Review for Exam 1.

- Sections 12.1-12.6.
- 50 minutes.
- 5 or 6 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Example

Consider the vectors $\mathbf{v} = 2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

1. Compute $\mathbf{v} \cdot \mathbf{w}$.

   Solution:

   $$\mathbf{v} \cdot \mathbf{w} = \langle 2, -2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 2 - 4 - 1 \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = -3.$$

   $\lhd$

2. Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

   $$|\mathbf{v}| = \sqrt{4 + 4 + 1} = 3, \quad |\mathbf{w}| = \sqrt{1 + 4 + 1} = \sqrt{6}.$$

   $$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{-3}{3\sqrt{6}} \quad \Rightarrow \quad \cos(\theta) = -\frac{1}{\sqrt{6}}.$$

   $\lhd$

Example

1. Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

   Solution:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{1+4+1} = \sqrt{6},$$

$$\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle.$$

2. Find $|\mathbf{u} - 2\mathbf{v}|$, where $\mathbf{u} = 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

   Solution: First: $\mathbf{u} - 2\mathbf{v} = \langle 1, 6, -1\rangle$. Then,

$$|\mathbf{u} - 2\mathbf{v}| = \sqrt{1+36+1}. \quad \Rightarrow \quad |\mathbf{u} - 2\mathbf{v}| = \sqrt{38}.$$

◁

Example

Find a unit vector $\mathbf{u}$ normal to both $\mathbf{v} = \langle 6, 2, -3\rangle$ and $\mathbf{w} = \langle -2, 2, 1\rangle$.

Solution:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ -2 & 2 & 1 \end{vmatrix} = (2+6)\mathbf{i} - (6-6)\mathbf{j} + (12+4)\mathbf{k} = \langle 8, 0, 16\rangle.$$

Since we look for a unit vector, the calculation is simpler with $\langle 1, 0, 2\rangle$ instead of $\langle 8, 0, 16\rangle$.

$$\mathbf{u} = \frac{\langle 1, 0, 2\rangle}{|\langle 1, 0, 2\rangle|} \quad \Rightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 0, 2\rangle.$$

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Example

Find the area of the parallelogram formed by **v** and **w** above.

Solution:
Since $\mathbf{v} \times \mathbf{w} = \langle 8, 0, 16 \rangle$, then

$$A = |\mathbf{v} \times \mathbf{w}| = |\langle 8, 0, 16 \rangle| = \sqrt{8^2 + 16^2} = \sqrt{8^2(1 + 4)}.$$

$$A = 8\sqrt{5}.$$

◁

Example

Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = \langle 6, 3, -1 \rangle$, $\mathbf{v} = \langle 0, 1, 2 \rangle$, and $\mathbf{w} = \langle 4, -2, 5 \rangle$.

Solution: We need to compute the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
We must start with the cross product.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = \langle (5 + 4), -(0 - 8), (0 - 4) \rangle$$

We obtain $\mathbf{v} \times \mathbf{w} = \langle 9, 8, -4 \rangle$. The triple product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle = 54 + 24 + 4 = 82.$$

Since $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$, we obtain $V = 82$.

Example

Does the line given by $\mathbf{r}(t) = \langle 0, 1, 1 \rangle + \langle 1, 2, 3 \rangle t$ intersects the plane given by $2x + y - z = 1$? If the answer is yes, then find the intersection point.

Solution: The line with parametric equation

$$x = t, \quad y = 1 + 2t, \quad z = 1 + 3t,$$

intersect the plane $2x + y - z = 1$ iff there is a solution t for the equation

$$2t + (1 + 2t) - (1 + 3t) = 1.$$

There is a solution given by $t = 1$. Therefore, the point of intersection has coordinates $x = 1$, $y = 3$, $z = 4$, then

$$P = (1, 3, 4).$$

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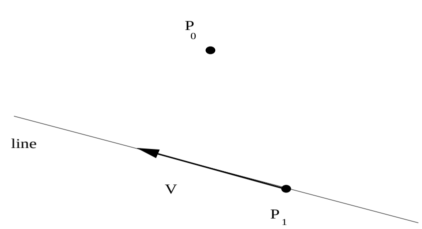
Example

Find the equation for the plane that contains the point $P_0 = (1, 2, 3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1 \rangle + \langle 1, 1, 2 \rangle t$.



A vector tangent to the line, and so to the plane, is $\mathbf{v} = \langle 1, 1, 2 \rangle$. The point $P_0 = (1, 2, 3)$ is in the plane. A second point in the plane is any point in the line, for example P_1 corresponding to the terminal point of $\mathbf{r}(0) = \langle -2, 0, -1 \rangle$.

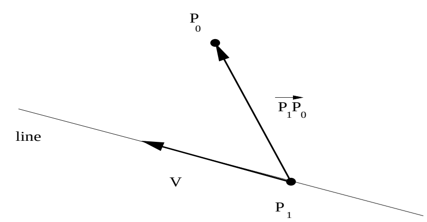
Then a second vector tangent to the plane is $\overrightarrow{P_1 P_0} = \langle 3, 2, 4 \rangle$.

