Review for Exam 4.

- Sections 16.1-16.5, 16.7, 16.8.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- **(16.3) Conservative fields, potential functions.**
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$. The equations for f are

$$\partial_x f = y\sin(z), \quad \partial_y f = x\sin(z), \quad \partial_z f = xy\cos(z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

We conclude that $f(x,y,z) = xy\sin(z) + c$. $\triangleleft$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz\big]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2) = 2\sin(\pi/2) - \sin(\pi/2) \;\Rightarrow\; I = 1.$$

Conservative fields, potential functions (16.3).

Example

Show that the differential form in the integral below is exact,

$$\int_C \Big[3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz\Big], \qquad y > 0.$$

Solution: We need to show that the field $\mathbf{F} = \Big\langle 3x^2, \frac{z^2}{y}, 2z\ln(y)\Big\rangle$ is conservative. It is, since,

$$\partial_y F_z = \frac{2z}{y} = \partial_z F_y, \quad \partial_x F_z = 0 = \partial_z F_x, \quad \partial_x F_y = 0 = \partial_y F_x.$$

Therefore, exists a scalar field f such that $\mathbf{F} = \nabla f$, or equivalently,

$$df = 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln(y)\,dz.$$

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- **(16.4) The Green Theorem in a plane.**
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

The Green Theorem in a plane (16.4).

Example

Use the Green Theorem in the plane to evaluate the line integral given by $\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big]$ on the circle C defined by $(x-1)^2 + (y-3)^2 = 4$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy$.

Here $\mathbf{F} = \langle (6y + x), (y + 2x)\rangle$. Since $\partial_x F_y = 2$ and $\partial_y F_x = 6$, Green's Theorem implies

$$\oint_C \big[(6y + x)\,dx + (y + 2x)\,dy\big] = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (2-6)\,dx\,dy.$$

Since the area of the disk $S = \{(x-1)^2 + (y-3)^2 \leqslant 4\}$ is $\pi(2^2)$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -4\iint_S dx\,dy = -4(4\pi) \quad\Rightarrow\quad \oint_C \mathbf{F}\cdot d\mathbf{r} = -16\pi.$$

Review for Exam 4.

- (16.1) Line integrals.
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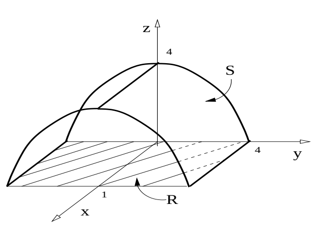
- (16.4) The Green Theorem in a plane.
- **(16.5) Surface area, surface integrals.**
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Surface area, surface integrals (16.5).

Example

Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution:



We must compute: $I = \iint_S g\, d\sigma$.

Recall $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dx\, dy$, with $\mathbf{k} \perp R$

and in this case $f(x, y, z) = y^2 + 4z - 16$.

$$\nabla f = \langle 0, 2y, 4\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{16 + 4y^2} = 2\sqrt{4 + y^2}.$$

Since $R = [0, 1] \times [-4, 4]$, its normal vector is $\mathbf{k}$ and $|\nabla f \cdot \mathbf{k}| = 4$. Then,

$$\iint_S g\, d\sigma = \iint_R \bigl(x\sqrt{4 + y^2}\bigr)\, \frac{2\sqrt{4 + y^2}}{4}\, dx\, dy.$$

Surface area, surface integrals (16.5).

Example

Integrate the function $g(x, y, z) = x\sqrt{4 + y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution: $\displaystyle\iint_S g\,d\sigma = \iint_R \big(x\sqrt{4+y^2}\big)\,\frac{2\sqrt{4+y^2}}{4}\,dx\,dy.$

$$\iint_S g\,d\sigma = \frac{1}{2}\iint_R x(4+y^2)\,dx\,dy = \frac{1}{2}\int_{-4}^{4}\int_0^1 x(4+y^2)\,dx\,dy$$

$$\iint_S g\,d\sigma = \frac{1}{2}\Big[\int_{-4}^{4}(4+y^2)\,dy\Big]\Big[\int_0^1 x\,dx\Big] = \frac{1}{2}\Big(4y+\frac{y^3}{3}\Big)\Big|_{-4}^{4}\Big(\frac{x^2}{2}\Big)\Big|_0^1$$

$$\iint_S g\,d\sigma = \frac{1}{2}2\Big(4^2+\frac{4^3}{3}\Big)\frac{1}{2} = 8\Big(1+\frac{4}{3}\Big) \quad\Rightarrow\quad \iint_S g\,d\sigma = \frac{56}{3}.$$

Review for Exam 4.

