Review for Exam 1.

- Sections 12.1-12.6.
- 50 minutes.
- 5 or 6 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.
Example
Consider the vectors \( \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

1. Compute \( \mathbf{v} \cdot \mathbf{w} \).
Example
Consider the vectors \( \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

1. Compute \( \mathbf{v} \cdot \mathbf{w} \).

Solution:

\[
\mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \Rightarrow \mathbf{v} \cdot \mathbf{w} = -3.
\]

\( \blacktriangle \)
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2. Find the cosine of the angle between \( \mathbf{v} \) and \( \mathbf{w} \).
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Consider the vectors \( \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

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   \]

2. Find the cosine of the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

   Solution:
   \[
   |\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.
   \]
   \[
   \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{-3}{3\sqrt{6}}
   \]
Example
Consider the vectors \( \mathbf{v} = 2 \mathbf{i} - 2 \mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = \mathbf{i} + 2 \mathbf{j} - \mathbf{k} \).

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   Solution:
   \[
   \mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = -3.
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2. Find the cosine of the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

   Solution:
   \[
   |\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.
   \]
   \[
   \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|} = \frac{-3}{3\sqrt{6}} \quad \Rightarrow \quad \cos(\theta) = -\frac{1}{\sqrt{6}}.
   \]
Example

1. Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
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1. Find a unit vector in the direction of \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

Solution:

\[
\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|},
\]
Example

1. Find a unit vector in the direction of $v = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution:

$$u = \frac{v}{|v|}, \quad |v| = \sqrt{1 + 4 + 1} = \sqrt{6},$$
Example

1. Find a unit vector in the direction of \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

   Solution:

   \[
   \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6},
   \]

   \[
   \mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.
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Example

1. Find a unit vector in the direction of \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

   Solution:
   
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   \]
   
   \[
   \mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.
   \]

2. Find \( |\mathbf{u} - 2\mathbf{v}| \), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).
Example

1. Find a unit vector in the direction of $v = \mathbf{i} - 2 \mathbf{j} + \mathbf{k}$.

   Solution:

   \[ u = \frac{v}{|v|}, \quad |v| = \sqrt{1 + 4 + 1} = \sqrt{6}, \]

   \[ u = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle. \]

2. Find $|u - 2v|$, where $u = 3 \mathbf{i} + 2 \mathbf{j} + \mathbf{k}$, $v = \mathbf{i} - 2 \mathbf{j} + \mathbf{k}$.

   Solution: First: $u - 2v = \langle 1, 6, -1 \rangle$. 
Example

1. Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

   Solution:
   
   $$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6},$$

   $$\mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.$$ 

2. Find $|\mathbf{u} - 2\mathbf{v}|$, where $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

   Solution: First: $\mathbf{u} - 2\mathbf{v} = \langle 1, 6, -1 \rangle$. Then,

   $$|\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1}.$$
Example

1. Find a unit vector in the direction of \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

   Solution:
   \[
   \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6},
   \]
   \[
   \mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle.
   \]

2. Find \( |\mathbf{u} - 2\mathbf{v}| \), where \( \mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \).

   Solution: First: \( \mathbf{u} - 2\mathbf{v} = \langle 1, 6, -1 \rangle \). Then,
   \[
   |\mathbf{u} - 2\mathbf{v}| = \sqrt{1 + 36 + 1}. \quad \Rightarrow \quad |\mathbf{u} - 2\mathbf{v}| = \sqrt{38}.
   \]
Example
Find a unit vector \( \mathbf{u} \) normal to both \( \mathbf{v} = \langle 6, 2, -3 \rangle \) and \( \mathbf{w} = \langle -2, 2, 1 \rangle \).
Example

Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

Solution:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2 + 6)\mathbf{i} - (6 - 6)\mathbf{j} + (12 + 4)\mathbf{k} = \langle 8, 0, 16 \rangle.$$
Example
Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

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Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2 \rangle$ instead of $\langle 8, 0, 16 \rangle$. 

$\Rightarrow$
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Solution:

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\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & 2 & -3 \\
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\end{vmatrix} = (2 + 6)\mathbf{i} - (6 - 6)\mathbf{j} + (12 + 4)\mathbf{k} = \langle 8, 0, 16 \rangle.
\]

Since we look for a unit vector, the calculation is simpler with \( \langle 1, 0, 2 \rangle \) instead of \( \langle 8, 0, 16 \rangle \).

\[
\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|}
\]
Example
Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3 \rangle$ and $\mathbf{w} = \langle -2, 2, 1 \rangle$.

Solution:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2 + 6)\mathbf{i} - (6 - 6)\mathbf{j} + (12 + 4)\mathbf{k} = \langle 8, 0, 16 \rangle.$$ 

Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2 \rangle$ instead of $\langle 8, 0, 16 \rangle$.

$$\mathbf{u} = \frac{\langle 1, 0, 2 \rangle}{|\langle 1, 0, 2 \rangle|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}} \langle 1, 0, 2 \rangle.$$
Example
Find the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ above.
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Solution:
Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = |\langle 8, 0, 16 \rangle| = \sqrt{8^2 + 16^2} = \sqrt{8^2(1 + 4)}.$$
Example
Find the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ above.

Solution:
Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = |\langle 8, 0, 16 \rangle| = \sqrt{8^2 + 16^2} = \sqrt{8^2(1 + 4)}.$$

$$A = 8\sqrt{5}.$$
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).
Example
Find the volume of the parallelepiped determined by the vectors $u = \langle 6, 3, -1 \rangle$, $v = \langle 0, 1, 2 \rangle$, and $w = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $u \cdot (v \times w)$. 
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle
\]

We obtain \( \mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle \).
Example

Find the volume of the parallelepiped determined by the vectors \( u = \langle 6, 3, -1 \rangle, \ v = \langle 0, 1, 2 \rangle, \) and \( w = \langle 4, -2, 5 \rangle. \)

Solution: We need to compute the triple product \( u \cdot (v \times w). \) We must start with the cross product.

\[
v \times w = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle
\]

We obtain \( v \times w = \langle 9, 8, -4 \rangle. \) The triple product is

\[
u \cdot (v \times w) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.
\]
Example

Find the volume of the parallelepiped determined by the vectors \( \mathbf{u} = \langle 6, 3, -1 \rangle \), \( \mathbf{v} = \langle 0, 1, 2 \rangle \), and \( \mathbf{w} = \langle 4, -2, 5 \rangle \).

Solution: We need to compute the triple product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \). We must start with the cross product.

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\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.
\]

Since \( V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \), we obtain \( V = 82 \).
Example

Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t$ intersects the plane given by $2x + y - z = 1$? If the answer is yes, then find the intersection point.

