Review for Exam 4.

- Sections 16.1-16.5, 16.7, 16.8.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- **(16.3) Conservative fields, potential functions.**
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

Conservative fields, potential functions (16.3).

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Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
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$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

Conservative fields, potential functions (16.3).

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Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
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$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y \cos(z) = \partial_z F_x,$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
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$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y \cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Conservative fields, potential functions (16.3).

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$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field,

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
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$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$. The equations for f are

$$\partial_x f = y\sin(z), \quad \partial_y f = x\sin(z), \quad \partial_z f = xy\cos(z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

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Integrating in x the first equation we get

$$f(x, y, z) = xy\sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z)$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
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Integrating in x the first equation we get

$$f(x, y, z) = xy\sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y, z) = 0,$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
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Solution: $\partial_x f = y \sin(z)$, $\partial_y f = x \sin(z)$, $\partial_z f = xy \cos(z)$.
Integrating in x the first equation we get

$$f(x, y, z) = xy \sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x \sin(z) + \partial_y g = x \sin(z) \quad \Rightarrow \quad \partial_y g(y, z) = 0,$$

so $g(y, z) = h(z)$.

Conservative fields, potential functions (16.3).

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Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

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Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.

Conservative fields, potential functions (16.3).

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Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
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$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z)$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

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Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

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$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

We conclude that $f(x,y,z) = xy\sin(z) + c$. $\triangleleft$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative,

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$,

Conservative fields, potential functions (16.3).

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Compute $I = \int_C y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz.$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0,1]$.

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$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0, 1]$.

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$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent,

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz \big]$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz\big]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2)$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

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We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \left[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz\right]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2) = 2\sin(\pi/2) - \sin(\pi/2)$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz \big]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2) = 2\sin(\pi/2) - \sin(\pi/2) \;\Rightarrow\; I = 1.$$

Conservative fields, potential functions (16.3).

Example

Show that the differential form in the integral below is exact,

$$\int_C \left[3x^2\, dx + \frac{z^2}{y}\, dy + 2z \ln(y)\, dz \right], \qquad y > 0.$$

Conservative fields, potential functions (16.3).

Example

Show that the differential form in the integral below is exact,

$$\int_C \Big[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz \Big], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \Big\langle 3x^2, \frac{z^2}{y}, 2z\ln(y) \Big\rangle$ is conservative.

Conservative fields, potential functions (16.3).

Example

Show that the differential form in the integral below is exact,

$$\int_C \left[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz \right], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \left\langle 3x^2, \frac{z^2}{y}, 2z\ln(y) \right\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y,$$

Conservative fields, potential functions (16.3).

Example

Show that the differential form in the integral below is exact,

$$\int_C \left[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz\right], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \left\langle 3x^2, \frac{z^2}{y}, 2z\ln(y)\right\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x,$$

Conservative fields, potential functions (16.3).

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Show that the differential form in the integral below is exact,

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Solution: We need to show that the field $\mathbf{F} = \Big\langle 3x^2, \frac{z^2}{y}, 2z\ln(y) \Big\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x, \quad \partial_x F_y = 0 = \partial_y F_x.$$

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Example

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Therefore, exists a scalar field f such that $\mathbf{F} = \nabla f$,

Conservative fields, potential functions (16.3).

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Show that the differential form in the integral below is exact,

$$\int_C \Big[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz\Big], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \Big\langle 3x^2, \frac{z^2}{y}, 2z\ln(y)\Big\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x, \quad \partial_x F_y = 0 = \partial_y F_x.$$

Therefore, exists a scalar field f such that $\mathbf{F} = \nabla f$, or equivalently,

$$df = 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz.$$

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- **(16.4) The Green Theorem in a plane.**
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

The Green Theorem in a plane (16.4).

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

The Green Theorem in a plane (16.4).

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$

The Green Theorem in a plane (16.4).

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy$.

Here $\mathbf{F} = \langle (6y + x), (y + 2x)\rangle$.

The Green Theorem in a plane (16.4).