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- (16.5) Surface area, surface integrals.
- **(16.7) The Stokes Theorem.**
- (16.8) The Divergence Theorem.

The Stokes Theorem (16.7).

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and $S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: Recall: $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

The surface S is the cylinder walls and its cover at $z = h$.
Therefore, the curve C is the circle $x^2 + y^2 = a^2$ at $z = 0$.
That circle can be parametrized (counterclockwise) as
$\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$.

The Stokes Theorem (16.7).

Example

Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2 \rangle$ and $S = \{x^2 + y^2 = a^2,\ z \in [0, h]\} \cup \{x^2 + y^2 \leqslant a^2,\ z = h\}$.

Solution: $\mathbf{F}(t) = \langle -a\sin(t), a\cos(t), a^2\cos^2(t) \rangle$ and
$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$. Hence

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \big(a^2\sin^2(t) + a^2\cos^2(t)\big)\, dt = \int_0^{2\pi} a^2\, dt.$$

We conclude that $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 2\pi a^2$.

Review for Exam 4.

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- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- **(16.8) The Divergence Theorem.**

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution: Recall: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv.$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2x - 2x + 3x \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 3x.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dv = \iint_D 3x\, dx\, dy\, dz.$$

It is convenient to use spherical coordinates:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

The Divergence Theorem (16.8).

Example

Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz \rangle$ across the boundary of the region $D = \{x^2 + y^2 + z^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$.

Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \big[3\rho \sin(\phi) \cos(\phi)\big]\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\int_0^{\pi/2} \cos(\theta)\, d\theta\Big] \Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big] \Big[\int_0^{2} 3\rho^3\, d\rho\Big]$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \Big[\sin(\theta)\Big|_0^{\pi/2}\Big] \Big[\frac{1}{2} \int_0^{\pi/2} \big(1 - \cos(2\phi)\big)\, d\phi\Big] \Big[\frac{3}{4}\rho^4\Big|_0^2\Big]$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = (1)\frac{1}{2}\Big(\frac{\pi}{2}\Big)(12) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 3\pi.$$

Review for the Final Exam.

- Monday, December 13, 10:00am - 12:00 noon. (2 hours.)
- Places:
  - Sctns 001, 002, 005, 006 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 003, 004, in 108 EBH (Ernst Bessey Hall);
  - Sctns 007, 008, in 339 CSE (Case Halls).
- Chapters 12-16.
- Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Final Exam.

- Chapter 16, Sections 16.1-16.5, 16.7, 16.8.
- Chapter 15, Sections 15.1-15.4, 15.6.
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1, 13.3.
- Chapter 12, Sections 12.1-12.6.

Chapter 16, Integration in vector fields.

Example

Use the Divergence Theorem to find the flux of $\mathbf{F} = \langle xy^2, x^2y, y\rangle$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$, and $z = 1$.

Solution: Recall: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F})\, dv$. We start with

$$\nabla \cdot \mathbf{F} = \partial_x(xy^2) + \partial_y(x^2y) + \partial_z(y) \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = y^2 + x^2.$$

The integration region is $D = \{x^2 + y^2 \leqslant 1,\ z \in [-1, 1]\}$. So,

$$I = \iiint_D (\nabla \cdot \mathbf{F})\, dv = \iiint_D (x^2 + y^2)\, dx\, dy\, dz.$$

We use cylindrical coordinates,

$$I = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2\, dz\, r\, dr\, d\theta = 2\pi \Big[\int_0^1 r^3\, dr\Big]\,(2) = 4\pi\Big(\frac{r^4}{4}\Big|_0^1\Big).$$

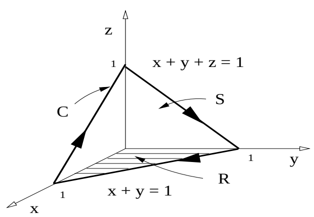
We conclude that $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \pi$. ◁

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz\rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:



Recall: $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma.$

The surface S is the level surface $f = 0$ of

$$f = x + y + z - 1$$

therefore, $\nabla f = \langle 1,1,1\rangle$, $|\nabla f| = \sqrt{3}$ and $|\nabla f\cdot\mathbf{k}| = 1$.

