The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- Stokes' Theorem in space.
- Idea of the proof of Stokes' Theorem.

The curl of a vector field in space.

Definition

The *curl* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ in $\mathbb{R}^3$ is the vector field

$$\operatorname{curl} \mathbf{F} = \big\langle (\partial_2 F_3 - \partial_3 F_2), (\partial_3 F_1 - \partial_1 F_3), (\partial_1 F_2 - \partial_2 F_1) \big\rangle.$$

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Remark: Since the following formula holds,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

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$$\operatorname{curl} \mathbf{F} = (\partial_2 F_3 - \partial_3 F_2)\,\mathbf{i} - (\partial_1 F_3 - \partial_3 F_1)\,\mathbf{j} + (\partial_1 F_2 - \partial_2 F_1)\,\mathbf{k},$$

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then one also uses the notation

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}.$$

The curl of a vector field in space.

Example

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$\big(\partial_y(-y^2) - \partial_z(xyz)\big)\,\mathbf{i}$

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$$= \big(-2y - xy\big)\,\mathbf{i} - \big(0 - x\big)\,\mathbf{j} + \big(yz - 0\big)\,\mathbf{k},$$

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We conclude that

$$\nabla \times \mathbf{F} = \langle -y(2+x), x, yz \rangle.$$

$\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- **The curl of conservative fields.**
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The curl of conservative fields.

Recall: A vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ is *conservative* iff there exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

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Proof of the Theorem:

$$\nabla \times \mathbf{F} = \big\langle \big(\partial_y \partial_z f - \partial_z \partial_y f\big), -\big(\partial_x \partial_z f - \partial_z \partial_x f\big), \big(\partial_x \partial_y f - \partial_y \partial_x f\big) \big\rangle$$

□

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Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not conservative. $\triangleleft$

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Since $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbb{R}^3$ is simple connected, then $\mathbf{F}$ is conservative,

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Since $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbb{R}^3$ is simple connected, then $\mathbf{F}$ is conservative, that is, there exists f in $\mathbb{R}^3$ such that $\mathbf{F} = \nabla f$. $\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
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- **Stokes' Theorem in space.**
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Stokes' Theorem in space.

Theorem

The circulation of a differentiable vector field $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *around the boundary* C *of the oriented surface* $S \subset D$ *satisfies the equation*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma,$$

where $d\mathbf{r}$ *points counterclockwise when the unit vector* $\mathbf{n}$ *normal to* S *points in the direction to the viewer (right-hand rule).*

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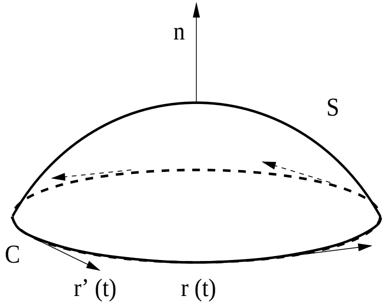

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

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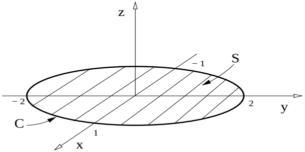

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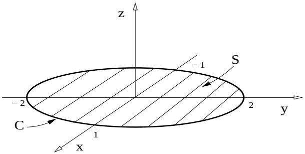

We start computing the circulation integral on the ellipse $x^2 + \frac{y^2}{2^2} = 1$.

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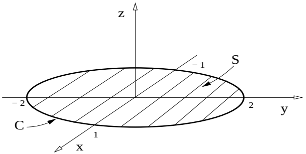

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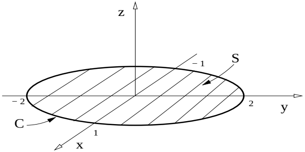

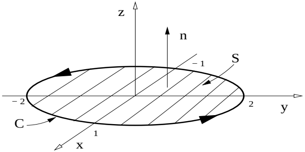

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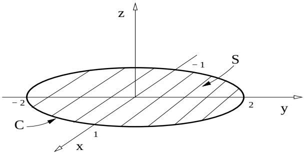

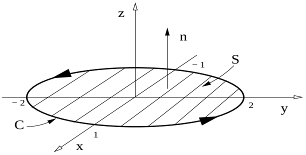

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We need to choose a counterclockwise parametrization, hence the normal to S points upwards.
We choose, for $t \in [0, 2\pi]$,

$$\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle.$$

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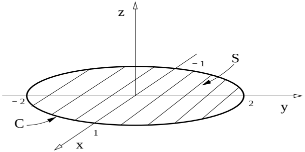

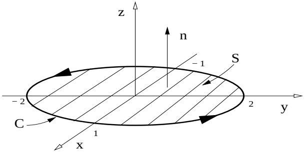

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$$\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle.$$

Therefore, the right-hand rule normal $\mathbf{n}$ to S is $\mathbf{n} = \langle 0, 0, 1 \rangle$.