Solution: The line with parametric equation $x = t$, $y = 1 + 2t$, $z = 1 + 3t$, intersect the plane $2x + y - z = 1$ iff there is a solution $t$ for the equation $2t + (1 + 2t) - (1 + 3t) = 1$. There is a solution given by $t = 1$. Therefore, the point of intersection has coordinates $x = 1$, $y = 3$, $z = 4$, then $P = (1, 3, 4)$.
Example
Does the line given by \( \mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t \) intersects the plane given by \( 2x + y - z = 1 \)? If the answer is yes, then find the intersection point.

Solution: The line with parametric equation

\[
x = t, \quad y = 1 + 2t, \quad z = 1 + 3t,
\]

intersect the plane \( 2x + y - z = 1 \) iff there is a solution \( t \) for the equation

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There is a solution given by \( t = 1 \). Therefore, the point of intersection has coordinates \( x = 1, y = 3, z = 4 \), then

\[
P = (1, 3, 4).
\]
Example
Find the equation for the plane that contains the point
$P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$. 
Example
Finding the equation for the plane that contains the point \( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, y = t, z = -1 + 2t \).

Solution:
The vector equation of the line is \( \mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t \).
Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$.

A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. 
Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is
$$\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t.$$ 

A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. The point $P_0 = (1, 2, 3)$ is in the plane. A second point in the plane is any point in the line, for example $P_1$ corresponding to the terminal point of $\mathbf{r}(0) = \langle -2, 0, -1 \rangle$. 
Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$.

A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. The point $P_0 = (1, 2, 3)$ is in the plane. A second point in the plane is any point in the line, for example $P_1$ corresponding to the terminal point of $\mathbf{r}(0) = \langle -2, 0, -1 \rangle$.

Then a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$. 
Example
Find the equation for the plane that contains the point
$P_0 = (1, 2, 3)$ and the line $x = -2 + t, y = t, z = -1 + 2t$. 

Solution:
The vector equation of the line is
$r(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$,
and a second vector tangent to the plane is
$\vec{P_1P_0} = \langle 3, 2, 4 \rangle$.

Then, a normal to the plane is given by
$n = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle 4 - 4, -(4 - 6), 2 - 3 \rangle = \langle 0, 2, -1 \rangle$.

So, the equation of the plane is
$0(x - 1) + 2(y - 2) - (z - 3) = 0$,
$\Rightarrow 2y - z = 1$. 
Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:
The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$, and a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

Then, a normal to the plane is given by $n = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle 4 - 4, -4 + 6, 2 - 3 \rangle \Rightarrow n = \langle 0, 2, -1 \rangle$.

So, the equation of the plane is $0(x - 1) + 2(y - 2) - (z - 3) = 0$, $\Rightarrow 2y - z = 1$.
Example
Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:
The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$, and a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4 \rangle$.

Then, a normal to the plane is given by

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle (4 - 4), -(4 - 6), (2 - 3) \rangle \Rightarrow \mathbf{n} = \langle 0, 2, -1 \rangle.$$

So, the equation of the plane is

$$0(x - 1) + 2(y - 2) - (z - 3) = 0, \quad \Rightarrow \quad 2y - z = 1.$$
Example
Find an equation for the plane that passes through the points 
\((1, 1, 1), (1, -1, 1), \) and \((0, 0, 2)\).
Example

Find an equation for the plane that passes through the points 
(1, 1, 1), (1, \(-1\), 1), and (0, 0, 2).

Solution: Denote \( P = (1, 1, 1) \), \( Q = (1, -1, 1) \), and \( R = (0, 0, 2) \).
Example

Find an equation for the plane that passes through the points 
(1, 1, 1), (1, −1, 1), and (0, 0, 2).

Solution: Denote \( P = (1, 1, 1), \ Q = (1, −1, 1), \) and \( R = (0, 0, 2) \). Then,

\[
PQ = \langle 0, −2, 0 \rangle, \quad PR = \langle −1, −1, 1 \rangle,
\]

that is,

\[
PQ \times PR = \begin{vmatrix}
  i & j & k \\
  0 & −2 & 0 \\
  −1 & −1 & 1
\end{vmatrix} = \langle −2, 0, −2 \rangle.
\]

Take \( n = \langle 2, 0, 2 \rangle \). With \( n = \langle 2, 0, 2 \rangle \) and a point \( R = (0, 0, 2) \), the equation of the plane is

\[
2(x − 0) + 0(y − 0) + 2(z − 2) = 0 \implies x + z = 2.
\]
Example

Find an equation for the plane that passes through the points \((1, 1, 1), (1, -1, 1),\) and \((0, 0, 2)\).

Solution: Denote \(P = (1, 1, 1),\) \(Q = (1, -1, 1),\) and \(R = (0, 0, 2)\). Then,

\[
PQ = \langle 0, -2, 0 \rangle, \quad PR = \langle -1, -1, 1 \rangle,
\]

\[
PQ \times PR = \begin{vmatrix}
i & j & k \\
0 & -2 & 0 \\
-1 & -1 & 1 \\
\end{vmatrix} = (-2 - 0)i - (0 - 0)j + (0 - 2)k,
\]

that is, \(PQ \times PR = \langle -2, 0, -2 \rangle\). Take \(n = \langle 2, 0, 2 \rangle\).
Example
Find an equation for the plane that passes through the points $(1, 1, 1)$, $(1, -1, 1)$, and $(0, 0, 2)$.

Solution: Denote $P = (1, 1, 1)$, $Q = (1, -1, 1)$, and $R = (0, 0, 2)$. Then,

$$
\vec{PQ} = \langle 0, -2, 0 \rangle, \quad \vec{PR} = \langle -1, -1, 1 \rangle,
$$

$$
\vec{PQ} \times \vec{PR} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & -2 & 0 \\
-1 & -1 & 1
\end{vmatrix}
= (-2 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (0 - 2)\mathbf{k},
$$

that is, $\vec{PQ} \times \vec{PR} = \langle -2, 0, -2 \rangle$. Take $\mathbf{n} = \langle 2, 0, 2 \rangle$. With $\mathbf{n} = \langle 2, 0, 2 \rangle$ and a point $R = (0, 0, 2)$, the equation of the plane is

$$
2(x - 0) + 0(y - 0) + 2(z - 2) = 0
$$
Example

Find an equation for the plane that passes through the points $(1, 1, 1)$, $(1, -1, 1)$, and $(0, 0, 2)$.

Solution: Denote $P = (1, 1, 1)$, $Q = (1, -1, 1)$, and $R = (0, 0, 2)$. Then,

$$
\mathbf{PQ} = \langle 0, -2, 0 \rangle, \quad \mathbf{PR} = \langle -1, -1, 1 \rangle,
$$

$$
\mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & -2 & 0 \\
-1 & -1 & 1 \\
\end{vmatrix} = (-2 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (0 - 2)\mathbf{k},
$$

that is, $\mathbf{PQ} \times \mathbf{PR} = \langle -2, 0, -2 \rangle$. Take $\mathbf{n} = \langle 2, 0, 2 \rangle$.