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle, \qquad d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}\,dx\,dy = \sqrt{3}\,dx\,dy.$$

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz\rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution: $\mathbf{n} = \dfrac{1}{\sqrt{3}}\langle 1,1,1\rangle$ and $d\sigma = \sqrt{3}\,dx\,dy$.

We now compute the curl of $\mathbf{F}$,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & xy & yz \end{vmatrix} = \langle (z-0), -(0-2x), (y-0)\rangle$$

so $\nabla\times\mathbf{F} = \langle z, 2x, y\rangle$. Therefore,

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma = \iint_R \Big(\langle z,2x,y\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,1\rangle\Big)\sqrt{3}\,dx\,dy$$

Chapter 16, Integration in vector fields.

Example

Use Stokes' Theorem to find the work done by the force $\mathbf{F} = \langle 2xz, xy, yz\rangle$ along the path C given by the intersection of the plane $x + y + z = 1$ with the first octant, counterclockwise when viewed from above.

Solution:

$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \Big(\langle z, 2x, y\rangle \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle\Big)\sqrt{3}\, dx\, dy.$$

$$I = \iint_R (z + 2x + y)\, dx\, dy, \qquad z = 1 - x - y,$$

$$I = \int_0^1 \int_0^{1-x} (1+x)\, dy\, dx = \int_0^1 (1+x)(1-x)\, dx = \int_0^1 (1-x^2)\, dx.$$

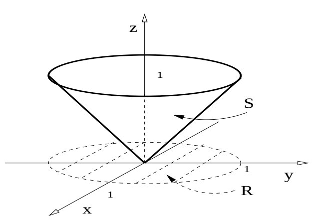
$$I = x\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \quad \Rightarrow \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}.$$

Chapter 16, Integration in vector fields.

Example

Find the area of the cone S given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0\rangle$ outward through S.

Solution:



Recall: $A(S) = \iint_S d\sigma$. The surface S is the level surface $f = 0$ of the function $f = x^2 + y^2 - z^2$. Also recall that

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy.$$

Since $\nabla f = 2\langle x, y, -z\rangle$, we get that

$$|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}, \quad z^2 = x^2 + y^2 \quad \Rightarrow \quad |\nabla f| = 2\sqrt{2}\, z.$$

Also $|\nabla f \cdot \mathbf{k}| = 2z$, therefore, $d\sigma = \sqrt{2}\, dx\, dy$, and then we obtain

$$A(S) = \iint_R \sqrt{2}\, dx\, dy = \int_0^{2\pi} \int_0^1 \sqrt{2} r\, dr\, d\theta = 2\pi\sqrt{2}\frac{r^2}{2}\Big|_0^1 = \sqrt{2}\,\pi.$$

Chapter 16, Integration in vector fields.

Example

Find the area of the cone S given by $z = \sqrt{x^2 + y^2}$ for $z \in [0, 1]$. Also find the flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ outward through S.

Solution: We now compute the outward flux $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$.

Since

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}\, z} \langle x, y, -z \rangle.$$

$$I = \iint_R \frac{1}{\sqrt{2}\, z} (x^2 + y^2) \sqrt{2}\, dx\, dy = \iint_R \sqrt{x^2 + y^2}\, dx\, dy.$$

Using polar coordinates, we obtain

$$I = \int_0^{2\pi} \int_0^1 r\, r\, dr\, d\theta = 2\pi \frac{r^3}{3}\Big|_0^1 \quad \Rightarrow \quad I = \frac{2\pi}{3}.$$

Review for Final Exam.