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Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$, with $\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle$, $t \in [0, 2\pi]$ and $\mathbf{n} = \langle 0, 0, 1 \rangle$.

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Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, with $\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle$, $t \in [0, 2\pi]$ and $\mathbf{n} = \langle 0, 0, 1 \rangle$.
The circulation integral is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

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$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 4\cos^2(t)\,dt = \int_0^{2\pi} 2\big[1 + \cos(2t)\big]\,dt.$$

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Since $\displaystyle \int_0^{2\pi} \cos(2t)\,dt = 0$, we conclude that $\displaystyle \oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi.$

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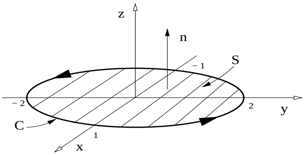
We now compute the right-hand side in Stokes' Theorem.



$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$

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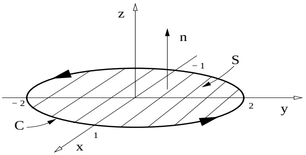

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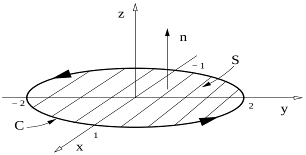

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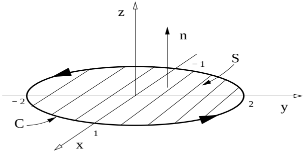

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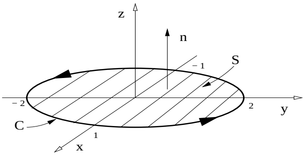

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Then, $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \langle 0, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle\, dy\, dx.$

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The right-hand side above is twice the area of the ellipse.

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This verifies Stokes' Theorem. $\lhd$

Stokes' Theorem in space.

Remark: Stokes' Theorem implies that for any smooth field $\mathbf{F}$ and any two surfaces S_1, S_2 having the same boundary curve C holds,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, d\sigma_1 = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2.$$

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Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \ : \ x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1, \ z \geqslant 0\}$.

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Solution: (The previous example was the case $a \to 0$.)

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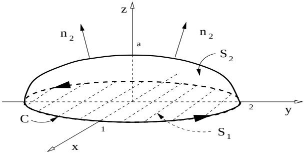

We must verify Stokes' Theorem on S_2,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2.$$

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Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \;:\; x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1,\; z \geqslant 0\}$.

Solution: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, $I = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2$.

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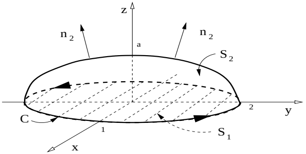

S_2 is the level surface $\mathbb{F} = 0$ of

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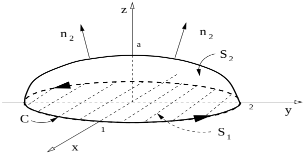

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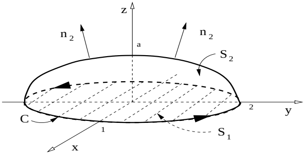

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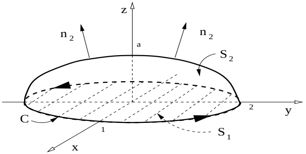
Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, $\displaystyle I = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2$.



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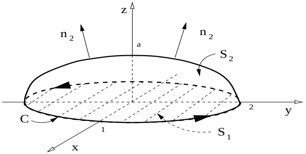

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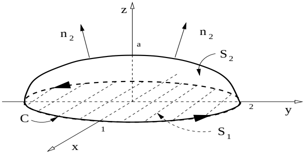

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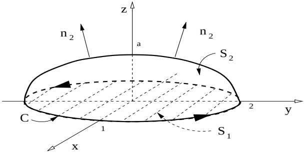

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$$\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = \iint_{S_1} 2 \, dx \, dy = 2(2\pi).$$

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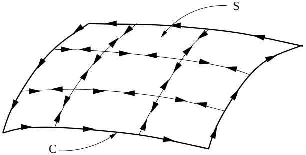
We conclude that $\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 4\pi$, no matter what is the value of $a > 0$. $\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- Stokes' Theorem in space.
- **Idea of the proof of Stokes' Theorem.**

Idea of the proof of Stokes' Theorem.

Split the surface S into n surfaces S_i, for $i = 1, \cdots, n$, as it is done in the figure for $n = 9$.