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

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- **(16.5) Surface area, surface integrals.**
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Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

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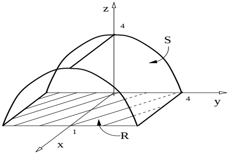

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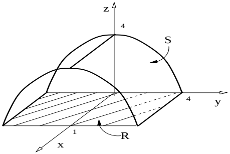

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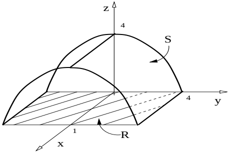

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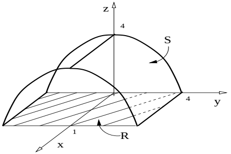

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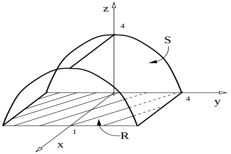

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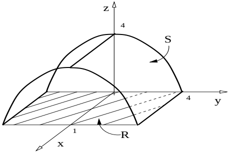

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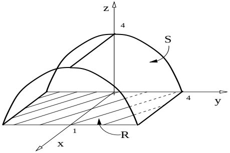

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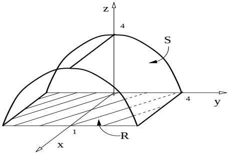

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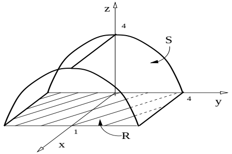

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Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
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$\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

The Stokes Theorem (16.7).

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
$S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: Recall: $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

The surface S is the cylinder walls and its cover at $z = h$.
Therefore, the curve C is the circle $x^2 + y^2 = a^2$ at $z = 0$.
That circle can be parametrized (counterclockwise) as
$\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$.

The Stokes Theorem (16.7).

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
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Solution: $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$. Hence

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

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Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
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$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \big(a^2\sin^2(t) + a^2\cos^2(t)\big)\, dt$$

The Stokes Theorem (16.7).

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and
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Solution: $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
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$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \big(a^2\sin^2(t) + a^2\cos^2(t)\big)\, dt = \int_0^{2\pi} a^2\, dt.$$

We conclude that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 2\pi a^2$.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- **(16.8) The Divergence Theorem.**

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

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$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\displaystyle \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 3x.$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

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The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\displaystyle \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 3x.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv = \iint_D 3x\, dx\, dy\, dz.$$

The Divergence Theorem (16.8).

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Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

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$$\nabla\cdot\mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla\cdot\mathbf{F} = 3x.$$

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It is convenient to use spherical coordinates:

The Divergence Theorem (16.8).

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It is convenient to use spherical coordinates:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution:

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$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\int_0^{\pi/2} \cos(\theta)\, d\theta\Big] \Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big] \Big[\int_0^2 3\rho^3\, d\rho\Big]$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi)\cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\int_0^{\pi/2} \cos(\theta)\, d\theta\Big] \Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big] \Big[\int_0^2 3\rho^3\, d\rho\Big]$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\sin(\theta)\Big|_0^{\pi/2}\Big] \Big[\frac{1}{2} \int_0^{\pi/2} \big(1 - \cos(2\phi)\big)\, d\phi\Big] \Big[\frac{3}{4}\rho^4\Big|_0^2\Big]$$

The Divergence Theorem (16.8).

Example

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Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\int_0^{\pi/2} \cos(\theta)\, d\theta\Big] \Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big] \Big[\int_0^2 3\rho^3\, d\rho\Big]$$

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$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = (1)\frac{1}{2}\Big(\frac{\pi}{2}\Big)(12)$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

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$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = (1)\frac{1}{2}\Big(\frac{\pi}{2}\Big)(12) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 3\pi.$$

Review for the Final Exam.

- Monday, December 13, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 005, 006 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 003, 004, in 108 EBH (Ernst Bessey Hall);
  - Sctns 007, 008, in 339 CSE (Case Halls).
- Chapters 12-16.
- Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Final Exam.

- Chapter 16, Sections 16.1-16.5, 16.7, 16.8.
- Chapter 15, Sections 15.1-15.4, 15.6.
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1, 13.3.
- Chapter 12, Sections 12.1-12.6.

Example

Use the Divergence Theorem to find the flux of $\mathbf{F} = \langle xy^2, x^2y, y \rangle$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of $\mathbf{F} = \langle xy^2, x^2y, y \rangle$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Solution: Recall: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F})\, dv$.