Example

Find the equation for the plane that contains the point $P_0 = (1,2,3)$ and the line $x = -2 + t$, $y = t$, $z = -1 + 2t$.

Solution:

The vector equation of the line is $\mathbf{r}(t) = \langle -2, 0, -1\rangle + \langle 1, 1, 2\rangle t$, and a second vector tangent to the plane is $\overrightarrow{P_1P_0} = \langle 3, 2, 4\rangle$.



Then, a normal to the plane is given by

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle (4-4), -(4-6), (2-3)\rangle \quad \Rightarrow \quad \mathbf{n} = \langle 0, 2, -1\rangle.$$

So, the equation of the plane is

$$0\,(x-1) + 2(y-2) - (z-3) = 0, \quad \Rightarrow \quad 2y - z = 1.$$

Example

Find an equation for the plane that passes through the points $(1,1,1)$, $(1,-1,1)$, and $(0,0,2)$.

Solution: Denote $P = (1,1,1)$, $Q = (1,-1,1)$, and $R = (0,0,2)$. Then,

$$\vec{PQ} = \langle 0, -2, 0\rangle, \quad \vec{PR} = \langle -1, -1, 1\rangle,$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = (-2-0)\mathbf{i} - (0-0)\mathbf{j} + (0-2)\mathbf{k},$$

that is, $\vec{PQ} \times \vec{PR} = \langle -2, 0, -2\rangle$. Take $\mathbf{n} = \langle 2, 0, 2\rangle$.
With $\mathbf{n} = \langle 2, 0, 2\rangle$ and a point $R = (0,0,2)$, the equation of the plane is

$$2(x-0) + 0(y-0) + 2(z-2) = 0 \quad \Rightarrow \quad x + z = 2.$$

◁

Example

Find the equation of the plane that is parallel to the plane $x - 2y + 3z = 1$ and passes through the center of the sphere $x^2 + 2x + y^2 + z^2 - 2z = 0$.

Solution: The plane is parallel to the plane $x - 2y + 3z = 1$, so their normal vectors are parallel. We choose $\mathbf{n} = \langle 1, -2, 3 \rangle$.
We need to find the center of the sphere. We complete squares:

$$\begin{aligned} 0 &= x^2 + 2x + y^2 + z^2 - 2z \\ &= (x^2 + 2x + 1) - 1 + y^2 + (z^2 - 2z + 1) - 1 = 0 \\ &= (x+1)^2 + y^2 + (z-1)^2 - 2. \end{aligned}$$

Therefore, the center of the sphere is at $P_0 = (-1, 0, 1)$.
The equation of the plane is

$$(x + 1) - 2(y - 0) + 3(z - 1) = 0 \quad \Rightarrow \quad x - 2y + 3z = 2.$$

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Example

Find the angle between the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: The angle between the planes is the angle between their normal vectors.
The normal vectors are $\mathbf{n} = \langle 2, -3, 2 \rangle$, $\mathbf{N} = \langle 1, 2, 2 \rangle$.
The cosine of the angle θ between these vectors is

$$\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}|\,|\mathbf{N}|}.$$

Since $\mathbf{n} \cdot \mathbf{N} = 2 - 6 + 4 = 0$, we conclude that $\mathbf{n} \perp \mathbf{N}$.
The angle θ is $\theta = \pi/2$. ◁

Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: We first find the vector tangent to the line. This is a vector $\mathbf{v}$ that belongs to both planes.
This means that $\mathbf{v}$ is perpendicular to both normal vectors $\mathbf{n} = \langle 2, -3, 2 \rangle$ and $\mathbf{N} = \langle 1, 2, 2 \rangle$.
One such vector is

$$\mathbf{v} = \mathbf{n} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \langle (-6-4), -(4-2), (4+3) \rangle.$$

So, $\mathbf{v} = \langle -10, -2, 7 \rangle$.

Example

Find the vector equation for the line of intersection of the planes $2x - 3y + 2z = 1$ and $x + 2y + 2z = 5$.

Solution: Recall $\mathbf{v} = \langle -10, -2, 7 \rangle$. Now we need a point in the intersection of the planes. From the first plane we compute z as follows: $2z = 1 - 2x + 3y$.
We introduce this equation for $2z$ into the second plane:

$$x + 2y + (1 - 2x + 3y) = 5 \quad \Rightarrow \quad -x + 5y = 4.$$

We need just one solution, so we choose: $y = 0$, then $x = -4$, and this implies $z = 9/2$. A point in the intersection of the planes is $P_0 = (-4, 0, 9/2)$. The vector equation of the line is:

$$\mathbf{r}(t) = \langle -4, -0, 9/2 \rangle + \langle -10, -2, 7 \rangle\, t.$$