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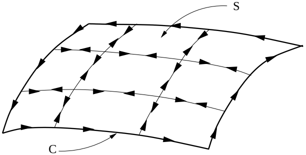

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \oint_{C_i} \mathbf{F} \cdot d\mathbf{r}_i$$

$$\simeq \sum_{i=1}^{n} \oint_{\tilde{C}_i} \mathbf{F} \cdot d\tilde{\mathbf{r}}_i \quad (\tilde{C}_i \text{ the border of small rectangles});$$

$$= \sum_{i=1}^{n} \iint_{\tilde{R}_i} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_i \, dA \text{ (Green's Theorem on a plane)};$$

$$\simeq \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$

□

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- The meaning of Curls and Divergences.
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The divergence of a vector field in space.

Definition

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y, F_z \rangle$ is the scalar field

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z.$$

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- A cooled gas contracts, so the divergence of its velocity field is negative.

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We conclude: $\nabla \times \mathbf{F} = \langle 0, 2xy, -(2xz+y) \rangle$. $\lhd$

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$$\nabla \cdot \mathbf{F} = \frac{3}{\rho^3} - 3\frac{(x^2 + y^2 + z^2)}{\rho^5}$$

The divergence of a vector field in space.

Example

Find the divergence of $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. (Notice: $|\mathbf{F}| = 1/\rho^2$.)

Solution: The field components are $F_x = \dfrac{x}{\rho^3}$, $F_y = \dfrac{y}{\rho^3}$, $F_z = \dfrac{z}{\rho^3}$.

$$\partial_x F_x = \partial_x \Big[x \big(x^2 + y^2 + z^2\big)^{-3/2} \Big]$$

$$\partial_x F_x = \big(x^2 + y^2 + z^2\big)^{-3/2} - \frac{3}{2} x \big(x^2 + y^2 + z^2\big)^{-5/2} (2x)$$

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We conclude: $\nabla \cdot \mathbf{F} = 0$. ◁

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- **The Divergence Theorem in space.**
- The meaning of Curls and Divergences.
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The Divergence Theorem in space.

Theorem

The flux of a differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *across a closed oriented surface* $S \subset \mathbb{R}^3$ *in the direction of the surface outward unit normal vector* $\mathbf{n}$ *satisfies the equation*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV,$$

where $V \subset \mathbb{R}^3$ *is the region enclosed by the surface* S*.*

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Remarks:

- The volume integral of the divergence of a field $\mathbf{F}$ in a volume V in space equals the outward flux (normal flow) of $\mathbf{F}$ across the boundary S of V.

The Divergence Theorem in space.

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Remarks:

- The volume integral of the divergence of a field $\mathbf{F}$ in a volume V in space equals the outward flux (normal flow) of $\mathbf{F}$ across the boundary S of V.
- The expansion part of the field $\mathbf{F}$ in V minus the contraction part of the field $\mathbf{F}$ in V equals the net normal flow of $\mathbf{F}$ across S out of the region V.

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\displaystyle \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$

We start with the flux integral across S.

The Divergence Theorem in space.

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Solution: Recall: $\displaystyle \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$

We start with the flux integral across S. The surface S is the level surface $f = 0$ of the function $f(x, y, z) = x^2 + y^2 + z^2 - R^2$.

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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$$\mathbf{n} = \frac{\nabla f}{|\nabla f|},$$

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$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z \rangle,$$

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$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z \rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2}$$

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z\rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z\rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R,$$

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$

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We conclude that $\mathbf{n} = \dfrac{1}{R} \langle x, y, z \rangle$, where $z = z(x, y)$.

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$,

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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We conclude that $\mathbf{n} = \dfrac{1}{R} \langle x, y, z \rangle$, where $z = z(x, y)$.

Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$, then $d\sigma = \dfrac{R}{z}\, dx\, dy$, with $z = z(x, y)$.

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \Big(\langle x, y, z \rangle \cdot \frac{1}{R} \langle x, y, z \rangle \Big) \, d\sigma.$$

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The Divergence Theorem in space.

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$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{R} \iint_S (x^2 + y^2 + z^2)\, d\sigma = R \iint_S d\sigma.$$

The integral on the sphere S can be written as the sum of the integral on the upper half plus the integral on the lower half, both integrated on the disk $R = \{x^2 + y^2 \leqslant R^2,\ z = 0\}$, that is,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2R \iint_R \frac{R}{z}\, dx\, dy.$$

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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Solution: $\displaystyle \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2R \iint_R \frac{R}{z}\, dx\, dy.$

Using polar coordinates on $\{z = 0\}$, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \int_0^{2\pi} \int_0^R \frac{R^2}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta.$$

The Divergence Theorem in space.