Example

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Solution: Recall: $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D (\nabla\cdot\mathbf{F})\,dv$. We start with

$$\nabla\cdot\mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y)$$

Example

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$$\nabla\cdot\mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad\Rightarrow\quad \nabla\cdot\mathbf{F} = y^2 + x^2.$$

Example

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The integration region is $D = \{x^2 + y^2 \leqslant 1,\ z \in [-1, 1]\}$.

Chapter 16, Integration in vector fields.

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$$I = \iiint_D (\nabla \cdot \mathbf{F})\, dv$$

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Chapter 16, Integration in vector fields.

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Chapter 16, Integration in vector fields.

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$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta$$

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$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = 2\pi\Big[\int_0^1 r^3\,dr\Big]\,(2)$$

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We use cylindrical coordinates,

$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = 2\pi\Big[\int_0^1 r^3\,dr\Big](2) = 4\pi\Big(\frac{r^4}{4}\Big|_0^1\Big).$$

Example

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The integration region is $D = \{x^2 + y^2 \leqslant 1,\ z \in [-1, 1]\}$. So,

$$I = \iiint_D (\nabla\cdot\mathbf{F})\,dv = \iiint_D (x^2 + y^2)\,dx\,dy\,dz.$$

We use cylindrical coordinates,

$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = 2\pi\Big[\int_0^1 r^3\,dr\Big]\,(2) = 4\pi\Big(\frac{r^4}{4}\Big|_0^1\Big).$$

We conclude that $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \pi$. $\triangleleft$

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz \rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

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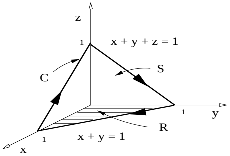

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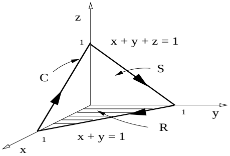

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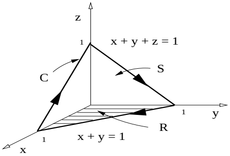

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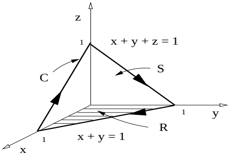

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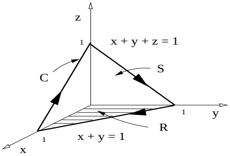

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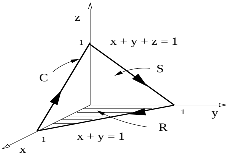

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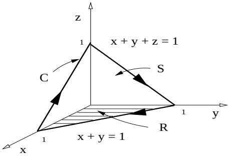

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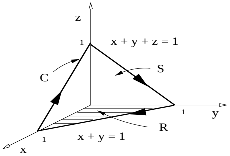

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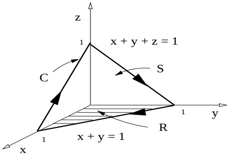

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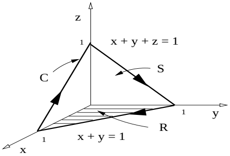

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Find the area of the cone S given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$.
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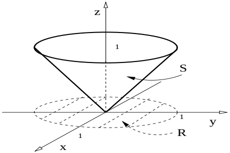

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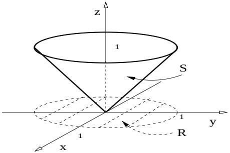

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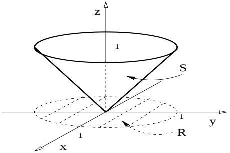

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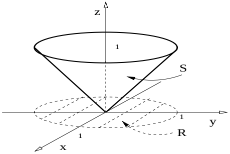

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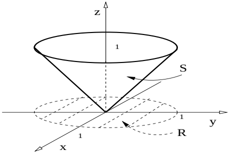

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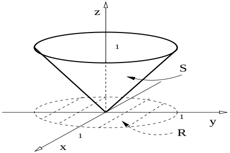

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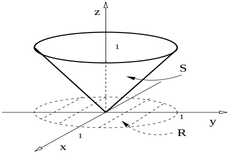

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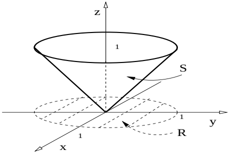

