Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

There are two main ways to introduce the dot product

Geometrical definition $\rightarrow$ Properties $\rightarrow$ Expression in components.

Geometrical expression $\leftarrow$ Properties $\leftarrow$ Definition in components.

We choose the first way, the textbook chooses the second way.

Dot product and vector projections (Sect. 12.3)

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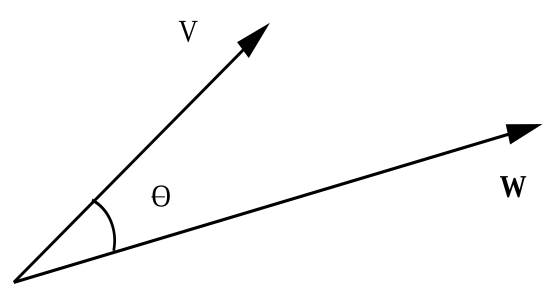
The dot product of two vectors is a scalar

Definition

Let $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^n$, with $n = 2, 3$, having length $|\mathbf{v}|$ and $|\mathbf{w}|$ with angle in between θ, where $0 \leq \theta \leq \pi$. The *dot product* of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \cdot \mathbf{w}$, is given by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos(\theta).$$



Initial points together.

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

$\triangleleft$

- The angle between two vectors is a usually not know in applications.
- It will be convenient to obtain a formula for the dot product involving the vector components.

Dot product and vector projections (Sect. 12.3)

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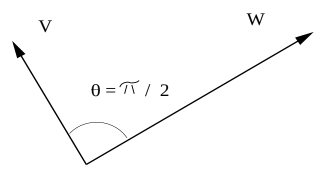
Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \cos(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right. \Leftrightarrow \quad \theta = \frac{\pi}{2}.$$

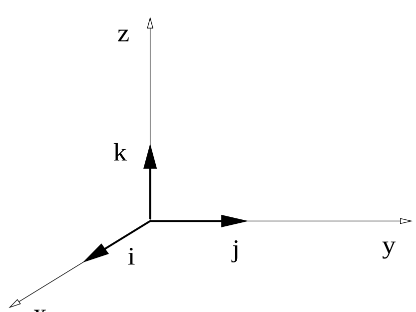
□

The dot product of i, j and k is simple to compute

Example

Compute all dot products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\begin{array}{ccc} \mathbf{i} \cdot \mathbf{i} = 1, & \mathbf{j} \cdot \mathbf{j} = 1, & \mathbf{k} \cdot \mathbf{k} = 1, \\ \mathbf{i} \cdot \mathbf{j} = 0, & \mathbf{j} \cdot \mathbf{i} = 0, & \mathbf{k} \cdot \mathbf{i} = 0, \\ \mathbf{i} \cdot \mathbf{k} = 0, & \mathbf{j} \cdot \mathbf{k} = 0, & \mathbf{k} \cdot \mathbf{j} = 0. \end{array}$$

◁

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
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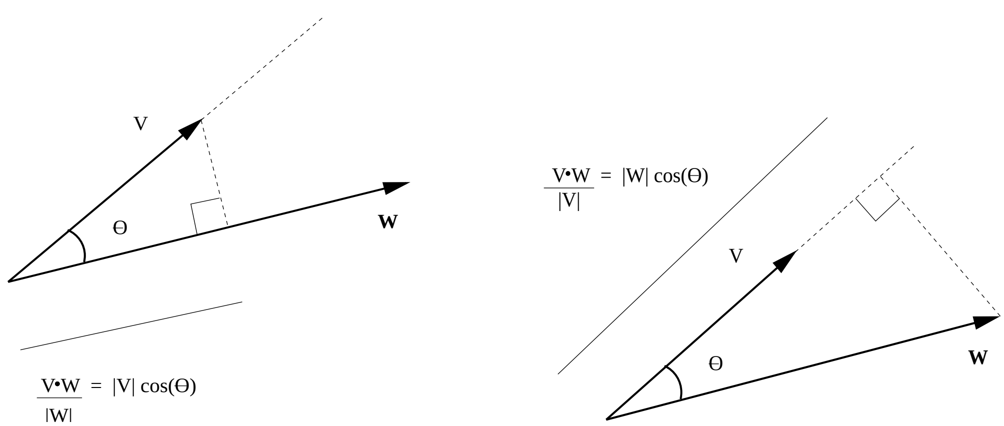
The dot product and orthogonal projections.