With $\mathbf{n} = \langle 2, 0, 2 \rangle$ and a point $R = (0, 0, 2)$, the equation of the plane is

$$
2(x - 0) + 0(y - 0) + 2(z - 2) = 0 \Rightarrow x + z = 2.
$$
Example
Find the equation of the plane that is parallel to the plane
\( x - 2y + 3z = 1 \) and passes through the center of the sphere
\( x^2 + 2x + y^2 + z^2 - 2z = 0 \).
Example
Find the equation of the plane that is parallel to the plane \( x - 2y + 3z = 1 \) and passes through the center of the sphere \( x^2 + 2x + y^2 + z^2 - 2z = 0 \).

Solution: The plane is parallel to the plane \( x - 2y + 3z = 1 \), so their normal vectors are parallel. We choose \( \mathbf{n} = \langle 1, -2, 3 \rangle \).
Example
Find the equation of the plane that is parallel to the plane 
\( x - 2y + 3z = 1 \) and passes through the center of the sphere 
\( x^2 + 2x + y^2 + z^2 - 2z = 0 \).

Solution: The plane is parallel to the plane \( x - 2y + 3z = 1 \), so 
their normal vectors are parallel. We choose \( \mathbf{n} = \langle 1, -2, 3 \rangle \). 
We need to find the center of the sphere. We complete squares:

\[
0 = x^2 + 2x + y^2 + z^2 - 2z \\
= (x^2 + 2x + 1) - 1 + y^2 + (z^2 - 2z + 1) - 1 = 0 \\
= (x + 1)^2 + y^2 + (z - 1)^2 - 2.
\]
Example
Find the equation of the plane that is parallel to the plane
\[x - 2y + 3z = 1\] and passes through the center of the sphere
\[x^2 + 2x + y^2 + z^2 - 2z = 0.\]

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\]

Therefore, the center of the sphere is at \( P_0 = (-1, 0, 1) \).
The equation of the plane is

\[
(x + 1) - 2(y - 0) + 3(z - 1) = 0 \quad \Rightarrow \quad x - 2y + 3z = 2.
\]
Example

Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$. 
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Solution: The angle between the planes is the angle between their normal vectors.
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Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: The angle between the planes is the angle between their normal vectors. The normal vectors are $\mathbf{n} = \langle 2, -3, 2 \rangle$, $\mathbf{N} = \langle 1, 2, 2 \rangle$. 
Example
Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: The angle between the planes is the angle between their normal vectors.
The normal vectors are $\mathbf{n} = \langle 2, -3, 2 \rangle$, $\mathbf{N} = \langle 1, 2, 2 \rangle$.
The cosine of the angle $\theta$ between these vectors is

$$\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}| |\mathbf{N}|}.$$
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The angle $\theta$ is $\theta = \pi/2$. 

\[
\square
\]
Example
Find the vector equation for the line of intersection of the planes
\(2x - 3y + 2z = 1\) and \(x + 2y + 2z = 5\).
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Solution: We first find the vector tangent to the line. This is a
vector \(v\) that belongs to both planes.
Example

Find the vector equation for the line of intersection of the planes $2x − 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: We first find the vector tangent to the line. This is a vector $\mathbf{v}$ that belongs to both planes. This means that $\mathbf{v}$ is perpendicular to both normal vectors $\mathbf{n} = \langle 2, -3, 2 \rangle$ and $\mathbf{N} = \langle 1, 2, 2 \rangle$. 
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Find the vector equation for the line of intersection of the planes
\[2x - 3y + 2z = 1\] and \[x + 2y + 2z = 5\].

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One such vector is
\[
\mathbf{v} = \mathbf{n} \times \mathbf{N} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & -3 & 2 \\
1 & 2 & 2 \\
\end{vmatrix} = \langle (-6 - 4), -(4 - 2), (4 + 3) \rangle.
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So, $\mathbf{v} = \langle -10, -2, 7 \rangle$. 
Example

Find the vector equation for the line of intersection of the planes 
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Example

Find the vector equation for the line of intersection of the planes

\(2x - 3y + 2z = 1\) and \(x + 2y + 2z = 5\).

Solution: Recall \( \mathbf{v} = \langle -10, -2, 7 \rangle \). Now we need a point in the intersection of the planes. From the first plane we compute \(z\) as follows: \(2z = 1 - 2x + 3y\).

We introduce this equation for \(2z\) into the second plane:

\[x + 2y + (1 - 2x + 3y) = 5 \quad \Rightarrow \quad -x + 5y = 4.\]
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We need just one solution, so we choose: $y = 0$, then $x = -4$, and this implies $z = 9/2$. 

The vector equation of the line is:

$$\mathbf{r}(t) = \langle -4, 0, 9/2 \rangle + \langle -10, -2, 7 \rangle t.$$
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$$\mathbf{r}(t) = \langle -4, -0, 9/2 \rangle + \langle -10, -2, 7 \rangle t.$$
Example

Sketch the surface $36x^2 + 4y^2 + 9z^2 = 36$. 

Solution:

We first rewrite the equation above in the standard form

$$x^2 + 4\frac{36}{36}y^2 + 9\frac{36}{36}z^2 = 1 \iff x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1.$$ 

This is the equation of an ellipsoid with principal radius of length 1, 3, and 2 on the $x$, $y$, and $z$ axis, respectively.
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Vector functions (Sect. 13.1).

- Definition of vector functions: \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.
Motion in space motivates to define vector functions.

**Definition**
A function $\mathbf{r} : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, is called a *vector function*, where the interval $I \subset \mathbb{R}$ is called the *domain* of the function.
Motion in space motivates to define vector functions.

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A function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is called a vector function, where the interval \( I \subset \mathbb{R} \) is called the domain of the function.

Remark: Given Cartesian coordinates in \( \mathbb{R}^3 \), the values of a vector function can be written in components as follows:

\[
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in I,
\]

where \( x(t), y(t), \) and \( z(t) \) are the values of three scalar functions.
Motion in space motivates to define vector functions.

Remarks:

- There is a natural association between a curve in $\mathbb{R}^n$ and the vector function values $\mathbf{r}(t)$.

![Diagram of a curve in $\mathbb{R}^3$ with terminal points $\mathbf{r}(0)$ and $\mathbf{r}(t)$.]
Motion in space motivates to define vector functions.

Remarks:

- There is a natural association between a curve in $\mathbb{R}^n$ and the vector function values $\mathbf{r}(t)$.

- The curve is determined by the terminal points of the vector function values $\mathbf{r}(t)$.
Motion in space motivates to define vector functions.

