derivate $f(x, y)$ keeping x constant.

Geometrical meaning of partial derivatives.

$f_x(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x,y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zx-plane.



$f_y(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x,y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zy-plane.

Partial derivatives can be computed on any point in D.

Example

Find the partial derivatives of $f(x,y) = \dfrac{2x - y}{x + 2y}$.

Solution:

$$f_x(x,y) = \frac{2(x+2y) - (2x - y)}{(x+2y)^2} \quad \Rightarrow \quad f_x(x,y) = \frac{5y}{(x+2y)^2}.$$

$$f_y(x,y) = \frac{(-1)(x+2y) - (2x-y)(2)}{(x+2y)^2} \Rightarrow f_y(x,y) = -\frac{5x}{(x+2y)^2}.$$

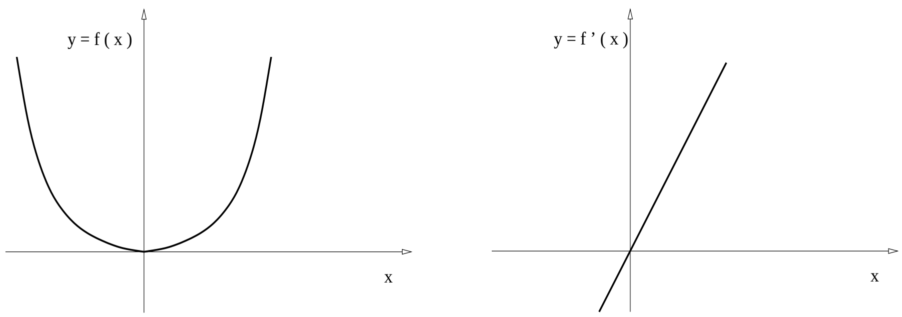
$\lhd$

The derivative of a function is a new function.

Recall: The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is itself a function.

Example

The derivative of function $f(x) = x^2$ at an arbitrary point x is the function $f'(x) = 2x$.



The same statement is true for partial derivatives.

The partial derivatives of a function are new functions.

Definition

Given a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$, the *functions partial derivatives of* $f(x,y)$ are denoted by $f_x(x,y)$ and $f_y(x,y)$, and they are given by the expressions

$$f_x(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x+h,y) - f(x,y)\right],$$

$$f_y(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x,y+h) - f(x,y)\right].$$

Notation:

Partial derivatives of $f(x,y)$ are denoted in several ways:

$$f_x(x,y), \qquad \frac{\partial f}{\partial x}(x,y), \qquad \partial_x f(x,y).$$

$$f_y(x,y), \qquad \frac{\partial f}{\partial y}(x,y), \qquad \partial_y f(x,y)$$

The partial derivatives of a paraboloid are planes

Example

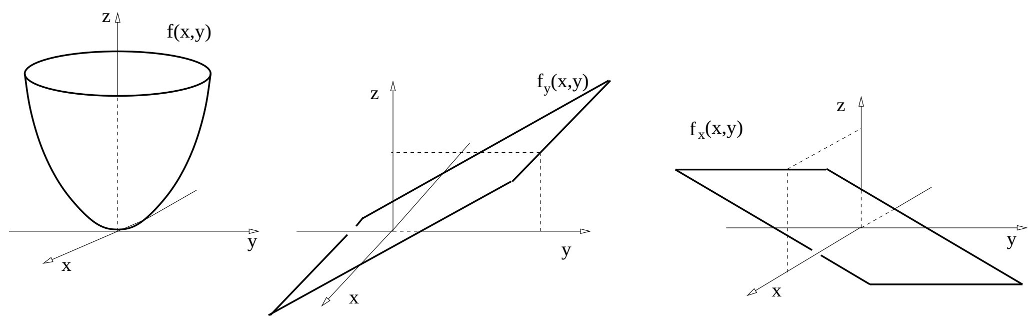
Find the functions partial derivatives of $f(x, y) = x^2 + y^2$.

Solution:

$f_x(x, y) = 2x \quad \Rightarrow \quad f_x(x, y) = 2x.$

$f_y(x, y) = 0 + 2y \quad \Rightarrow \quad f_y(x, y) = 2y.$ $\triangleleft$

The partial derivatives of a paraboloid are planes.



The partial derivatives of a function are new functions.

Example

Find the partial derivatives of $f(x, y) = x^2 \ln(y)$.

Solution:

$$f_x(x, y) = 2x \ln(y), \qquad f_y(x, y) = \frac{x^2}{y}.$$

$\triangleleft$

Example

Find the partial derivatives of $f(x, y) = x^2 + \dfrac{y^2}{4}$.