The dot product is closely related to orthogonal projections of one vector onto the other. Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.



Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
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Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric);*

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear);*

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear);*

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive);*

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

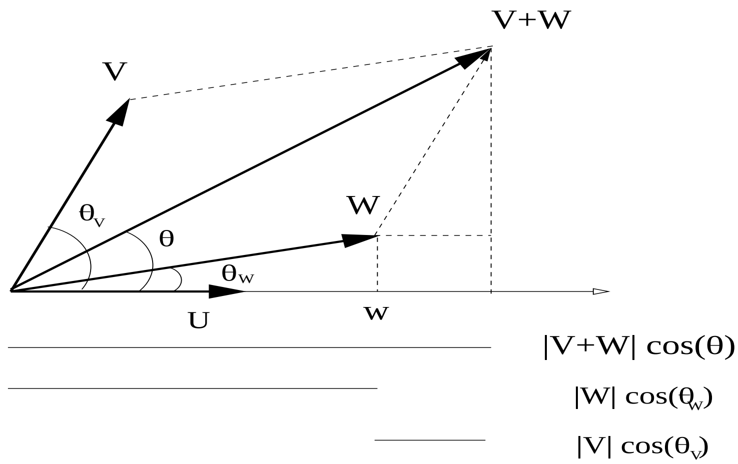
Proof.
Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

$$\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta) = a\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = a\,(\mathbf{v} \cdot \mathbf{w}).$$

□

Properties of the dot product.

(c), $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$, is non-trivial. The proof is:



$$\left.\begin{aligned} |\mathbf{v}+\mathbf{w}|\cos(\theta) &= \frac{\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})}{|\mathbf{u}|},\\ |\mathbf{w}|\cos(\theta_w) &= \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}|},\\ |\mathbf{v}|\cos(\theta_v) &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}, \end{aligned}\right\} \Rightarrow \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$$

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
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The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x\, \mathbf{i} + v_y\, \mathbf{j}$ and $\mathbf{w} = w_x\, \mathbf{i} + w_y\, \mathbf{j}$. The linear property of the dot product implies

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (v_x\, \mathbf{i} + v_y\, \mathbf{j}) \cdot (w_x\, \mathbf{i} + w_y\, \mathbf{j}) \\ &= v_x w_x\, \mathbf{i} \cdot \mathbf{i} + v_x w_y\, \mathbf{i} \cdot \mathbf{j} + v_y w_x\, \mathbf{j} \cdot \mathbf{i} + v_y w_y\, \mathbf{j} \cdot \mathbf{j}. \end{aligned}$$

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. We conclude that

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

□

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.
- The geometrical meaning of the dot product is simple to see from the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta)$.

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1)(2) + (2)(1) \\ |\mathbf{v}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |\mathbf{w}| &= \sqrt{2^2 + 1^2} = \sqrt{5}, \end{aligned}\right\} \quad \Rightarrow \quad \cos(\theta) = \frac{4}{5}.$$

$\triangleleft$

Dot product and vector projections (Sect. 12.3)

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Scalar and vector projection formulas.

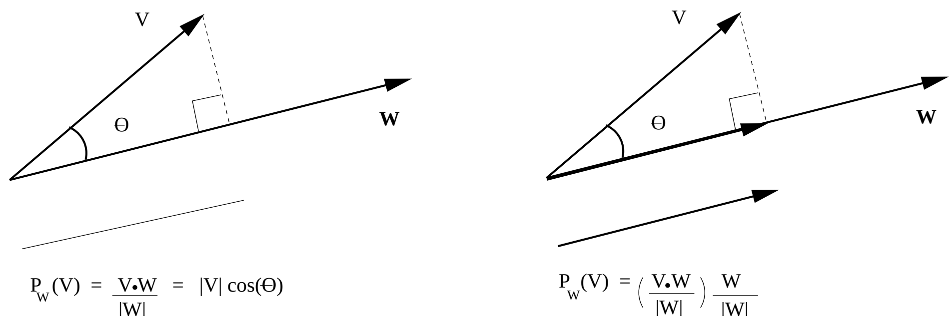
Theorem

The scalar projection of vector $\mathbf{v}$ *along the vector* $\mathbf{w}$ *is the number* $p_w(v)$ *given by*