Remarks:

- There is a natural association between a curve in $\mathbb{R}^n$ and the vector function values $\mathbf{r}(t)$.

- The curve is determined by the terminal points of the vector function values $\mathbf{r}(t)$.

- The independent variable $t$ is called the parameter of the curve.
Example

Graph the vector function \( r(t) = \langle \cos(t), \sin(t), t \rangle \).
Example

Graph the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution:

The curve given by \( \mathbf{r}(t) \) lies on a vertical cylinder with radius one, since

\[
x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.
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Example
Graph the vector function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

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The curve given by $\mathbf{r}(t)$ lies on a vertical cylinder with radius one, since

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The $z(t)$ coordinate of the curve increases with $t$, so the terminal point $\mathbf{r}(t)$ moves up on the cylinder surface when $t$ increases.
Vector functions.

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Graph the vector function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

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Graph the vector function \( \mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle \).

Solution:

The curve given by \( \mathbf{r}(t) \) lies on a horizontal cylinder with radius one, since

\[
x^2 + z^2 = \sin^2(t) + \cos^2(t) = 1.
\]
Vector functions.

Example

Graph the vector function \( \mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle \).

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\[
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\]

The \( y(t) \) coordinate of the curve increases with \( t \), so the terminal point \( \mathbf{r}(t) \) moves to the right on the cylinder surface when \( t \) increases.
**Vector functions.**

**Example**

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\( \triangle \)
Vector functions (Sect. 13.1).

- Definition of vector functions: \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.
Limits and continuity of vector functions.

Definition
The vector function \( r : I \to \mathbb{R}^n \), with \( n = 2, 3 \), has a limit given by the vector \( \mathbf{L} \) when \( t \) approaches \( t_0 \), denoted as \( \lim_{t \to t_0} r(t) = \mathbf{L} \), iff the following holds: For every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that

\[
|t - t_0| < \delta \quad \Rightarrow \quad |r(t) - \mathbf{L}| < \epsilon.
\]
Limits and continuity of vector functions.

Definition
The vector function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), has a limit given by the vector \( L \) when \( t \) approaches \( t_0 \), denoted as \( \lim_{t \to t_0} r(t) = L \), iff the following holds: For every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that

\[
|t - t_0| < \delta \quad \Rightarrow \quad |r(t) - L| < \epsilon.
\]

Remark:
- The limit of \( r(t) = \langle x(t), y(t), z(t) \rangle \) as \( t \to t_0 \) is the limit of its components \( x(t), y(t), z(t) \) in Cartesian coordinates.
Limits and continuity of vector functions.

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The vector function $\mathbf{r} : I \rightarrow \mathbb{R}^n$, with $n = 2, 3$, has a limit given by the vector $\mathbf{L}$ when $t$ approaches $t_0$, denoted as $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$, iff the following holds: For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|t - t_0| < \delta \implies |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

Remark:

- The limit of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ as $t \to t_0$ is the limit of its components $x(t), y(t), z(t)$ in Cartesian coordinates.
- That is:

$$\lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle.$$
\[ \lim_{t \to t_0} r(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle. \]

**Example**

Given \( r(t) = \langle \cos(t), \sin(t)/t, t^2 + 2 \rangle \), compute \( \lim_{t \to 0} r(t) \).
\[ \lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle. \]

**Example**

Given \( \mathbf{r}(t) = \langle \cos(t), \sin(t)/t, t^2 + 2 \rangle \), compute \( \lim_{t \to 0} \mathbf{r}(t) \).

**Solution:**

Notice that the vector function \( \mathbf{r} \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists.
\[
\lim_{t \to t_0} r(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle.
\]

Example
Given \( r(t) = \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle \), compute \( \lim_{t \to 0} r(t) \).

Solution:
Notice that the vector function \( r \) is not defined at \( t = 0 \), however its limit at \( t = 0 \) exists. Indeed,

\[
\lim_{t \to 0} r(t) = \lim_{t \to 0} \left\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2 \right\rangle
= \left\langle \lim_{t \to 0} \cos(t), \lim_{t \to 0} \frac{\sin(t)}{t}, \lim_{t \to 0} (t^2 + 2) \right\rangle
= \langle 1, 1, 2 \rangle.
\]

We conclude that \( \lim_{t \to 0} r(t) = \langle 1, 1, 2 \rangle \).
Definition
A vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is \textit{continuous at} \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \). The function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \) is \textit{continuous} if it is continuous at every \( t \) in its domain interval \( I \).
Limits and continuity of vector functions.

Definition
A vector function \( r : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is \textit{continuous at} \( t = t_0 \in I \) iff holds \( \lim_{t \to t_0} r(t) = r(t_0) \). The function \( r : I \rightarrow \mathbb{R}^n \) is \textit{continuous} if it is continuous at every \( t \) in its domain interval \( I \).

Remark: A vector function with Cartesian components \( r = \langle x, y, z \rangle \) is continuous iff each component is continuous.
Limits and continuity of vector functions.

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Example
The function \( r(t) = \langle \sin(t), t, \cos(t) \rangle \) is continuous for \( t \in \mathbb{R} \). \( \triangle \)
Limits and continuity of vector functions.

Definition
A vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is continuous at $t = t_0 \in I$ iff holds $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function $\mathbf{r} : I \to \mathbb{R}^n$ is continuous if it is continuous at every $t$ in its domain interval $I$.

Remark: A vector function with Cartesian components $\mathbf{r} = \langle x, y, z \rangle$ is continuous iff each component is continuous.

Example
The function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$ is continuous for $t \in \mathbb{R}$.

Remark: Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.
Vector functions (Sect. 13.1).

- Definition of vector functions: \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- **Derivatives and motion.**
- Differentiation rules.
- Integrals of vector functions.
Derivatives and motion.

Definition
The vector function \( \mathbf{r} : I \rightarrow \mathbb{R}^n \), with \( n = 2, 3 \), is differentiable at \( t = t_0 \), denoted as \( \mathbf{r}'(t) \) or \( \frac{d\mathbf{r}}{dt} \), iff the following limit exists,

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}.
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Derivatives and motion.

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Derivatives and motion.

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Remarks:
- A vector function $\mathbf{r} : I \rightarrow \mathbb{R}^n$ is differentiable if it is differentiable for each $t \in I$.

- If a vector function with Cartesian components $\mathbf{r} = \langle x, y, z \rangle$ is differentiable, then

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$
Derivatives and motion.

Theorem

*If a vector function with Cartesian components* \( \mathbf{r} = \langle x, y, z \rangle \) *is differentiable, then* \( \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).
Derivatives and motion.

Theorem
If a vector function with Cartesian components \( \mathbf{r} = \langle x, y, z \rangle \) is differentiable, then \( \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).

Proof.