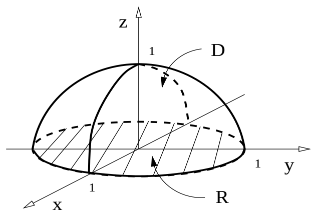
- Chapter 16, Sections 16.1-16.5, 16.7, 16.8.
- **Chapter 15, Sections 15.1-15.4, 15.6.**
- Chapter 14, Sections 14.1-14.7.
- Chapter 13, Sections 13.1, 13.3.
- Chapter 12, Sections 12.1-12.6.

Chapter 15, Multiple integrals.

Example

Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:



So, $D = \{x^2 + y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2 - y^2\}$, and $R = \{x^2 + y^2 \leqslant 1,\ z = 0\}$. We know that

$$V(D) = \iiint_D dv = \iint_R \int_0^{1-x^2-y^2} dz\,dx\,dy.$$

Using cylindrical coordinates (r, θ, z), we get

$$V(D) = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} dz\,r\,dr\,d\theta = 2\pi \int_0^1 (1 - r^2)\,r\,dr.$$

Substituting $u = 1 - r^2$, so $du = -2r\,dr$, we obtain

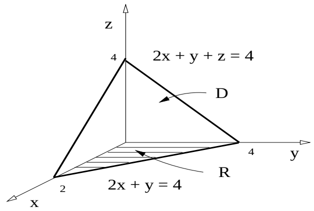
$$V(D) = 2\pi \int_1^0 u\,\frac{(-du)}{2} = \pi \int_0^1 u\,du = \pi \frac{u^2}{2}\Big|_0^1 \quad \Rightarrow \quad V(D) = \frac{\pi}{2}.$$

Chapter 15, Multiple integrals.

Example

Set up the integrals needed to compute the average of the function $f(x, y, z) = z\sin(x)$ on the bounded region D in the first octant bounded by the plane $z = 4 - 2x - y$. Do not evaluate the integrals.

Solution: Recall: $\overline{f} = \dfrac{1}{V(D)} \iiint_D f\,dv$.



Since $V(D) = \displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx$,

we conclude that

$$\overline{f} = \frac{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z\sin(x)\,dz\,dy\,dx}{\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx}.$$

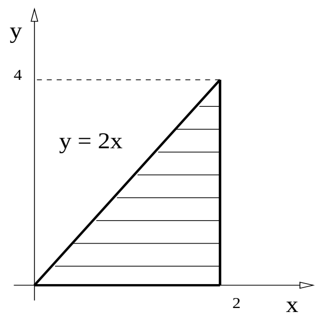
Chapter 15, Multiple integrals.

Example

Reverse the order of integration and evaluate the double integral

$$I = \int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy.$$

Solution: We see that $y \in [0, 4]$ and $x \in [0, y/2]$, that is,



Therefore, reversing the integration order means

$$I = \int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx.$$

This integral is simple to compute,

$$I = \int_0^2 e^{x^2}\, x\, dx, \qquad u = x^2, \quad du = 2x\, dx,$$

$$I = \int_0^4 e^u\, du \quad \Rightarrow \quad I = e^4 - 1.$$

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  - Sctns 001, 002, 005, 006 in E-100 VMC (Vet. Medical Ctr.),
  - Sctns 003, 004, in 108 EBH (Ernst Bessey Hall);
  - Sctns 007, 008, in 339 CSE (Case Halls).
- Chapters 12-16.
- $\sim$ 12 Problems, similar to homework problems.
- No calculators, no notes, no books, no phones.

Plan for today: Practice final exam: April 30, 2001.

Remark on Chapter 16.

Remark: The normal form of Green's Theorem is a two-dimensional restriction of the Divergence Theorem.

- The Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{F})\, dv.$
- Normal form of Green's Thrm: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S (\nabla \cdot \mathbf{F})\, dA.$

Remark: The tangential form of Green's Theorem is a particular case of the Stokes Theorem when C, S are flat (on $z = 0$ plane).

- The Stokes Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$
- Tang. form of Green's Thrm: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$

Practice final exam: April 30, 2001. Prbl. 1.