$$p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of vector $\mathbf{v}$ *along the vector* $\mathbf{w}$ *is the vector* $\mathbf{p}_w(v)$ *given by*

$$\mathbf{p}_w(v) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|}.$$



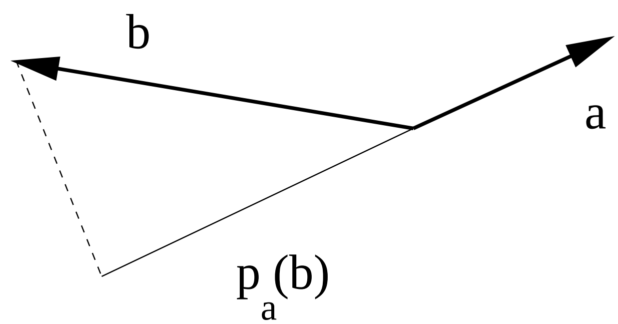
Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1\rangle$ onto $\mathbf{a} = \langle 1, 2\rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

We therefore obtain $p_a(b) = -\frac{2}{\sqrt{5}}$.



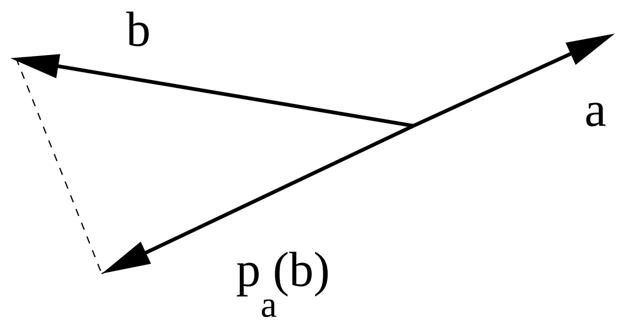
Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(-\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle,$$

we therefore obtain $\mathbf{p}_a(b) = -\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.



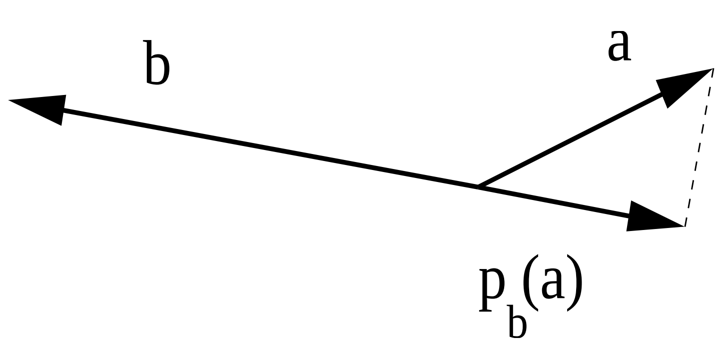
Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(-\frac{2}{\sqrt{17}} \right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle,$$

we therefore obtain $\mathbf{p}_a(b) = \left\langle \frac{8}{17}, -\frac{2}{17} \right\rangle$.



Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Parallel vectors.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

There are two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.
Geometrical expression	$\longleftarrow$	Properties	$\longleftarrow$	Definition in components.

We choose the first way, like the textbook.

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
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- Parallel vectors.
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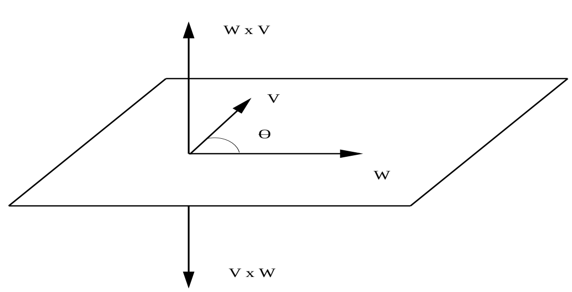
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

The cross product of two vectors is another vector

Definition

Let $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$ having length $|\mathbf{v}|$ and $|\mathbf{w}|$ with angle in between θ, where $0 \leq \theta \leq \pi$. The *cross product* of $\mathbf{v}$ and $\mathbf{w}$, denoted as $\mathbf{v} \times \mathbf{w}$, is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin(\theta).$$