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h},
\]

\[
= \lim_{h \to 0} \left\langle \frac{x(t + h) - x(t)}{h}, \frac{y(t + h) - y(t)}{h}, \frac{z(t + h) - z(t)}{h} \right\rangle
\]

\[
= \left\langle \lim_{h \to 0} \frac{x(t + h) - x(t)}{h}, \lim_{h \to 0} \frac{y(t + h) - y(t)}{h}, \lim_{h \to 0} \frac{z(t + h) - z(t)}{h} \right\rangle
\]

\[
= \langle x'(t), y'(t), z'(t) \rangle.
\]
Derivatives and motion.

Example
Find the derivative of the vector function
\[ r(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle. \]
Derivatives and motion.

Example

Find the derivative of the vector function

\[ r(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle. \]

Solution: We differentiate each component of \( r \), that is,

\[ r'(t) = \langle -\sin(t), \cos(t), (2t + 3) \rangle. \]
Derivatives and motion.

Example
Find the derivative of the vector function
\( r(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle \).

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\[
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Example
Find the derivative of the vector function \( r(t) = \langle \cos(2t), e^{3t}, 1/t \rangle \).
Example
Find the derivative of the vector function
\[ r(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle. \]

Solution: We differentiate each component of \( r \), that is,
\[ r'(t) = \langle -\sin(t), \cos(t), (2t + 3) \rangle. \]

Example
Find the derivative of the vector function \( r(t) = \langle \cos(2t), e^{3t}, 1/t \rangle \).

Solution: We differentiate each component of \( r \), that is,
\[ r'(t) = \langle -2\sin(2t), 3e^{3t}, -1/t^2 \rangle. \]
Remark: The vector $\mathbf{r}'(t)$ is tangent to the curve given by $\mathbf{r}$ at the point $\mathbf{r}(t)$. 
Geometrical property of the derivative.

Remark: The vector $r'(t)$ is tangent to the curve given by $r$ at the point $r(t)$.

Remark: If $r(t)$ represents the vector position of a particle, then:

\[ \frac{dr}{dt} = v(t) \]

\[ \frac{dv}{dt} = a(t) \]
Geometrical property of the derivative.

Remark: The vector $r'(t)$ is tangent to the curve given by $\mathbf{r}$ at the point $\mathbf{r}(t)$.

Remark: If $\mathbf{r}(t)$ represents the vector position of a particle, then:

- The derivative of the position function is the velocity function, $\mathbf{v}(t) = r'(t)$. The speed is $|\mathbf{v}(t)|$. 
**Remark:** The vector $\mathbf{r}'(t)$ is tangent to the curve given by $\mathbf{r}$ at the point $\mathbf{r}(t)$.

**Remark:** If $\mathbf{r}(t)$ represents the vector position of a particle, then:

- The derivative of the position function is the velocity function, $\mathbf{v}(t) = \mathbf{r}'(t)$. The speed is $|\mathbf{v}(t)|$.
- The derivative of the velocity function is the acceleration function, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$. 
Example

Compute the derivative of the position function \( r(t) = \langle \cos(t), \sin(t), 0 \rangle \). Graph the curve given by \( r \), and explicitly show the position vector \( r(0) \) and velocity vector \( v(0) \).
Example

Compute the derivative of the position function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \). Graph the curve given by \( \mathbf{r} \), and explicitly show the position vector \( \mathbf{r}(0) \) and velocity vector \( \mathbf{v}(0) \).

Solution:

The derivative of \( \mathbf{r} \) is:

\[
\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.
\]
Example

Compute the derivative of the position function
\( r(t) = \langle \cos(t), \sin(t), 0 \rangle \). Graph the curve given by \( r \), and explicitly show the position vector \( r(0) \) and velocity vector \( v(0) \).

Solution:

The derivative of \( r \) is:

\[
v(t) = \langle -\sin(t), \cos(t), 0 \rangle.
\]

\( r(0) = \langle 1, 0, 0 \rangle \), \( v(0) = \langle 0, 1, 0 \rangle \).
Derivatives and motion.

Example
Compute the derivative of the position function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \). Graph the curve given by \( \mathbf{r} \), and explicitly show the position vector \( \mathbf{r}(0) \) and velocity vector \( \mathbf{v}(0) \).

Solution:
The derivative of \( \mathbf{r} \) is:
\[
\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.
\]
\( \mathbf{r}(0) = \langle 1, 0, 0 \rangle, \mathbf{v}(0) = \langle 0, 1, 0 \rangle. \)
Differentiation rules are the same as for scalar functions

**Theorem**

If \( \mathbf{v} \) and \( \mathbf{w} \) are differentiable vector functions, then holds:

- \( [\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t), \) (addition);
- \( [c\mathbf{v}(t)]' = c\mathbf{v}'(t), \) (product rule);
- \( [\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t), \) (chain rule);
- \( [f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t), \) (product rule);
- \( [\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t), \) (dot product);
- \( [\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t), \) (cross product).
Higher derivatives can also be computed.

**Remark:** The $m$-derivative of a vector function $\mathbf{r}$ is denoted as $\mathbf{r}^{(m)}$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$. 

Example: Compute the third derivative of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle$.

Solution:

$r'(t) = \langle -\sin(t), \cos(t), 2t + 2 \rangle$,

$r''(t) = (r'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle$,

$r'''(t) = (r''(t))' = \langle \sin(t), -\cos(t), 0 \rangle$.

Recall: If $\mathbf{r}(t)$ is the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $\mathbf{a}(t) = \mathbf{r}''(t)$ is the acceleration of the particle.
Higher derivatives can also be computed.

**Remark:** The $m$-derivative of a vector function $\mathbf{r}$ is denoted as $\mathbf{r}^{(m)}$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$. 

**Example**

Compute the third derivative of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle$. 

Recall: If $\mathbf{r}(t)$ is the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $\mathbf{a}(t) = \mathbf{r}''(t)$ is the acceleration of the particle.
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**Remark:** The $m$-derivative of a vector function $\mathbf{r}$ is denoted as $\mathbf{r}^{(m)}$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$.

**Example**
Compute the third derivative of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle$.

**Solution:**
\[
\begin{align*}
\mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
\mathbf{r}^{(2)}(t) &= (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\
\mathbf{r}^{(3)}(t) &= (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle.
\end{align*}
\]
Higher derivatives can also be computed.

**Remark:** The \( m \)-derivative of a vector function \( \mathbf{r} \) is denoted as \( \mathbf{r}^{(m)} \) and is given by the expression \( \mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]' \).