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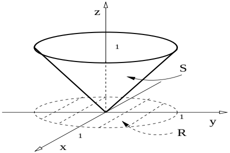

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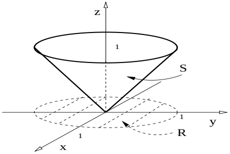

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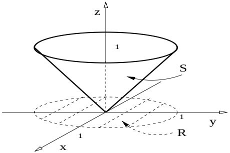

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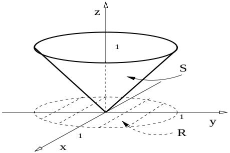

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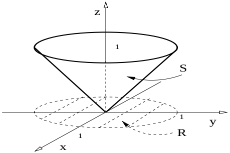

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Review for Final Exam.

- Chapter 16, Sections 16.1-16.5, 16.7, 16.8.
- **Chapter 15, Sections 15.1-15.4, 15.6.**
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1, 13.3.
- Chapter 12, Sections 12.1-12.6.

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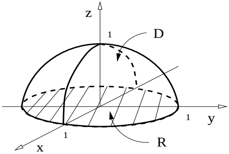

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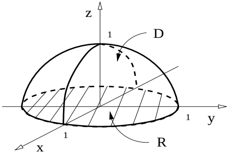

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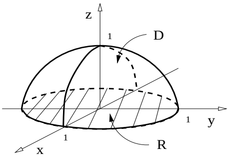

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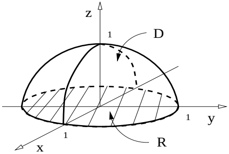

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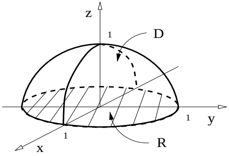

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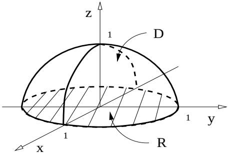

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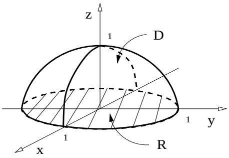

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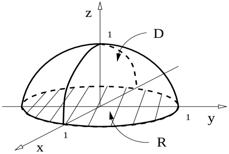

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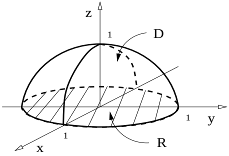

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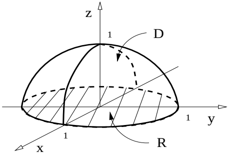

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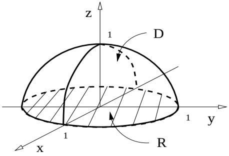

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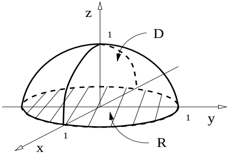

So, $D = \{x^2 + y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leqslant 1,\ z = 0\}$. We know that

$$V(D) = \iiint_D dv = \iint_R \int_0^{1-x^2-y^2} dz\, dx\, dy.$$

Using cylindrical coordinates (r, θ, z), we get

$$V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz\, r\, dr\, d\theta = 2\pi \int_0^1 (1 - r^2)\, r\, dr.$$

Substituting $u = 1 - r^2$, so $du = -2r\, dr$, we obtain

$$V(D) = 2\pi \int_1^0 u\, \frac{(-du)}{2} = \pi \int_0^1 u\, du = \pi \frac{u^2}{2}\Big|_0^1$$

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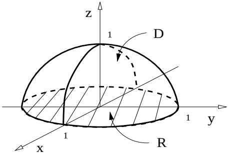

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Chapter 15, Multiple integrals.

Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z\sin(x)$ on the bounded region D in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

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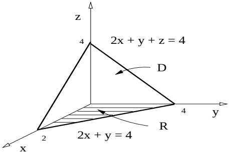

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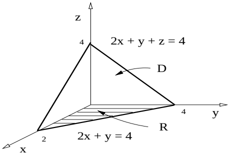

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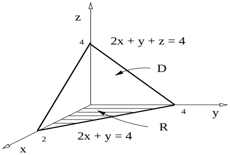

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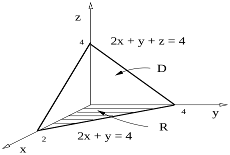

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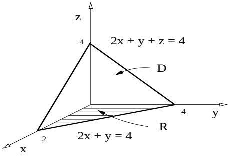

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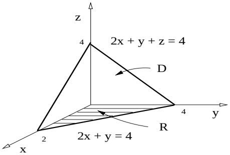

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we conclude that

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Chapter 15, Multiple integrals.