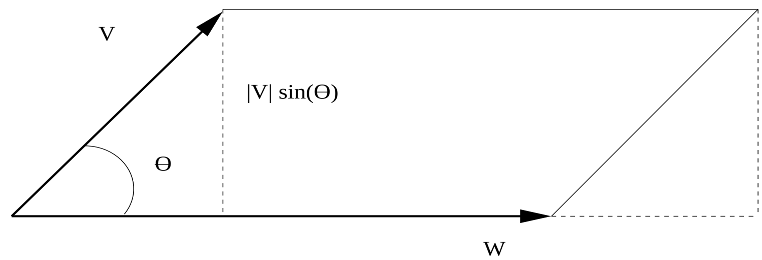
Cross product vectors are perpendicular to the original vectors.

$|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$.

Proof.



The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is given by

$$A = |\mathbf{w}|\big(|\mathbf{v}|\,\sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

□

Cross product and determinants (Sect. 12.4)

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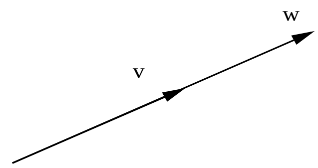
Parallel vectors have zero cross product.

Definition
Two vectors are *parallel* iff the angle in between them is $\theta = 0$.



Theorem
The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel iff* $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

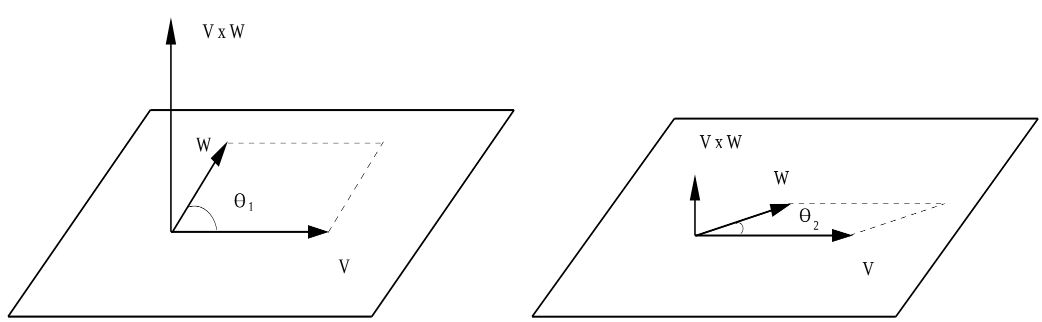
Proof.
Recall: Vector $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ iff its length $|\mathbf{v} \times \mathbf{w}| = 0$, then

$$\left.\begin{array}{r} |\mathbf{v}|\,|\mathbf{w}|\,\sin(\theta) = 0 \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \sin(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{c} \theta = 0, \\ \text{or} \\ \theta = \pi. \end{array}\right.$$

□

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram

Example
The closer the vectors $\mathbf{v}$, $\mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.

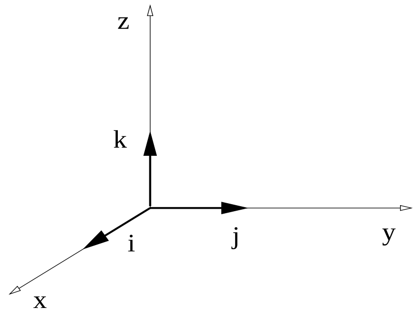


◁

Example

Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

◁

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Parallel vectors.
- **Properties of the cross product.**
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

Main properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(Skew-symmetric);*

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\,\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\,\mathbf{w}) = a\,(\mathbf{v} \times \mathbf{w})$, *(linear);*

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear);*

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative).*

Proof.

Part (a) results from the right hand rule. Part (b) comes from part (a). Parts (b) and (c) are proven in a similar ways as the linear property of the dot product. Part (d) is proven by giving an example. □

The cross product is not associative, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Example

Show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$.

Solution:

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

◁

Recall: The cross product of two vectors vanishes when the vectors are parallel

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Parallel vectors.
- Properties of the cross product.
- **Cross product in vector components.**
- Determinants to compute cross products.
- Triple product and volumes.

The cross product vector in vector components.

Theorem

If the vector components of $\mathbf{v}$ *and* $\mathbf{w}$ *in a Cartesian coordinate system are* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$*, then holds*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

For the proof, recall the non-zero cross products

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and their skew-symmetric products, while all the other cross products vanish, and then use the properties of the cross product.