**Example**
Compute the third derivative of \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle \).

**Solution:**
\[
\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
\mathbf{r}^{(2)}(t) = (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\
\mathbf{r}^{(3)}(t) = (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle.
\]

**Recall:** If \( \mathbf{r}(t) \) is the position of a particle, then \( \mathbf{v}(t) = \mathbf{r}'(t) \) is the velocity and \( \mathbf{a}(t) = \mathbf{r}^{(2)}(t) \) is the acceleration of the particle.
Vector functions (Sect. 13.1).

- Definition of vector functions: \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \).
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.
Integrals of vector functions.

Definition
The *indefinite integral*, also called the *antiderivative*, of a vector function \( \mathbf{v} \) is denoted as \( \int \mathbf{v}(t) \, dt \) and given by

\[
\int \mathbf{v}(t) \, dt = \mathbf{V}(t) + \mathbf{C},
\]

where \( \mathbf{V}'(t) = \mathbf{v}(t) \) and \( \mathbf{C} \) is a constant vector.

Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

Solution:
The position function is the primitive of the velocity function, \( \mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} \), that satisfies the initial condition \( \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} \). This initial condition fixes the constant vector \( \mathbf{C} \).
Integrals of vector functions.

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The *indefinite integral*, also called the *antiderivative*, of a vector function \( \mathbf{v} \) is denoted as \( \int \mathbf{v}(t) \, dt \) and given by

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Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).
Integrals of vector functions.

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Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

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Example

Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$. 

\[ \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \]

\[ \mathbf{r}(0) = \langle 1, 1, 1 \rangle \]
Integrals of vector functions.

Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

Solution: The position function is a primitive of the velocity,

\[
\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,
\]

with \( \mathbf{C} = \langle c_x, c_y, c_z \rangle \) a constant vector.
Example
Find the position function \( \mathbf{r} \) knowing that the velocity function is \( \mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle \) and the initial position is \( \mathbf{r}(0) = \langle 1, 1, 1 \rangle \).

Solution: The position function is a primitive of the velocity,

\[
\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,
\]

with \( \mathbf{C} = \langle c_x, c_y, c_z \rangle \) a constant vector. The initial condition determines the vector \( \mathbf{C} \):

\[
\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,
\]

that is, \( c_x = 1, c_y = 1, c_z = 2 \).
Example
Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t) \rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} = \langle t^2, \sin(t), -\cos(t) \rangle + \langle c_x, c_y, c_z \rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1 \rangle = \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} = \langle 0, 0, -1 \rangle + \langle c_x, c_y, c_z \rangle,$$

that is, $c_x = 1$, $c_y = 1$, $c_z = 2$.
The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2 \rangle$. \hspace{1cm} \triangleleft
Example

Find the position function of a particle with acceleration $\mathbf{a}(t) = \langle 0, 0, -10 \rangle$ having an initial velocity $\mathbf{v}(0) = \langle 0, 1, 1 \rangle$ and initial position $\mathbf{r}(0) = \langle 1, 0, 1 \rangle$. 
Example
Find the position function of a particle with acceleration $a(t) = \langle 0, 0, -10 \rangle$ having an initial velocity $v(0) = \langle 0, 1, 1 \rangle$ and initial position $r(0) = \langle 1, 0, 1 \rangle$.

Solution: The velocity is $v(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle$. 
**Integrals of vector functions.**

**Example**

Find the position function of a particle with acceleration
\( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

**Solution:** The velocity is \( \mathbf{v}(t) = \langle v_0x, v_0y, (-10t + v_0z) \rangle \).

The initial condition implies \( \langle 0, 1, 1 \rangle = \mathbf{v}(0) = \langle v_0x, v_0y, v_0z \rangle \), that is \( v_0x = 0, v_0y = 1, v_0z = 1 \).
Example

Find the position function of a particle with acceleration \( a(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

Solution: The velocity is \( \mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle \). The initial condition implies \( \langle 0, 1, 1 \rangle = \mathbf{v}(0) = \langle v_{0x}, v_{0y}, v_{0z} \rangle \), that is \( v_{0x} = 0, \ v_{0y} = 1, \ v_{0z} = 1 \). The velocity function is

\[
\mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle.
\]
Integrals of vector functions.

Example

Find the position function of a particle with acceleration \( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity \( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position \( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

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The initial condition implies \( \langle 0, 1, 1 \rangle = \mathbf{v}(0) = \langle v_{0x}, v_{0y}, v_{0z} \rangle \), that is \( v_{0x} = 0 \), \( v_{0y} = 1 \), \( v_{0z} = 1 \). The velocity function is

\[
\mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle.
\]

The position is \( \mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle \).
Integrals of vector functions.

Example

Find the position function of a particle with acceleration \(a(t) = \langle 0, 0, -10 \rangle\) having an initial velocity \(v(0) = \langle 0, 1, 1 \rangle\) and initial position \(r(0) = \langle 1, 0, 1 \rangle\).

Solution: The velocity is \(v(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle\).

The initial condition implies \(\langle 0, 1, 1 \rangle = v(0) = \langle v_{0x}, v_{0y}, v_{0z} \rangle\), that is \(v_{0x} = 0, v_{0y} = 1, v_{0z} = 1\). The velocity function is

\[
v(t) = \langle 0, 1, (-10t + 1) \rangle.
\]

The position is \(r(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle\).

The initial condition implies \(\langle 1, 0, 1 \rangle = r(0) = \langle r_{0x}, r_{0y}, r_{0z} \rangle\), that is \(r_{0x} = 1, r_{0y} = 0, r_{0z} = 1\).
Integrals of vector functions.

Example

Find the position function of a particle with acceleration 
\( \mathbf{a}(t) = \langle 0, 0, -10 \rangle \) having an initial velocity 
\( \mathbf{v}(0) = \langle 0, 1, 1 \rangle \) and initial position 
\( \mathbf{r}(0) = \langle 1, 0, 1 \rangle \).

Solution: The velocity is 
\( \mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z}) \rangle \).

The initial condition implies 
\( \langle 0, 1, 1 \rangle = \mathbf{v}(0) = \langle v_{0x}, v_{0y}, v_{0z} \rangle \), that is 
\( v_{0x} = 0, \ v_{0y} = 1, \ v_{0z} = 1 \). The velocity function is

\[ \mathbf{v}(t) = \langle 0, 1, (-10t + 1) \rangle. \]

The position is 
\( \mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z}) \rangle. \)

The initial condition implies 
\( \langle 1, 0, 1 \rangle = \mathbf{r}(0) = \langle r_{0x}, r_{0y}, r_{0z} \rangle \), that is 
\( r_{0x} = 1, \ r_{0y} = 0, \ r_{0z} = 1 \). The velocity function is

\[ \mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1) \rangle. \]
Integrals of vector functions.