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Reverse the order of integration and evaluate the double integral

$$I = \int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy.$$

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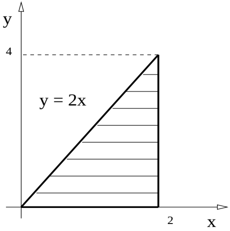

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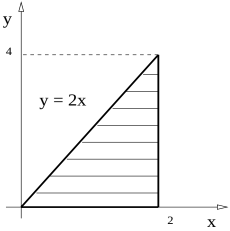

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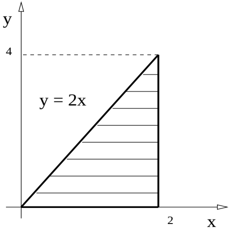

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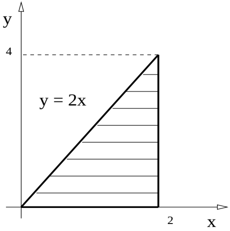

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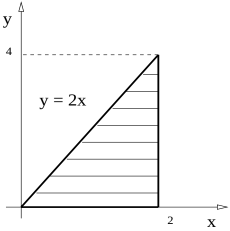

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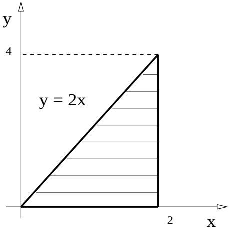

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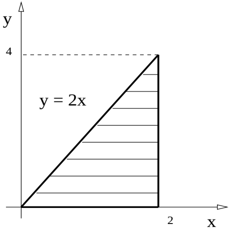

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$$I = \int_0^4 e^u\, du \quad \Rightarrow \quad I = e^4 - 1.$$

Review for the Final Exam.

- Monday, December 13, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 005, 006 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 003, 004, in 108 EBH (Ernst Bessey Hall);
  - Sctns 007, 008, in 339 CSE (Case Halls).
- Chapters 12-16.
- $\sim$ 12 Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.

Plan for today: Practice final exam: April 30, 2001.

Remark on Chapter 16.

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Practice final exam: April 30, 2001. Prbl. 1.

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Given $A = (1, 2, 3)$, $B = (6, 5, 4)$ and $C = (8, 9, 7)$, find the following:

- $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Practice final exam: April 30, 2001. Prbl. 1.

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  Solution: $\overrightarrow{AB} = \langle (6 - 1), (5 - 2), (4 - 3) \rangle$,

Practice final exam: April 30, 2001. Prbl. 1.

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Practice final exam: April 30, 2001. Prbl. 2.

Example

Find the parametric equation of the line through the point $(1, 0, -1)$ and perpendicular to the plane $2x - 3y + 5x = 35$. Then find the intersection of the line and the plane.

Practice final exam: April 30, 2001. Prbl. 2.

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Practice final exam: April 30, 2001. Prbl. 2.

Example

Find the parametric equation of the line through the point $(1, 0, -1)$ and perpendicular to the plane $2x - 3y + 5x = 35$. Then find the intersection of the line and the plane.

Solution: The normal vector to the plane $\langle 2, -3, 5\rangle$ is the tangent vector to the line. Therefore,

$$\mathbf{r}(t) = \langle 1, 0, -1\rangle + t\,\langle 2, -3, 5\rangle,$$

so the parametric equations of the line are

$$x(t) = 1 + 2t, \quad y(t) = -3t, \quad z(t) = -1 + 5t.$$

The intersection point has a t solution of

$$2(1+2t)-3(-3t)+5(-1+5t) = 35 \quad \Rightarrow \quad 2+4t+9t-5+25t = 35$$

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Practice final exam: April 30, 2001. Prbl. 3.

Example

The velocity of a particle is given by $\mathbf{v}(t) = \langle t^2, (t^3 + 1)\rangle$, and the particle is at $\langle 2, 1\rangle$ for $t = 0$.