Example

Given $A = (1, 2, 3)$, $B = (6, 5, 4)$ and $C = (8, 9, 7)$, find the following:

- $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

  Solution: $\overrightarrow{AB} = \langle (6-1), (5-2), (4-3) \rangle$, hence $\overrightarrow{AB} = \langle 5, 3, 1 \rangle$. In the same way $\overrightarrow{AC} = \langle 7, 7, 4 \rangle$.
- $\overrightarrow{AB} + \overrightarrow{AC} = \langle 12, 10, 5 \rangle$.
- $\overrightarrow{AB} \cdot \overrightarrow{AC} = 35 + 21 + 4$.
- $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 1 \\ 7 & 7 & 4 \end{vmatrix} = \langle (12-7), -(20-7), (35-21) \rangle,$

  hence $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 5, -13, 14 \rangle$.

Practice final exam: April 30, 2001. Prbl. 2.

Example

Find the parametric equation of the line through the point $(1, 0, -1)$ and perpendicular to the plane $2x - 3y + 5x = 35$. Then find the intersection of the line and the plane.

Solution: The normal vector to the plane $\langle 2, -3, 5\rangle$ is the tangent vector to the line. Therefore,

$$\mathbf{r}(t) = \langle 1, 0, -1\rangle + t\,\langle 2, -3, 5\rangle,$$

so the parametric equations of the line are

$$x(t) = 1 + 2t, \quad y(t) = -3t, \quad z(t) = -1 + 5t.$$

The intersection point has a t solution of

$$2(1+2t)-3(-3t)+5(-1+5t) = 35 \quad \Rightarrow \quad 2+4t+9t-5+25t = 35$$

$$38t = 38 \quad \Rightarrow \quad t = 1 \quad \Rightarrow \quad \mathbf{r}(1) = \langle 3, -3, 4\rangle.$$

Practice final exam: April 30, 2001. Prbl. 3.

Example

The velocity of a particle is given by $\mathbf{v}(t) = \langle t^2, (t^3 + 1)\rangle$, and the particle is at $\langle 2, 1\rangle$ for $t = 0$.

- Where is the particle at $t = 2$?

  Solution: $\mathbf{r}(t) = \Big\langle \Big(\dfrac{t^3}{3} + r_x\Big), \Big(\dfrac{t^4}{4} + t + r_y\Big)\Big\rangle$. Since $\mathbf{r}(0) = \langle 2, 1\rangle$, we get that $\mathbf{r}(t) = \Big\langle \Big(\dfrac{t^3}{3} + 2\Big), \Big(\dfrac{t^4}{4} + t + 1\Big)\Big\rangle$. Hence $\mathbf{r}(2) = \langle 8/3 + 2, 7\rangle$.

- Find an expression for the particle arc length for $t \in [0, 2]$.

  Solution: $s(t) = \displaystyle\int_0^t \sqrt{\tau^4 + (\tau^3 + 1)^2}\, d\tau$.

- Find the particle acceleration.

  Solution: $\mathbf{a}(t) = \langle 2t, 3t^2\rangle$.

Practice final exam: April 30, 2001. Prbl. 4.

Example

- Draw a rough sketch of the surface $z = 2x^2 + 3y^2 + 5$.

  Solution: This is a paraboloid along the vertical direction, opens up, with vertex at $z = 5$ on the z-axis, and the x-radius is a bit longer than the y-radius.

- Find the equation of the tangent plane to the surface at the point $(1, 1, 10)$.

  Solution: Introduce $f(x, y) = 2x^2 + 3y^2 + 5$, then

  $$L_{(1,1)}(x, y) = \partial_x f(1, 1)\,(x - 1) + \partial_y f(1, 1)\,(y - 1) + f(1, 1).$$

  Since $f(1, 1) = 10$, and $\partial_x f = 4x$, $\partial_y f = 6y$, then

  $$z = L_{(1,1)}(x, y) = 4(x - 1) + 6(y - 1) + 10.$$

Practice final exam: April 30, 2001. Prbl. 5.

Example

Let $w = f(x, y)$ and $x = s^2 + t^2$, $y = st^2$. If $\partial_x f = x - y$ and $\partial_y f = y - x$, find $\partial_s w$ and $\partial_t w$ in terms of s and t.