Definition
If the components of \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) are integrable functions on the interval \([a, b]\), then the definite integral of \( \mathbf{r} \) is given by

\[
\int_{a}^{b} \mathbf{r}(t)\,dt = \left\langle \int_{a}^{b} x(t)\,dt, \int_{a}^{b} y(t)\,dt, \int_{a}^{b} z(t)\,dt \right\rangle.
\]
**Integrals of vector functions.**

**Definition**  
If the components of \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) are integrable functions on the interval \([a, b] \), then the *definite integral* of \( \mathbf{r} \) is given by

\[
\int_a^b \mathbf{r}(t) \, dt = \langle \int_a^b x(t) \, dt, \int_a^b y(t) \, dt, \int_a^b z(t) \, dt \rangle.
\]

**Example**  
Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).
Example
Compute $\int_0^\pi r(t) \, dt$ for the function $r(t) = \langle \cos(t), \sin(t), t \rangle$. 
Example

Compute \( \int_0^\pi \mathbf{r}(t) \, dt \) for the function \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

Solution:

\[
\int_0^\pi \mathbf{r}(t) \, dt = \int_0^\pi \langle \cos(t), \sin(t), t \rangle \, dt \\
= \left\langle \int_0^\pi \cos(t) \, dt, \int_0^\pi \sin(t) \, dt, \int_0^\pi t \, dt \right\rangle, \\
= \left\langle \sin(t) \bigg|_0^\pi, -\cos(t) \bigg|_0^\pi, \frac{t^2}{2} \bigg|_0^\pi \right\rangle, \\
= \langle 0, 2, \frac{\pi^2}{2} \rangle, \quad \Rightarrow \quad \int_0^\pi \mathbf{r}(t) \, dt = \langle 0, 2, \frac{\pi^2}{2} \rangle.
\]
The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- The arc length function.
- Parametrizations of a curve.
- The arc length parametrization of a curve.
The length of a curve is called its arc length.

Definition
The arc length of a continuously differentiable curve \( r : [a, b] \rightarrow \mathbb{R}^n \), with \( n=2,3 \), is the number given by

\[
\ell_{ba} = \int_a^b |r'(t)| \, dt.
\]
The length of a curve is called its arc length.

**Definition**

The arc length of a continuously differentiable curve \( \mathbf{r} : [a, b] \rightarrow \mathbb{R}^n \), with \( n=2,3 \), is the number given by

\[
\ell_{ba} = \int_a^b |\mathbf{r}'(t)| \, dt.
\]

**Remark:**

- If the curve \( \mathbf{r} \) is the path traveled by a particle in space, then \( \mathbf{r}' = \mathbf{v} \) is the velocity of the particle.
The length of a curve is called its arc length.

**Definition**

The arc length of a continuously differentiable curve \( r : [a, b] \rightarrow \mathbb{R}^n \), with \( n=2,3 \), is the number given by

\[
\ell_{ba} = \int_a^b \left| r'(t) \right| \, dt.
\]

**Remark:**

- If the curve \( r \) is the path traveled by a particle in space, then \( r' = v \) is the velocity of the particle.
- The arc length is the integral in time of the particle speed \( |v(t)| \).
The length of a curve is called its arc length.

Definition
The **arc length** of a continuously differentiable curve \( r : [a, b] \to \mathbb{R}^n \), with \( n=2,3 \), is the number given by

\[
\ell_{ba} = \int_a^b |r'(t)| \, dt.
\]

Remark:
- If the curve \( r \) is the path traveled by a particle in space, then \( r' = v \) is the velocity of the particle.
- The arc length is the integral in time of the particle speed \( |v(t)| \).
- Therefore, the arc length of the curve is the distance traveled by the particle.
The length of a curve is called its arc length.

Recall:
The arc length of a curve \( \mathbf{r} : [a, b] \rightarrow \mathbb{R}^3 \)

\[
\ell_{ba} = \int_{a}^{b} |\mathbf{r}'(t)| 
\]

\[
dt.
\]
The length of a curve is called its arc length.

Recall:
The arc length of a curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$

$$\ell_{ba} = \int_a^b |\mathbf{r}'(t)| \, dt.$$  

Remark:
In Cartesian coordinates the functions $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$
The length of a curve is called its arc length.

Recall:
The arc length of a curve \( \mathbf{r} : [a, b] \to \mathbb{R}^3 \)

\[
\ell_{ba} = \int_{a}^{b} |\mathbf{r}'(t)| \, dt.
\]

Remark:
In Cartesian coordinates the functions \( \mathbf{r} \) and \( \mathbf{r}' \) are given by

\[
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.
\]

Therefore the arc length of the curve is given by the expression

\[
\ell_{ba} = \int_{a}^{b} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]
The arc length of a curve in a plane.

Example
Find the arc length of the curve $r(t) = \langle \cos(t), \sin(t) \rangle$, for $t \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right]$. 

Solution:

The derivative vector function is $r'(t) = \langle -\sin(t), \cos(t) \rangle$.

The arc length formula is

$$
\ell = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt.
$$

This result is reasonable, since the curve is a circle and we are computing the length of quarter a circle.

$\Rightarrow$
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\[ \mathbf{r}'(t) = \langle -t \sin(t) + \cos(t), t \cos(t) + \sin(t) \rangle. \]

\[ |\mathbf{r}'(t)|^2 = \left[ t^2 \sin^2(t) + \cos^2(t) - 2t \sin(t) \cos(t) \right] \]
\[ + \left[ t^2 \cos^2(t) + \sin^2(t) + 2t \sin(t) \cos(t) \right] = t^2 + 1. \]
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\[
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The arc length is \( \ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2} \, dt \).
The arc length of a curve in a plane.

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The arc length is \( \ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2} \, dt = \ln(t + \sqrt{1 + t^2}) \bigg|_0^{t_0} \).
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The arc length is \( \ell(t_0) = \int_0^{t_0} \sqrt{1 + t^2} \, dt = \ln(t + \sqrt{1 + t^2}) \bigg|_0^{t_0} \).

We conclude: \( \ell(t_0) = \ln(t_0 + \sqrt{1 + t_0^2}) \). \( \triangle \)
The arc length of a curve in space.

Example

Find the arc length of
\[ \mathbf{r}(t) = \langle 6 \cos(2t), 6 \sin(2t), 5t \rangle, \] for \( t \in [0, \pi] \).
The arc length of a curve in space.

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for \( t \in [0, \pi] \).