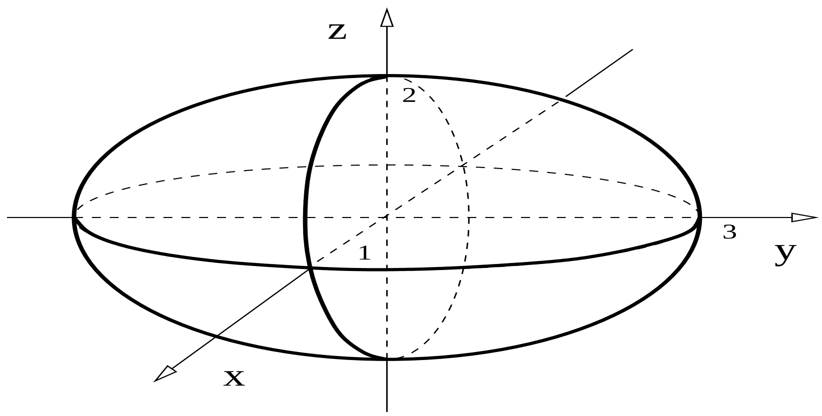
Example

Sketch the surface $36x^2 + 4y^2 + 9z^2 = 36$.

Solution: We first rewrite the equation above in the standard form

$$x^2 + \frac{4}{36}\,y^2 + \frac{9}{36}\,z^2 = 1 \quad \Leftrightarrow \quad x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1.$$

This is the equation of an ellipsoid with principal radius of length 1, 3, and 2 on the x, y and z axis, respectively. Therefore



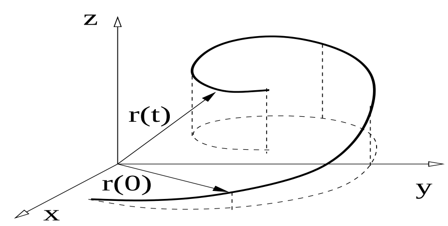
Vector functions (Sect. 13.1).

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- Limits and continuity of vector functions.
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.

Motion in space motivates to define vector functions.

Definition

A function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is called a *vector function*, where the interval $I \subset \mathbb{R}$ is called the *domain* of the function.



Remark: Given Cartesian coordinates in $\mathbb{R}^3$, the values of a vector function can be written in components as follows:

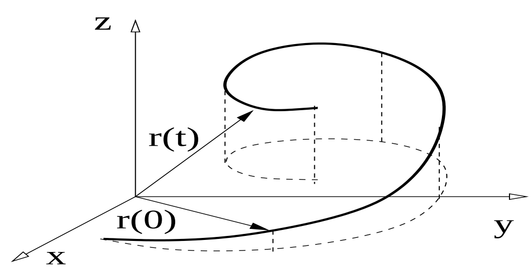
$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \qquad t \in I,$$

where $x(t)$, $y(t)$, and $z(t)$ are the values of three scalar functions.

Motion in space motivates to define vector functions.

Remarks:

- There is a natural association between a curve in $\mathbb{R}^n$ and the vector function values $\mathbf{r}(t)$.



- The curve is determined by the terminal points of the vector function values $\mathbf{r}(t)$.
- The independent variable t is called the parameter of the curve.

Vector functions.

Example

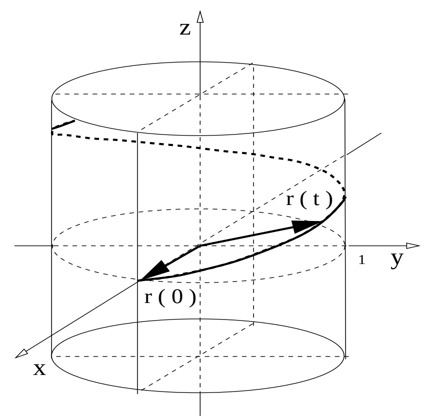
Graph the vector function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

The curve given by $\mathbf{r}(t)$ lies on a vertical cylinder with radius one, since

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.$$

The $z(t)$ coordinate of the curve increases with t, so the terminal point $\mathbf{r}(t)$ moves up on the cylinder surface when t increases. $\triangleleft$



Vector functions.

Example

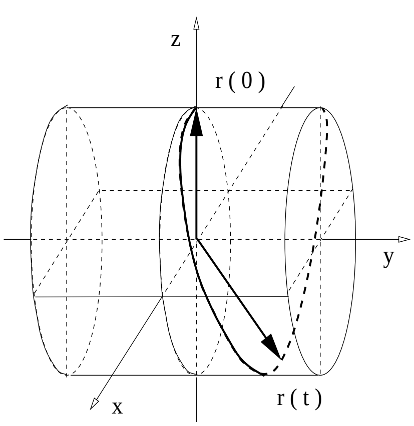
Graph the vector function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$.