Solution:

$$f_x(x, y) = 2x, \qquad f_y(x, y) = \frac{y}{2}.$$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- **Higher-order partial derivatives.**
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Higher-order partial derivatives.

Remark:
Higher derivatives of a function are partial derivatives of its partial derivatives.
The second partial derivatives of $f(x,y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$

$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x,y+h) - f_x(x,y)\right],$$

$$f_{yx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x+h,y) - f_y(x,y)\right].$$

Notation: f_{xx}, $\dfrac{\partial^2 f}{\partial x^2}$, $\partial_{xx} f$, and also f_{xy}, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\partial_{xy} f$.

Higher-order partial derivatives.

Example

Find all second order derivatives of the function $f(x,y) = x^3 e^{2y} + 3y$.

Solution:

$$f_x(x,y) = 3x^2 e^{2y}, \qquad f_y(x,y) = 2x^3 e^{2y} + 3.$$

$$f_{xx}(x,y) = 6x e^{2y}, \qquad f_{yy}(x,y) = 4x^3 e^{2y}.$$

$$f_{xy} = 6x^2 e^{2y}, \qquad f_{yx} = 6x^2 e^{2y}.$$

$\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
- **The Mixed Derivative Theorem.**
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Higher-order partial derivatives sometimes commute.

Theorem

If the partial derivatives f_x, f_y, f_{xy} and f_{yx} of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ exist and all are continuous functions, then holds

$$f_{xy} = f_{yx}.$$

Example

Find f_{xy} and f_{yx} for $f(x, y) = \cos(xy)$.

Solution:

$$f_x = -y \sin(xy), \qquad f_{xy} = -\sin(xy) - yx \cos(xy).$$

$$f_y = -x \sin(xy), \qquad f_{yx} = -\sin(xy) - xy \cos(xy).$$

$\triangleleft$

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- **Examples of implicit partial differentiation.**
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Examples of implicit partial differentiation.

Remark: Implicit differentiation rules for partial derivatives are similar to those for functions of one variable.

Example

Find $\partial_x z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

Solution:

$$yz + xy\,(\partial_x z) + \frac{2}{y}\,(\partial_x z)e^{2z/y} - (\partial_x z)\,\sin(z) = 0.$$

Compute $\partial_x z$ as a function of x, y and $z(x, y)$, as follows,

$$(\partial_x z)\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big] = -yz,$$

that is, $(\partial_x z) = -\dfrac{yz}{\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big]}$. ◁

Examples of implicit partial differentiation.

Remark: Implicit differentiation rules for partial derivatives are similar to those for functions of one variable.

Example

Find $\partial_y z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

Solution:

$$xz + xy\,(\partial_y z) + \Big(\frac{2}{y}\,(\partial_y z) - \frac{2}{y^2}\,z\Big)e^{2z/y} - (\partial_y z)\,\sin(z) = 0.$$

Compute $\partial_y z$ as a function of x, y and $z(x, y)$, as follows,

$$(\partial_y z)\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big] = -xz + \frac{2}{y^2}\,z\,e^{2z/y},$$

that is, $(\partial_y z) = \dfrac{\big[-xz + \frac{2}{y^2}\,z\,e^{2z/y}\big]}{\big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\big]}$. ◁

Partial derivatives and differentiability (Sect. 14.3).

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- **Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.**

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Definition

Given a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, the *partial derivative of $f(x_1, \cdots, x_n)$ with respect to x_i*, with $i = 1, \cdots, n$, at a point $(x_1, \cdots, x_n) \in D$ is given by

$$f_{x_i} = \lim_{h \to 0} \frac{1}{h} \big[f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n) \big].$$

Remark: To compute f_{x_i} derivate f with respect to x_i keeping all other variables x_j constant.

Notation: $f_{x_i}, \qquad f_i, \qquad \dfrac{\partial f}{\partial x_i}, \qquad \partial_{x_i} f, \qquad \partial_i f.$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Example

Compute all first partial derivatives of the function $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

Solution:

$$\phi_x = -\frac{1}{2}\frac{2x}{(x^2 + y^2 + z^2)^{3/2}} \quad \Rightarrow \quad \phi_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}.$$

Analogously, the other partial derivatives are given by

$$\phi_y = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \phi_z = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}.$$

$\triangleleft$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Example

Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation : $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$.

Solution: Recall: $\phi_x = -x/(x^2 + y^2 + z^2)^{3/2}$. Then,

$$\phi_{xx} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{2}\frac{2x^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $\phi_{xx} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$.

Analogously, $\phi_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$, and $\phi_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$. Then,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3r^2}{r^5}.$$

We conclude that $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$. $\triangleleft$