- Where is the particle at $t = 2$?

Practice final exam: April 30, 2001. Prbl. 3.

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Practice final exam: April 30, 2001. Prbl. 3.

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  Solution: $s(t) = \displaystyle\int_0^t \sqrt{\tau^4 + (\tau^3 + 1)^2}\, d\tau$.

- Find the particle acceleration.

  Solution: $\mathbf{a}(t) = \langle 2t, 3t^2\rangle$.

Practice final exam: April 30, 2001. Prbl. 4.

Example

- Draw a rough sketch of the surface $z = 2x^2 + 3y^2 + 5$.

Practice final exam: April 30, 2001. Prbl. 4.

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  Solution: This is a paraboloid along the vertical direction, opens up, with vertex at $z = 5$ on the z-axis, and the x-radius is a bit longer than the y-radius.

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- Find the equation of the tangent plane to the surface at the point $(1, 1, 10)$.

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  Solution: Introduce $f(x, y) = 2x^2 + 3y^2 + 5$, then

$$L_{(1,1)}(x, y) = \partial_x f(1, 1)\,(x - 1) + \partial_y f(1, 1)\,(y - 1) + f(1, 1).$$

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  $$z = L_{(1,1)}(x, y) = 4(x - 1) + 6(y - 1) + 10.$$

Practice final exam: April 30, 2001. Prbl. 5.

Example

Let $w = f(x, y)$ and $x = s^2 + t^2$, $y = st^2$. If $\partial_x f = x - y$ and $\partial_y f = y - x$, find $\partial_s w$ and $\partial_t w$ in terms of s and t.

Practice final exam: April 30, 2001. Prbl. 5.

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$$\partial_s w = \partial_x f\, \partial_s x + \partial_y f\, \partial_s y = (x-y)2s + (y-x)t^2 = (x-y)(2s-t^2).$$

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Therefore, $\partial_t w = (s^2 + t^2 - st^2)2t(1 - s)$.

Practice final exam: April 30, 2001. Prbl. 6.

Example

Find all critical points of the function $f(x, y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Practice final exam: April 30, 2001. Prbl. 6.

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Solution:

$$\nabla f = \langle (4x + 8y), (8x + 4y^3) \rangle$$

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Since $f_{xx} = 4$, $f_{yy} = 12y^2$, and $f_{xy} = 8$,

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Since $f_{xx} = 4$, $f_{yy} = 12y^2$, and $f_{xy} = 8$, we conclude that $D = 3(16)y^2 - 4(16)$.

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$P_0 = (0, 0)$, $P_1 = (4, -2)$, $P_2 = (-4, 2)$, $D = 3(16)y^2 - 4(16)$.

$$D(0, 0) = -4(16) < 0$$

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$P_0 = (0, 0)$, $P_1 = (4, -2)$, $P_2 = (-4, 2)$, $D = 3(16)y^2 - 4(16)$.

$$D(0, 0) = -4(16) < 0 \quad \Rightarrow \quad P_0 = (0, 0) \text{ saddle point.}$$

Practice final exam: April 30, 2001. Prbl. 6.

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Practice final exam: April 30, 2001. Prbl. 7.

Example

Evaluate the integral $I = \int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx$ by reversing the order of integration.

Practice final exam: April 30, 2001. Prbl. 7.

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Solution: The integration region is the set in the square $[0,1] \times [0,1]$ in between the curves $y = x$ and $y = \sqrt{x}$.

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Practice final exam: April 30, 2001. Prbl. 8.

Example

Find the work done by the force $\mathbf{F} = \langle yz, xz, -xy \rangle$ on a particle moving along the path $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

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Solution:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

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Therefore, $W = 2^7/3$.

Practice final exam: April 30, 2001. Prbl. 9.

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The potential function is a scalar function f solution of

$$\partial_x f = y\cos(z) - yze^x, \ \partial_y f = x\cos(z) - ze^x, \ \partial_z f = -xy\sin(z) - ye^x.$$

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$f = xy\cos(z) - yze^x + g(y,z)$.