Solution:

The curve given by $\mathbf{r}(t)$ lies on a horizontal cylinder with radius one, since

$$x^2 + z^2 = \sin^2(t) + \cos^2(t) = 1.$$

The $y(t)$ coordinate of the curve increases with t, so the terminal point $\mathbf{r}(t)$ moves to the right on the cylinder surface when t increases. $\triangleleft$



Vector functions (Sect. 13.1).

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- **Limits and continuity of vector functions.**
- Derivatives and motion.
- Differentiation rules.
- Integrals of vector functions.

Limits and continuity of vector functions.

Definition

The vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, has a *limit* given by the vector $\mathbf{L}$ when t approaches t_0, denoted as $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L}$, iff the following holds: For every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|t - t_0| < \delta \quad \Rightarrow \quad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

Remark:

- The limit of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ as $t \to t_0$ is the limit of its components $x(t)$, $y(t)$, $z(t)$ in Cartesian coordinates.
- That is:
  $\lim_{t\to t_0} \mathbf{r}(t) = \big\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\big\rangle.$

$\lim_{t\to t_0} \mathbf{r}(t) = \big\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\big\rangle.$

Example

Given $\mathbf{r}(t) = \langle \cos(t), \sin(t)/t, t^2 + 2\rangle$, compute $\lim_{t\to 0} \mathbf{r}(t)$.

Solution:

Notice that the vector function $\mathbf{r}$ is not defined at $t = 0$, however its limit at $t = 0$ exists. Indeed,

$$\begin{aligned}
\lim_{t\to 0} \mathbf{r}(t) &= \lim_{t\to 0}\Big\langle \cos(t), \frac{\sin(t)}{t}, t^2 + 2\Big\rangle \\
&= \Big\langle \lim_{t\to 0} \cos(t), \lim_{t\to 0} \frac{\sin(t)}{t}, \lim_{t\to 0}(t^2 + 2)\Big\rangle \\
&= \langle 1, 1, 2\rangle.
\end{aligned}$$

We conclude that $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 1, 2\rangle$. $\lhd$

Limits and continuity of vector functions.

Definition

A vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is *continuous at* $t = t_0 \in I$ iff holds $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function $\mathbf{r} : I \to \mathbb{R}^n$ is *continuous* if it is continuous at every t in its domain interval I.

Remark: A vector function with Cartesian components $\mathbf{r} = \langle x, y, z\rangle$ is continuous iff each component is continuous.

Example

The function $\mathbf{r}(t) = \langle \sin(t), t, \cos(t)\rangle$ is continuous for $t \in \mathbb{R}$. $\lhd$

Remark: Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.

Vector functions (Sect. 13.1).

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- Limits and continuity of vector functions.
- **Derivatives and motion.**
- Differentiation rules.
- Integrals of vector functions.

Derivatives and motion.

Definition

The vector function $\mathbf{r} : I \to \mathbb{R}^n$, with $n = 2, 3$, is *differentiable at* $t = t_0$, denoted as $\mathbf{r}'(t)$ or $\dfrac{d\mathbf{r}}{dt}$, iff the following limit exists,

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Remarks:

- A vector function $\mathbf{r} : I \to \mathbb{R}^n$ is *differentiable* if it is differentiable for each $t \in I$.
- If a vector function with Cartesian components $\mathbf{r} = \langle x, y, z \rangle$ is differentiable, then

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Derivatives and motion.

Theorem

If a vector function with Cartesian components $\mathbf{r} = \langle x, y, z \rangle$ *is differentiable, then* $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

Proof.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}, \\
&= \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle \\
&= \left\langle \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h \to 0} \frac{z(t+h) - z(t)}{h} \right\rangle \\
&= \langle x'(t), y'(t), z'(t) \rangle.
\end{aligned}$$

□

Derivatives and motion.

Example

Find the derivative of the vector function
$\mathbf{r}(t) = \langle \cos(t), \sin(t), (t^2 + 3t - 1) \rangle$.

Solution: We differentiate each component of $\mathbf{r}$, that is,

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), (2t + 3) \rangle.$$

Example

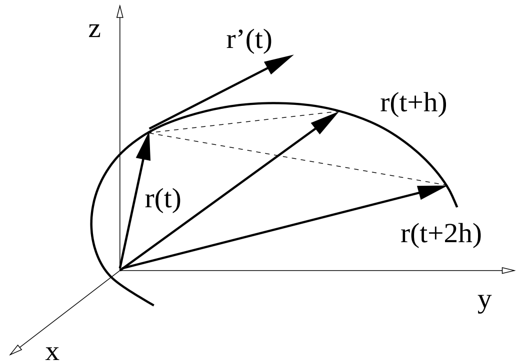
Find the derivative of the vector function $\mathbf{r}(t) = \langle \cos(2t), e^{3t}, 1/t \rangle$.

Solution: We differentiate each component of $\mathbf{r}$, that is,

$$\mathbf{r}'(t) = \langle -2\sin(2t), 3e^{3t}, -1/t^2 \rangle.$$

Geometrical property of the derivative.