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The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

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Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F} \, dV$.

The Divergence Theorem in space.

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Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3$.

We now compute the volume integral $\iiint_V \nabla \cdot \mathbf{F} \, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$,

The Divergence Theorem in space.

Example

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Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F} \, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$.

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F} \, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$. Therefore

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The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$. Therefore

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = 3\Big(\frac{4}{3}\pi R^3\Big)$$

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$. Therefore

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = 3\Big(\frac{4}{3}\pi R^3\Big)$$

We obtain $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV = 4\pi R^3$.

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We obtain $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV = 4\pi R^3$.

We have verified the Divergence Theorem in this case. $\lhd$

The Divergence Theorem in space.

Example

Find the flux of the field $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$ across the boundary of the region between the spheres of radius $R_1 > R_0 > 0$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

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$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV.$$

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Since $\nabla \cdot \mathbf{F} = 0$, then $\displaystyle\iiint_V (\nabla \cdot \mathbf{F}) \, dV = 0$. Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

The flux along any surface S vanishes as long as $\mathbf{0}$ is not included in the region surrounded by S. ($\mathbf{F}$ is not differentiable at $\mathbf{0}$.) $\triangleleft$

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- **The meaning of Curls and Divergences.**
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The meaning of Curls and Divergences.

Remarks: The meaning of the Curl and the Divergence of a vector field $\mathbf{F}$ is best given through the Stokes and Divergence Theorems.

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- $\nabla \times \mathbf{F} = \lim_{S \to \{P\}} \dfrac{1}{A(S)} \oint_C \mathbf{F} \cdot d\mathbf{r}$,

  where S is a surface containing the point P with boundary given by the loop C and $A(S)$ is the area of that surface.

The meaning of Curls and Divergences.

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  where S is a surface containing the point P with boundary given by the loop C and $A(S)$ is the area of that surface.

- $\nabla \cdot \mathbf{F} = \lim_{R \to \{P\}} \frac{1}{V(R)} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$,

  where R is a region in space containing the point P with boundary given by the closed orientable surface S and $V(R)$ is the volume of that region.

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- The meaning of Curls and Divergences.
- **Applications in electromagnetism:**
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

Applications in electromagnetism: Gauss' Law.

Gauss' law: Let $q : \mathbb{R}^3 \to \mathbb{R}$ be the charge density in space, and $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ be the electric field generated by that charge. Then

$$\iiint_R q\, dV = k \iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma,$$

that is, the total charge in a region R in space with closed orientable surface S is proportional to the integral of the electric field $\mathbf{E}$ on this surface S.

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The Divergence Theorem relates surface integrals with volume integrals, that is, $\iint_S \mathbf{E} \cdot \mathbf{n} \, d\sigma = \iiint_R (\nabla \cdot \mathbf{E}) \, dV.$

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The Divergence Theorem relates surface integrals with volume integrals, that is, $\displaystyle\iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \iiint_R (\nabla \cdot \mathbf{E})\, dV$.

Using the Divergence Theorem we obtain the differential form of Gauss' law,

$$\nabla \cdot \mathbf{E} = \frac{1}{k}\, q.$$

Applications in electromagnetism: Faraday's Law.

Faraday's law: Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be the magnetic field across an orientable surface S with boundary given by the loop C, and let $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ measured on that loop. Then

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = - \oint_C \mathbf{E} \cdot d\mathbf{r},$$

that is, the time variation of the magnetic flux across S is the negative of the electromotive force on the loop.

Applications in electromagnetism: Faraday's Law.

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The Stokes Theorem relates line integrals with surface integrals, that is, $\displaystyle\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma$.

Applications in electromagnetism: Faraday's Law.

Faraday's law: Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be the magnetic field across an orientable surface S with boundary given by the loop C, and let $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ measured on that loop. Then

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The Stokes Theorem relates line integrals with surface integrals, that is, $\displaystyle\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, d\sigma$.

Using the Stokes Theorem we obtain the differential form of Faraday's law,

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}.$$

Review for Exam 4.

- Sections 16.1-16.5, 16.7, 16.8.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Line integrals (16.1).

Example

Integrate the function $f(x, y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0, 2]$, from the point $(0, 0)$ to $(2, 2)$.

Line integrals (16.1).

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Solution: We have to compute $I = \int_C f \, ds$,

Line integrals (16.1).

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Solution: We have to compute $I = \int_C f\, ds$, by that we mean

$$\int_C f\, ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\, |\mathbf{r}'(t)|\, dt,$$

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where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is a parametrization of the path C.