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The x-integral of the first equation implies
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Introduce f into the equation $\partial_z f = -xy\sin(z) - ye^{x}$, that is,

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$$-xy\sin(z) - e^x + h'(z) = -xy\sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0.$$

So, $h(z) = c$, a constant, hence $f = xy\cos(z) - yze^x + c$.

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Solution: Recall: $f = xy\cos(z) - yze^x + h(z)$.

Introduce f into the equation $\partial_z f = -xy\sin(z) - ye^x$, that is,

$$-xy\sin(z) - e^x + h'(z) = -xy\sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0.$$

So, $h(z) = c$, a constant, hence $f = xy\cos(z) - yze^x + c$.

Finally $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,1,\pi)} df$

Practice final exam: April 30, 2001. Prbl. 9.

Example

Show that the force field $\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x) \rangle$ is conservative. Then find its potential function. Then evaluate $I = \int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3 \rangle$.

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Practice final exam: April 30, 2001. Prbl. 9.

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So we conclude that $\int_C \mathbf{F}\cdot d\mathbf{r} = -(1 + \pi e)$.

Practice final exam: April 30, 2001. Prbl. 10.

Example

Use the Green Theorem to evaluate the integral $\int_C F_x\,dx + F_y\,dy$ where $F_x = y + e^x$ and $F_y = 2x^2 + \cos(y)$ and C is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,1)$ traversed counterclockwise.

Practice final exam: April 30, 2001. Prbl. 10.

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Use the Green Theorem to evaluate the integral $\int_C F_x\,dx + F_y\,dy$ where $F_x = y + e^x$ and $F_y = 2x^2 + \cos(y)$ and C is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,1)$ traversed counterclockwise.

Solution: Denoting $\mathbf{F} = \langle F_x, F_y \rangle$,

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Solution: Denoting $\mathbf{F} = \langle F_x, F_y \rangle$, Green's Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Practice final exam: April 30, 2001. Prbl. 10.

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Practice final exam: April 30, 2001. Prbl. 10.

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$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (4x - 1)\,dx\,dy$$

Practice final exam: April 30, 2001. Prbl. 10.

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$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (4x-1)\,dx\,dy = \int_0^1\int_y^{2-y}(4x-1)\,dx\,dy.$$

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A straightforward calculation gives $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 3$.

Practice final exam: April 30, 2001. Prbl. 11.

Example

Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Use polar coordinates to evaluate the integral.

Practice final exam: April 30, 2001. Prbl. 11.

Example

Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Use polar coordinates to evaluate the integral.

Solution:

$$A(S) = \iint_S d\sigma,$$

Practice final exam: April 30, 2001. Prbl. 11.

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$$A(S) = \iint_S d\sigma, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$$

Practice final exam: April 30, 2001. Prbl. 11.

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where $f = x^2 + y^2 + z - 4$.

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$$\nabla f = \langle 2x, 2y, 1 \rangle$$

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$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta,$$

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The finally obtain $A(S) = (\pi/6)(17^{3/2} - 1)$.

Practice final exam: April 30, 2001. Prbl. 12.

Example

Use the Stokes Theorem to evaluate $I = \iint_S \big[\nabla \times (y\mathbf{i})\big] \cdot \mathbf{n}\, d\sigma$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geqslant 0$.

Practice final exam: April 30, 2001. Prbl. 12.

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Solution: $\mathbf{F} = \langle y, 0, 0 \rangle$.

Practice final exam: April 30, 2001. Prbl. 12.

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Solution: $\mathbf{F} = \langle y, 0, 0\rangle$. The border of the hemisphere is given by the circle $x^2 + y^2 = 1$, with $z = 0$.

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$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle$$

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and we also have $\mathbf{F}(t) = \langle \sin(t), 0, 0 \rangle$.

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$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\frac{1}{2}\int_0^{2\pi} \big[1 - \cos(2t)\big]\, dt.$$

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Recall that

$$\int_0^{2\pi} \cos(2t)\, dt = \frac{1}{2}\left(\sin(2t)\Big|_0^{2\pi}\right) = 0.$$

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Recall that

$$\int_0^{2\pi} \cos(2t)\, dt = \frac{1}{2} \left(\sin(2t) \Big|_0^{2\pi} \right) = 0.$$

Therefore, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\pi.$$

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