Remark: The vector $\mathbf{r}'(t)$ is tangent to the curve given by $\mathbf{r}$ at the point $\mathbf{r}(t)$.



Remark: If $\mathbf{r}(t)$ represents the vector position of a particle, then:

- The derivative of the position function is the velocity function, $\mathbf{v}(t) = \mathbf{r}'(t)$. The speed is $|\mathbf{v}(t)|$.
- The derivative of the velocity function is the acceleration function, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Derivatives and motion.

Example

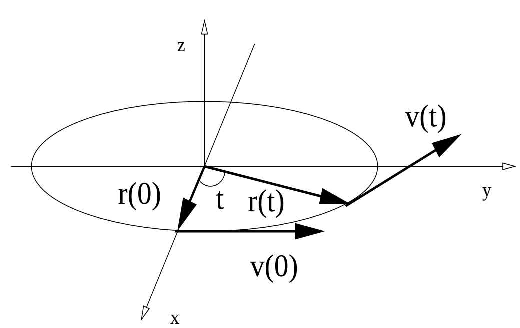
Compute the derivative of the position function $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Graph the curve given by $\mathbf{r}$, and explicitly show the position vector $\mathbf{r}(0)$ and velocity vector $\mathbf{v}(0)$.

Solution:

The derivative of $\mathbf{r}$ is:

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 0 \rangle.$$

$\mathbf{r}(0) = \langle 1, 0, 0 \rangle$, $\mathbf{v}(0) = \langle 0, 1, 0 \rangle$.



Differentiation rules are the same as for scalar functions

Theorem

If $\mathbf{v}$ *and* $\mathbf{w}$ *are differentiable vector functions, then holds:*

- $[\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)$, *(addition);*
- $[c\mathbf{v}(t)]' = c\mathbf{v}'(t)$, *(product rule);*
- $[\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)$, *(chain rule);*
- $[f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$, *(product rule);*
- $[\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$, *(dot product);*
- $[\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)$, *(cross product).*

Higher derivatives can also be computed.

Remark: The m-derivative of a vector function $\mathbf{r}$ is denoted as $\mathbf{r}^{(m)}$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$.

Example

Compute the third derivative of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle$.

Solution:

$$\begin{aligned} \mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\ \mathbf{r}^{(2)}(t) &= (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\ \mathbf{r}^{(3)}(t) &= (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle. \end{aligned}$$

$\triangleleft$

Recall: If $\mathbf{r}(t)$ is the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $\mathbf{a}(t) = \mathbf{r}^{(2)}(t)$ is the acceleration of the particle.

Vector functions (Sect. 13.1).

- Definition of vector functions: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
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- **Integrals of vector functions.**

Integrals of vector functions.

Definition

The *indefinite integral*, also called the *antiderivative*, of a vector function $\mathbf{v}$ is denoted as $\int \mathbf{v}(t)\,dt$ and given by

$$\int \mathbf{v}(t)\,dt = \mathbf{V}(t) + \mathbf{C},$$

where $\mathbf{V}'(t) = \mathbf{v}(t)$ and $\mathbf{C}$ is a constant vector.

Example

Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t)\rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1\rangle$.

Solution: The position function is the primitive of the velocity function, $\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C}$, that satisfies the initial condition $\mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C}$. This initial condition fixes the constant vector $\mathbf{C}$.

Integrals of vector functions.

Example

Find the position function $\mathbf{r}$ knowing that the velocity function is $\mathbf{v}(t) = \langle 2t, \cos(t), \sin(t)\rangle$ and the initial position is $\mathbf{r}(0) = \langle 1, 1, 1\rangle$.

Solution: The position function is a primitive of the velocity,

$$\mathbf{r}(t) = \mathbf{V}(t) + \mathbf{C} = \langle t^2, \sin(t), -\cos(t)\rangle + \langle c_x, c_y, c_z\rangle,$$

with $\mathbf{C} = \langle c_x, c_y, c_z\rangle$ a constant vector. The initial condition determines the vector $\mathbf{C}$:

$$\langle 1, 1, 1\rangle = \mathbf{r}(0) = \mathbf{V}(0) + \mathbf{C} = \langle 0, 0, -1\rangle + \langle c_x, c_y, c_z\rangle,$$

that is, $c_x = 1$, $c_y = 1$, $c_z = 2$.
The position function is $\mathbf{r}(t) = \langle t^2 + 1, \sin(t) + 1, -\cos(t) + 2\rangle$. $\lhd$

Integrals of vector functions.

Example

Find the position function of a particle with acceleration $\mathbf{a}(t) = \langle 0, 0, -10\rangle$ having an initial velocity $\mathbf{v}(0) = \langle 0, 1, 1\rangle$ and initial position $\mathbf{r}(0) = \langle 1, 0, 1\rangle$.