Solution:

$$\partial_s w = \partial_x f\, \partial_s x + \partial_y f\, \partial_s y = (x - y)2s + (y - x)t^2 = (x - y)(2s - t^2).$$

Therefore, $\partial_s w = (s^2 + t^2 - st^2)(2s - t^2)$.

$$\partial_t w = \partial_x f\, \partial_t x + \partial_y f\, \partial_t y = (x - y)2t + (y - x)2st = (x - y)(2t - 2st).$$

Therefore, $\partial_t w = (s^2 + t^2 - st^2)2t(1 - s)$.

Practice final exam: April 30, 2001. Prbl. 6.

Example

Find all critical points of the function $f(x,y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

$$\nabla f = \langle (4x+8y), (8x+4y^3)\rangle = \langle 0,0\rangle \quad \Rightarrow \quad \begin{cases} x + 2y = 0, \\ 2x + y^3 = 0. \end{cases}$$

$$-4y + y^3 = 0 \Rightarrow \begin{cases} y = 0 \Rightarrow x = 0 & \Rightarrow \quad P_0 = (0,0) \\ y = \pm 2 \Rightarrow x = \mp 4 & \Rightarrow \quad \begin{cases} P_1 = (4,-2) \\ P_2 = (-4,2) \end{cases} \end{cases}$$

Since $f_{xx} = 4$, $f_{yy} = 12y^2$, and $f_{xy} = 8$, we conclude that $D = 3(16)y^2 - 4(16)$.

Practice final exam: April 30, 2001. Prbl. 6.

Example

Find all critical points of the function $f(x,y) = 2x^2 + 8xy + y^4$ and determine whether they re local maximum, minimum of saddle points.

Solution:

$P_0 = (0,0)$, $P_1 = (4,-2)$, $P_2 = (-4,2)$, $D = 3(16)y^2 - 4(16)$.

$$D(0,0) = -4(16) < 0 \quad \Rightarrow \quad P_0 = (0,0) \text{ saddle point.}$$

$$D(4,-2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (4,-2) \text{ min.}$$

$$D(-4,2) = 12(16) - 4(16) > 0, \quad f_{xx} = 4 \Rightarrow P_1 = (-4,2) \text{ min.}$$

Practice final exam: April 30, 2001. Prbl. 7.

Example

Evaluate the integral $I = \int_0^1 \int_x^{\sqrt{x}} y\, dy\, dx$ by reversing the order of integration.

Solution: The integration region is the set in the square $[0,1] \times [0,1]$ in between the curves $y = x$ and $y = \sqrt{x}$. Therefore,

$$I = \int_0^1 \int_{y^2}^{y} y\, dx\, dy = \int_0^1 y(y - y^2)\, dy = \int_0^1 (y^2 - y^3)\, dy$$

$$I = \Big(\frac{y^3}{3} - \frac{y^4}{4}\Big)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} \quad \Rightarrow \quad I = \frac{1}{12}.$$

Practice final exam: April 30, 2001. Prbl. 8.

Example

Find the work done by the force $\mathbf{F} = \langle yz, xz, -xy \rangle$ on a particle moving along the path $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0,2]$.

Solution:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle$ and $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$. Hence

$$W = \int_0^2 (3t^5 + 2t^5 - t^5)\, dt = \int_0^2 4t^5\, dt = \frac{4}{6}\, t^6\Big|_0^2 = \frac{2}{3}\, 2^6.$$

Therefore, $W = 2^7/3$.

Practice final exam: April 30, 2001. Prbl. 9.

Example

Show that the force field
$\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x)\rangle$ is conservative. Then find its potential function. Then evaluate
$I = \int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3\rangle$.

Solution: The field $\mathbf{F}$ is conservative, since

$$\partial_x F_y = \cos(z) - ze^x = \partial_y F_x,$$

$$\partial_x F_z = -xy\sin(z) - ye^x = \partial_z F_x,$$

$$\partial_y F_z = -x\sin(z) - e^x = \partial_z F_y.$$

The potential function is a scalar function f solution of

$$\partial_x f = y\cos(z) - yze^x,\ \partial_y f = x\cos(z) - ze^x,\ \partial_z f = -xy\sin(z) - ye^x.$$

Practice final exam: April 30, 2001. Prbl. 9.

Example

Show that the force field
$\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x)\rangle$ is conservative. Then find its potential function. Then evaluate
$I = \int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3\rangle$.