Cross product in vector components.

Proof.
Recall:

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}, \qquad \mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}.$$

Then, it holds

$$\mathbf{v} \times \mathbf{w} = (v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) \times (w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}).$$

Use the linearity property. The only non-zero terms are those with products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. The result is

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

□

Cross product in vector components.

Example
Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0\rangle$ and $\mathbf{w} = \langle 3, 2, 1\rangle$,

Solution: We use the formula

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1)\rangle \\ &= \langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)]\rangle \\ &= \langle (2 - 0), (-1), (2 - 6)\rangle \quad \Rightarrow \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4\rangle.\end{aligned}$$

◁

Exercise: Find the angle between **v** and **w** above, and then check that this angle is correct using the dot product of these vectors.

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Parallel vectors.
- Properties of the cross product.
- Cross product in vector components.
- **Determinants to compute cross products.**
- Triple product and volumes.

Determinants help to compute cross products.

We use determinants only as a tool to remember the components of $\mathbf{v} \times \mathbf{w}$. Let us recall here the definition of determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix can be computed using three 2×2 determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Determinants help to compute cross products.

Claim

If the vector components of $\mathbf{v}$ and $\mathbf{w}$ in a Cartesian coordinate system are $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then holds

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

A straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} - (v_1 w_3 - v_3 w_1)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

Determinants help to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9 - 2)\,\mathbf{i} - (-6 - 1)\,\mathbf{j} + (-4 - 3)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$

From the properties of the determinant we know that $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, therefore $\mathbf{v} \times \mathbf{w} = \langle -7, -7, 7 \rangle$. $\triangleleft$

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Parallel vectors.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- **Triple product and volumes.**

The triple product of three vectors is a number

Definition
Given vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, the triple product is the number given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The parentheses are important. First do the cross product, and only then dot the resulting vector with the first vector.

Property of the triple product.

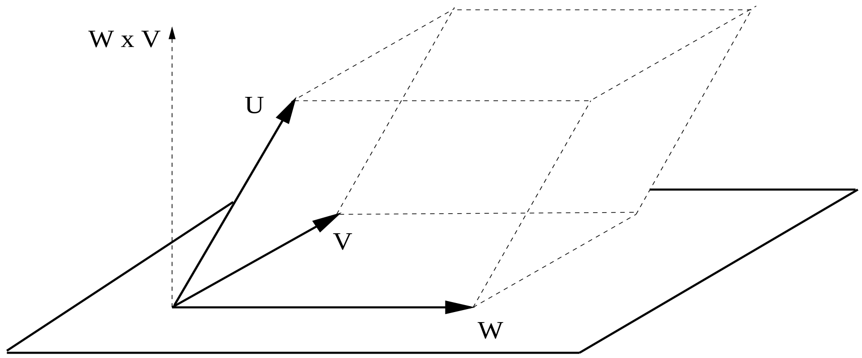
Theorem
The triple product of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *satisfies*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

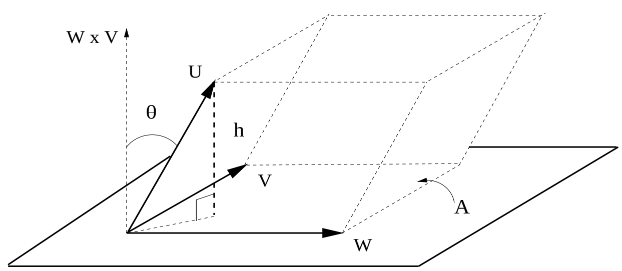
The triple product is related to the volume of the parallelepiped formed by the three vectors

Theorem

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *are vectors in* $\mathbb{R}^3$, *then* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



The triple product and volumes



Proof.

Recall the definition of a dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,\cos(\theta) = h\,|\mathbf{v} \times \mathbf{w}|.$$

$|\mathbf{v} \times \mathbf{w}|$ is the area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h\,A,$$

which is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. □

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. We conclude that $V = 24$. $\triangleleft$

The triple product is computed with a determinant

Theorem

The triple product of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}.$$

The triple product is computed with a determinant

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}.$$

The result is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (1)(2+2) - (2)(3-1) + (3)(-6-2), = 4 - 4 - 24,$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. We conclude that $V = 24$. $\triangleleft$