Solution: The derivative vector is

\[ \mathbf{r}'(t) = \langle -12 \sin(2t), 12 \cos(2t), 5 \rangle, \]

\[ |\mathbf{r}'(t)|^2 = 144[\sin^2(2t) + \cos^2(2t)] + 25 = 169 = (13)^2. \]
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The arc length is
\[
  \ell = \int_0^\pi 13 \, dt = 13 \left. t \right|_0^\pi \quad \Rightarrow \quad \ell = 13\pi.
\]
The arc length formula can be obtained as a limit procedure. One adds up the lengths of a polygonal line that approximates the original curve.

\[ \ell_N = \sum_{n=0}^{N-1} |\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)|, \quad \{a = t_0, t_1, \ldots, t_{N-1}, t_N = b\}, \]

\[ \simeq \sum_{n=0}^{N-1} |\mathbf{r}'(t_n)| (t_{n+1} - t_n) \xrightarrow{N \to \infty} \int_a^b |\mathbf{r}'(t)| \, dt \]
The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- **The arc length function.**
- Parametrizations of a curve.
- The arc length parametrization of a curve.
Definition

The arc length function. The arc *length function* of a continuously differentiable vector function $\mathbf{r}$ is given by

$$\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau.$$
The arc length function.

Definition
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Remarks:

- The value $\ell(t)$ of the arc length function represents the length along the curve $\mathbf{r}$ from $t_0$ to $t$. 
The arc length function.

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Remarks:

\begin{itemize}
\item The value \( \ell(t) \) of the arc length function represents the length along the curve \( \mathbf{r} \) from \( t_0 \) to \( t \).
\item If the function \( \mathbf{r} \) is the position of a moving particle as function of time, then the arc length \( \ell(t) \) is the distance traveled by the particle from the time \( t_0 \) to \( t \).
\end{itemize}
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Solution: We have found that \( |\mathbf{r}'(t)| = 13 \). Therefore,

\[
\ell(t) = \int_0^t 13 \, d\tau \quad \Rightarrow \quad \ell(t) = 13 \, t.
\]
The arc length function.

Example

Given the position function in time \( r(t) = \langle 6 \cos(2t), 6 \sin(2t), 5t \rangle \), find the position vector \( r(t_0) \) located at a length \( \ell_0 = 20 \) from the initial position \( r(0) \).

Solution:
We have found that the arc length function for the vector function \( r(t) \) is \( \ell(t) = 13t \).
Since \( t = \ell/13 \), the time at \( \ell = \ell_0 = 20 \) is \( t_0 = 13/20 \).
Therefore, the position vector at \( \ell_0 = 20 \) is given by \( r(t_0) = \langle 6 \cos(13/10), 6 \sin(13/10), 13/4 \rangle \).
The arc length function.

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Given the position function in time $r(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$, find the position vector $r(t_0)$ located at a length $\ell_0 = 20$ from the initial position $r(0)$.

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The arc length of a curve in space (Sect. 13.3).

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- **Parametrizations of a curve.**
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Parametrizations of a curve.

Remark:
A curve in space can be represented by different vector functions.
Parametrizations of a curve.

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Example
The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

- $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$;
- $\mathbf{r}_2(t) = \langle \cos(5t), \sin(5t) \rangle$;
- $\mathbf{r}_3(t) = \langle \cos(e^t), \sin(e^t) \rangle$.
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**Remark:**
The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.
Remarks:
▶ If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion: The time $t$. 

Another parameter that is useful to describe a moving particle is the distance traveled by the particle, the arc length $\ell$. 

A common problem is the following: Given a vector function parametrized by the time $t$, switch the curve parameter to the arc length $\ell$. 

The problem above is called the arc length parametrization of a curve.
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The arc length parametrization of a curve.

Problem:
Given vector function $\mathbf{r}$ in terms of a parameter $t$, find the arc length parametrization of that curve.
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The arc length parametrization of a curve.

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The function values \( \mathbf{r}(\ell) \) are the parametrization of the function values \( \mathbf{r}(t) \) using the arc length as the new parameter.
The arc length parametrization of a curve.

Example

Find the arc length parametrization of the vector function \( r(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \) starting at \( t = 0 \).
The arc length parametrization of a curve.

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Hence, \( |\mathbf{r}'(t)|^2 = 4^2 \sin^2(t) + 4^2 \cos^2(t) + 3^2 = 16 + 9 = 5^2 \).
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Reparametrize the original curve:

\[ r(\ell) = \langle 4 \cos(\ell/5), 4 \sin(\ell/5), 3\ell/5 \rangle. \]
The arc length parametrization of a curve.

Theorem
A unit tangent vector to a curve given by the vector function values $\mathbf{r}(t)$ is given by $\mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell}$, where $\ell$ is the arc length of the curve.
The arc length parametrization of a curve.

Theorem
A unit tangent vector to a curve given by the vector function values \( r(t) \) is given by \( u(\ell) = \frac{dr}{d\ell} \), where \( \ell \) is the arc length of the curve.

Proof.
Given the function values \( r(t) \), let \( r(\ell) \) be the reparametrization of \( r(t) \) with the arc length function \( \ell(t) = \int_{t_0}^{t} |r'(\tau)| \, d\tau \).
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Notice that $\frac{d\ell}{dt} = |r'(t)|$ and $\frac{dt}{d\ell} = \frac{1}{|r'(t)|}$. 
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Therefore, \( u(\ell) = \frac{dr(\ell)}{d\ell} = \frac{dr(t)}{dt} \frac{dt}{d\ell} = \frac{r'(t)}{|r'(t)|} \).
The arc length parametrization of a curve.

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Therefore, $\mathbf{u}(\ell) = \frac{d\mathbf{r}(\ell)}{d\ell} = \frac{d\mathbf{r}(t)}{dt} \frac{dt}{d\ell} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

We conclude that $|\mathbf{u}(\ell)| = 1$. \hfill \qed
The arc length parametrization of a curve.

Example

Find a unit vector tangent to the curve given by \( \mathbf{r}(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \) for \( t \geq 0 \).
The arc length parametrization of a curve.

Example
Find a unit vector tangent to the curve given by
\[ r(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \text{ for } t \geq 0. \]

Solution: Reparametrize the curve using the arc length.
The arc length parametrization of a curve.

Example
Find a unit vector tangent to the curve given by \( r(t) = \langle 4 \cos(t), 4 \sin(t), 3t \rangle \) for \( t \geq 0 \).

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Therefore, a unit tangent vector is
\[ \mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell} \Rightarrow \mathbf{u}(\ell) = \left\langle -\frac{4}{5} \sin(\ell/5), \frac{4}{5} \cos(\ell/5), \frac{3}{5} \right\rangle. \]
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\[ \mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell} \Rightarrow \mathbf{u}(\ell) = \langle -\frac{4}{5} \sin(\ell/5), \frac{4}{5} \cos(\ell/5), \frac{3}{5} \rangle. \]

We can verify that this is a unit vector, since
\[ |\mathbf{u}(\ell)|^2 = \left(\frac{4}{5}\right)^2 \left[ \sin^2(\ell/5) + \cos^2(\ell/5) \right] + \left(\frac{3}{5}\right)^2 \Rightarrow |\mathbf{u}(\ell)| = 1. \]