Solution: The velocity is $\mathbf{v}(t) = \langle v_{0x}, v_{0y}, (-10t + v_{0z})\rangle$.
The initial condition implies $\langle 0, 1, 1\rangle = \mathbf{v}(0) = \langle v_{0x}, v_{0y}, v_{0z}\rangle$, that is $v_{0x} = 0$, $v_{0y} = 1$, $v_{0z} = 1$. The velocity function is

$$\mathbf{v}(t) = \langle 0, 1, (-10t + 1)\rangle.$$

The position is $\mathbf{r}(t) = \langle r_{0x}, (t + r_{0y}), (-5t^2 + t + r_{0z})\rangle$.
The initial condition implies $\langle 1, 0, 1\rangle = \mathbf{r}(0) = \langle r_{0x}, r_{0y}, r_{0z}\rangle$, that is $r_{0x} = 1$, $r_{0y} = 0$, $r_{0z} = 1$. The velocity function is

$$\mathbf{r}(t) = \langle 1, t, (-5t^2 + t + 1)\rangle.$$

$\lhd$

Integrals of vector functions.

Definition

If the components of $\mathbf{r}(t) = \langle \mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t) \rangle$ are integrable functions on the interval $[a, b]$, then the *definite integral* of $\mathbf{r}$ is given by

$$\int_a^b \mathbf{r}(t)dt = \Big\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \Big\rangle.$$

Example

Compute $\int_0^\pi \mathbf{r}(t)\,dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Integrals of vector functions.

Example

Compute $\int_0^\pi \mathbf{r}(t)\,dt$ for the function $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:

$$\begin{aligned}
\int_0^\pi \mathbf{r}(t)\,dt &= \int_0^\pi \langle \cos(t), \sin(t), t \rangle dt \\
&= \Big\langle \int_0^\pi \cos(t)dt, \int_0^\pi \sin(t)dt, \int_0^\pi t dt \Big\rangle, \\
&= \Big\langle \sin(t)\Big|_0^\pi, -\cos(t)\Big|_0^\pi, \frac{t^2}{2}\Big|_0^\pi, \Big\rangle \\
&= \Big\langle 0, 2, \frac{\pi^2}{2} \Big\rangle, \quad \Rightarrow \quad \int_0^\pi \mathbf{r}(t)\,dt = \Big\langle 0, 2, \frac{\pi^2}{2} \Big\rangle.
\end{aligned}$$

◁

The arc length of a curve in space (Sect. 13.3).

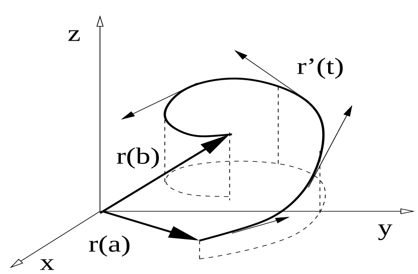
- The arc length of a curve in space.
- The arc length function.
- Parametrizations of a curve.
- The arc length parametrization of a curve.

The length of a curve is called its arc length.

Definition

The arc length of a continuously differentiable curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$, with n=2,3, is the number given by

$$\ell_{ba} = \int_a^b |\mathbf{r}'(t)|\, dt.$$



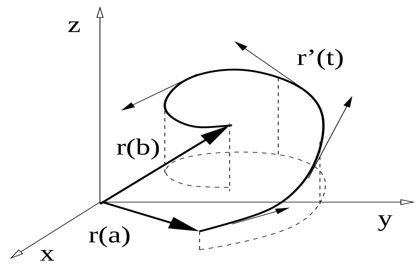
Remark:

- If the curve $\mathbf{r}$ is the path traveled by a particle in space, then $\mathbf{r}' = \mathbf{v}$ is the velocity of the particle.
- The arc length is the integral in time of the particle speed $|\mathbf{v}(t)|$.
- Therefore, the arc length of the curve is the distance traveled by the particle.

The length of a curve is called its arc length.

Recall:
The arc length of a curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$

$$\ell_{ba} = \int_a^b \left|\mathbf{r}'(t)\right| \, dt.$$



Remark:
In Cartesian coordinates the functions $\mathbf{r}$ and $\mathbf{r}'$ are given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \qquad \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Therefore the arc length of the curve is given by the expression

$$\ell_{ba} = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2} \, dt.$$

The arc length of a curve in a plane.

Example
Find the arc length of the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $t \in [\pi/4, 3\pi/4]$.

Solution: The derivative vector function is $\mathbf{r}'(t) = \langle -\sin(t), \cos(t)\rangle$. The arc length formula is

$$\ell = \int_{\pi/4}^{3\pi/4} \sqrt{\left[-\sin(t)\right]^2 + \left[\cos(t)\right]^2} \, dt$$

$$= \int_{\pi/4}^{3\pi/4} dt \quad \Rightarrow \quad \ell = \frac{\pi}{2}.$$

This result is reasonable, since the curve is a circle and we are computing the length of quarter a circle. ◁

The arc length of a curve in a plane.

Example

Find the arc length of the spiral $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$, for $t \in [0, t_0]$.