Solution: Recall:

$$\partial_x f = y\cos(z) - yze^x,\ \partial_y f = x\cos(z) - ze^x,\ \partial_z f = -xy\sin(z) - ye^x.$$

The x-integral of the first equation implies
$f = xy\cos(z) - yze^x + g(y,z)$. Introduce f into the second equation above,

$$x\cos(z) - ze^x + \partial_y g = x\cos(z) - ze^x \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so we conclude $g(y,z) = h(z)$, hence $f = xy\cos(z) - yze^x + h(z)$.

Practice final exam: April 30, 2001. Prbl. 9.

Example

Show that the force field
$\mathbf{F} = \langle (y\cos(z) - yze^x), (x\cos(z) - ze^x), (-xy\sin(z) - ye^x)\rangle$ is conservative. Then find its potential function. Then evaluate $I = \int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{r}(t) = \langle t, t^2, \pi t^3\rangle$.

Solution: Recall: $f = xy\cos(z) - yze^x + h(z)$.

Introduce f into the equation $\partial_z f = -xy\sin(z) - ye^x$, that is,

$$-xy\sin(z) - e^x + h'(z) = -xy\sin(z) - ye^x \quad \Rightarrow \quad h'(z) = 0.$$

So, $h(z) = c$, a constant, hence $f = xy\cos(z) - yze^x + c$.

Finally $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,1,\pi)} df = f(1,1,\pi) - f(0,0,0)$.

So we conclude that $\int_C \mathbf{F}\cdot d\mathbf{r} = -(1+\pi e)$.

Practice final exam: April 30, 2001. Prbl. 10.

Example

Use the Green Theorem to evaluate the integral $\int_C F_x\,dx + F_y\,dy$ where $F_x = y + e^x$ and $F_y = 2x^2 + \cos(y)$ and C is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,1)$ traversed counterclockwise.

Solution: Denoting $\mathbf{F} = \langle F_x, F_y\rangle$, Green's Theorem says

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA = \iint_S (\partial_x F_y - \partial_y F_x)\,dA.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (4x-1)\,dx\,dy = \int_0^1 \int_y^{2-y} (4x-1)\,dx\,dy.$$

A straightforward calculation gives $\int_C \mathbf{F}\cdot d\mathbf{r} = 3$.

Practice final exam: April 30, 2001. Prbl. 11.

Example

Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Use polar coordinates to evaluate the integral.

Solution:

$$A(S) = \iint_S d\sigma, \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$$

where $f = x^2 + y^2 + z - 4$. Therefore,

$$\nabla f = \langle 2x, 2y, 1\rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{1 + 4x^2 + 4y^2}, \quad \nabla f \cdot \mathbf{k} = 1.$$

$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta, \quad u = 1 + 4r^2, \quad du = 8r\, dr.$$

The finally obtain $A(S) = (\pi/6)(17^{3/2} - 1)$.

Practice final exam: April 30, 2001. Prbl. 12.

Example

Use the Stokes Theorem to evaluate $I = \iint_S \big[\nabla \times (y\mathbf{i})\big] \cdot \mathbf{n}\, d\sigma$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geqslant 0$.

Solution: $\mathbf{F} = \langle y, 0, 0\rangle$. The border of the hemisphere is given by the circle $x^2 + y^2 = 1$, with $z = 0$. This circle can be parametrized for $t \in [0, 2\pi]$ as

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle,$$

and we also have $\mathbf{F}(t) = \langle \sin(t), 0, 0\rangle$. Therefore,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = -\int_0^{2\pi} \sin^2(t)\, dt$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\frac{1}{2} \int_0^{2\pi} \big[1 - \cos(2t)\big]\, dt.$$

Practice final exam: April 30, 2001. Prbl. 12.

Example

Use the Stokes Theorem to evaluate $I = \iint_S \big[\nabla \times (y\mathbf{i})\big] \cdot \mathbf{n}\, d\sigma$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geqslant 0$.

Solution: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\frac{1}{2} \int_0^{2\pi} \big[1 - \cos(2t)\big]\, dt.$

Recall that

$$\int_0^{2\pi} \cos(2t)\, dt = \frac{1}{2} \left(\sin(2t)\Big|_0^{2\pi}\right) = 0.$$

Therefore, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\pi.$$

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