Solution: The derivative vector is

$$\mathbf{r}'(t) = \big\langle \big[-t\sin(t) + \cos(t)\big], \big[t\cos(t) + \sin(t)\big]\big\rangle.$$

$$|\mathbf{r}'(t)|^2 = \big[t^2\sin^2(t) + \cos^2(t) - 2t\sin(t)\cos(t)\big] + \big[t^2\cos^2(t) + \sin^2(t) + 2t\sin(t)\cos(t)\big] = t^2 + 1.$$

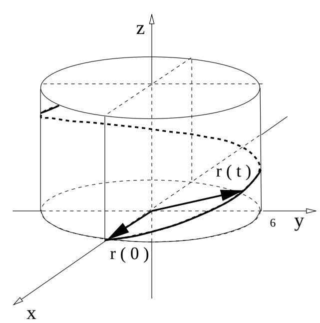
The arc length is $\ell(t_0) = \displaystyle\int_0^{t_0} \sqrt{1+t^2}\,dt = \ln\big(t + \sqrt{1+t^2}\big)\Big|_0^{t_0}$.

We conclude: $\ell(t_0) = \ln\big(t_0 + \sqrt{1+t_0^2}\big)$. ◁

The arc length of a curve in space.

Example

Find the arc length of $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, for $t \in [0, \pi]$.



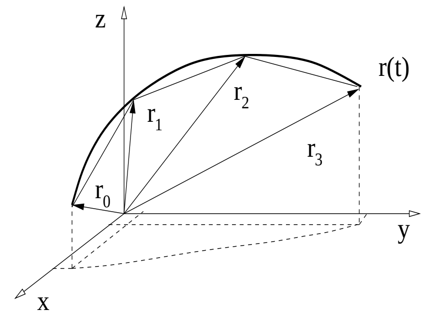
Solution: The derivative vector is

$$\mathbf{r}'(t) = \langle -12\sin(2t), 12\cos(2t), 5\rangle,$$
$$|\mathbf{r}'(t)|^2 = 144\big[\sin^2(2t) + \cos^2(2t)\big] + 25 = 169 = (13)^2.$$

The arc length is $\ell = \int_0^{\pi} 13\,dt = 13\,t\big|_0^{\pi} \quad\Rightarrow\quad \ell = 13\pi.$ ◁

Idea behind the arc length formula.

The arc length formula can be obtained as a limit procedure One adds up the lengths of a polygonal line that approximates the original curve.



$$\ell_N = \sum_{n=0}^{N-1} |\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)|, \qquad \{a = t_0, t_1, \cdots, t_{N-1}, t_N = b\},$$

$$\simeq \sum_{n=0}^{N-1} |\mathbf{r}'(t_n)|\,(t_{n+1} - t_n) \xrightarrow{N\to\infty} \int_a^b |\mathbf{r}'(t)|\,dt$$

The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- **The arc length function.**
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The arc length function.

Definition

function. The arc *length function* of a continuously differentiable vector function **r** is given by

$$\ell(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau.$$

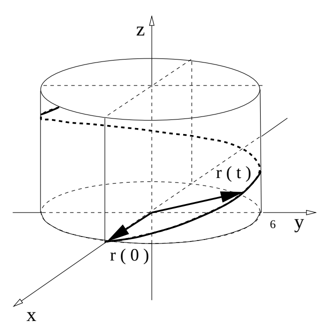
Remarks:

- The value $\ell(t)$ of the arc length function represents the length along the curve **r** from t_0 to t.
- If the function **r** is the position of a moving particle as function of time, then the arc length $\ell(t)$ is the distance traveled by the particle from the time t_0 to t.

The arc length function.

Example

Find the arc length function for the curve $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t\rangle$, starting at $t = 0$.



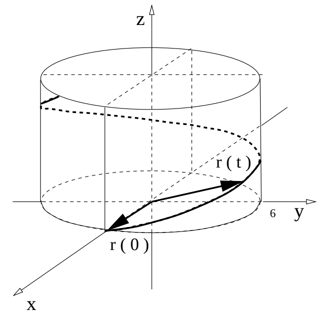
Solution: We have found that $|\mathbf{r}'(t)| = 13$. Therefore,

$$\ell(t) = \int_0^t 13\, d\tau \quad \Rightarrow \quad \ell(t) = 13\, t.$$

The arc length function.

Example

Given the position function in time $\mathbf{r}(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$, find the position vector $\mathbf{r}(t_0)$ located at a length $\ell_0 = 20$ from the initial position $\mathbf{r}(0)$.



Solution: We have found that the arc length function for the vector function $\mathbf{r}$ is $\ell(t) = 13\,t$.
Since $t = \ell/13$, the time at $\ell = \ell_0 = 20$ is $t_0 = 13/20$.
Therefore, the position vector at $\ell_0 = 20$ is given by

$$\mathbf{r}(t_0) = \langle 6\cos(13/10), 6\sin(13/10), 13/4 \rangle.$$

◁

The arc length of a curve in space (Sect. 13.3).

- The arc length of a curve in space.
- The arc length function.
- **Parametrizations of a curve.**
- The arc length parametrization of a curve.

Parametrizations of a curve.

Remark:
A curve in space can be represented by different vector functions.

Example
The unit circle in $\mathbb{R}^2$ is the curve represented by the following vector functions:

- $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$;
- $\mathbf{r}_2(t) = \langle \cos(5t), \sin(5t) \rangle$;
- $\mathbf{r}_3(t) = \langle \cos(e^t), \sin(e^t) \rangle$.

Remark:
The curve in space is the same for all three functions above. The vector $\mathbf{r}$ moves along the curve at different speeds for the different parametrizations.

Parametrizations of a curve.

Remarks:

- If the vector function $\mathbf{r}$ represents the position in space of a moving particle, then there is a preferred parameter to describe the motion: The time t.
- Another parameter that is useful to describe a moving particle is the distance traveled by the particle, the arc length ℓ.
- A common problem is the following: Given a vector function parametrized by the time t, switch the curve parameter to the arc length ℓ.
- The problem above is called the arc length parametrization of a curve.

The arc length of a curve in space (Sect. 13.3).

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The arc length parametrization of a curve.

Problem:
Given vector function $\mathbf{r}$ in terms of a parameter t, find the arc length parametrization of that curve.

Solution:

- With the function values $\mathbf{r}(t)$ compute the arc length function $\ell(t)$, starting at some $t = t_0$.
- Invert the function values $\ell(t)$ to find the function values $t(\ell)$.
- Example: If $\ell(t) = 3e^{t/2}$, then $t(\ell) = 2\ln(\ell/3)$.
- Compute the composition function $\mathbf{r}(\ell) = \mathbf{r}(t(\ell))$. That is, replace t by $t(\ell)$ in the function values $\mathbf{r}(t)$.

The function values $\mathbf{r}(\ell)$ are the parametrization of the function values $\mathbf{r}(t)$ using the arc length as the new parameter.

The arc length parametrization of a curve.

Example

Find the arc length parametrization of the vector function $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ starting at $t = 0$.

Solution: First find the derivative function:

$$\mathbf{r}'(t) = \langle -4\sin(t), 4\cos(t), 3\rangle.$$

Hence, $|\mathbf{r}'(t)|^2 = 4^2\sin^2(t) + 4^2\cos^2(t) + 3^2 = 16 + 9 = 5^2$.

Find the arc length function: $\ell(t) = \int_0^t 5\, d\tau \quad \Rightarrow \quad \ell(t) = 5t$.

Invert the equation above: $t = \ell/5$.

Reparametrize the original curve:

$$\mathbf{r}(\ell) = \langle 4\cos(\ell/5), 4\sin(\ell/5), 3\ell/5\rangle.$$

The arc length parametrization of a curve.

Theorem

A unit tangent vector to a curve given by the vector function values $\mathbf{r}(t)$ *is given by* $\mathbf{u}(\ell) = \dfrac{d\mathbf{r}}{d\ell}$, *where* ℓ *is the arc length of the curve.*

Proof.

Given the function values $\mathbf{r}(t)$, let $\mathbf{r}(\ell)$ be the reparametrization of $\mathbf{r}(t)$ with the arc length function $\ell(t) = \int_{t_0}^t |\mathbf{r}'(\tau)|\, d\tau$.

Notice that $\dfrac{d\ell}{dt} = |\mathbf{r}'(t)|$ and $\dfrac{dt}{d\ell} = \dfrac{1}{|\mathbf{r}'(t)|}$.

Therefore, $\mathbf{u}(\ell) = \dfrac{d\mathbf{r}(\ell)}{d\ell} = \dfrac{d\mathbf{r}(t)}{dt}\,\dfrac{dt}{d\ell} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

We conclude that $|\mathbf{u}(\ell)| = 1$. $\square$

The arc length parametrization of a curve.

Example

Find a unit vector tangent to the curve given by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ for $t \geqslant 0$.

Solution: Reparametrize the curve using the arc length. We get

$$\mathbf{r}(\ell) = \langle 4\cos(\ell/5), 4\sin(\ell/5), 3\ell/5\rangle.$$

Therefore, a unit tangent vector is

$$\mathbf{u}(\ell) = \frac{d\mathbf{r}}{d\ell} \quad \Rightarrow \quad \mathbf{u}(\ell) = \Big\langle -\frac{4}{5}\sin(\ell/5), \frac{4}{5}\cos(\ell/5), \frac{3}{5}\Big\rangle.$$

$\lhd$

We can verify that this is a unit vector, since

$$|\mathbf{u}(\ell)|^2 = \Big(\frac{4}{5}\Big)^2 \big[\sin^2(\ell/5) + \cos^2(\ell/5)\big] + \Big(\frac{3}{5}\Big)^2 \quad \Rightarrow \quad |\mathbf{u}(\ell)| = 1.$$