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$$\mathbf{r}(t) = \left\langle t, \frac{t^2}{2} \right\rangle, \quad t \in [0, 2] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, t\rangle.$$

Line integrals (16.1).

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Integrate the function $f(x, y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0, 2]$, from the point $(0, 0)$ to $(2, 2)$.

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Line integrals (16.1).

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$$\int_C f\,ds = \int_1^5 u^{1/2}\,du$$

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Line integrals (16.1).

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$$\int_C f\, ds = \int_1^5 u^{1/2}\, du = \frac{2}{3}\, u^{3/2}\Big|_1^5 = \frac{2}{3}\big(5^{3/2} - 1\big).$$

Line integrals (16.1).

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We conclude that $\displaystyle\int_C f\, ds = \frac{2}{3}\big(5\sqrt{5} - 1\big)$. ◁

Review for Exam 4.

- (16.1) Line integrals.
- **(16.2) Vector fields, work, circulation, flux (plane).**
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
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Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Vector fields, work, circulation, flux (plane) (16.2).

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Solution: The formula for the work done by a force on a particle moving along C given by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Vector fields, work, circulation, flux (plane) (16.2).

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Vector fields, work, circulation, flux (plane) (16.2).

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In this case $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Solution: The formula for the work done by a force on a particle moving along C given by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

In this case $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$. We now need to evaluate $\mathbf{F}$ along the curve, that is,

$$\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t), z(t)\big)$$

Vector fields, work, circulation, flux (plane) (16.2).

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$$\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t), z(t)\big) = \langle (t^2)t, t(t^3), -(t^3)t^2 \rangle$$

We obtain $\mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle$.

Vector fields, work, circulation, flux (plane) (16.2).

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We conclude that $W = 2^7/3$.

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Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz \rangle$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

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We conclude that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}$. $\triangleleft$

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Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

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so we obtain $\mathbf{F} \cdot \mathbf{n}\, ds = \big[\mathbf{F}_x y'(t) - F_y x'(t) \big]\, dt$.

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We evaluate $\mathbf{F}$ along the loop,

$$\mathbf{F}(t) = \big\langle -a\cos(t), a\big[\cos(t) - \sin(t)\big]\big\rangle,$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[\mathbf{F}_x y'(t) - F_y x'(t) \big]\, dt.$

We evaluate $\mathbf{F}$ along the loop,

$$\mathbf{F}(t) = \langle -a\cos(t), a\big[\cos(t) - \sin(t)\big] \rangle,$$

and compute $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

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We evaluate $\mathbf{F}$ along the loop,

$$\mathbf{F}(t) = \big\langle -a\cos(t), a\big[\cos(t) - \sin(t)\big] \big\rangle,$$

and compute $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$. Therefore,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[-a\cos(t) a\cos(t) - a\big(\cos(t) - \sin(t)\big)(-a)\sin(t) \big]\, dt$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

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$$\mathbf{F}(t) = \big\langle -a\cos(t), a\big[\cos(t) - \sin(t)\big]\big\rangle,$$

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$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a\cos(t)a\cos(t) - a\big(\cos(t) - \sin(t)\big)(-a)\sin(t)\big]\,dt$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\,dt$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[-a^2 \cos^2(t) + a^2 \sin(t)\cos(t) - a^2 \sin^2(t) \right] dt.$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[-a^2 \cos^2(t) + a^2 \sin(t)\cos(t) - a^2 \sin^2(t) \right] dt.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \left[-1 + \sin(t)\cos(t) \right] dt,$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t) \right] dt.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \left[-1 + \sin(t)\cos(t) \right] dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \left[-1 + \frac{1}{2}\sin(2t) \right] dt.$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x - y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\right] dt.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \left[-1 + \sin(t)\cos(t)\right] dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \left[-1 + \frac{1}{2}\sin(2t)\right] dt.$$

Since $\displaystyle\int_0^{2\pi} \sin(2t)\, dt = 0,$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\, dt.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \big[-1 + \sin(t)\cos(t)\big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = a^2 \int_0^{2\pi} \Big[-1 + \frac{1}{2}\sin(2t)\Big]\, dt.$$

Since $\int_0^{2\pi} \sin(2t)\, dt = 0$, we obtain $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -2\pi a^2$. $\lhd$

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- **(16.3) Conservative fields, potential functions.**
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

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$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y \cos(z) = \partial_z F_x,$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field,

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative? If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative? If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x \cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y \cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$. The equations for f are

$$\partial_x f = y \sin(z), \quad \partial_y f = x \sin(z), \quad \partial_z f = xy \cos(z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x, y, z) = xy\sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z)$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y \sin(z)$, $\partial_y f = x \sin(z)$, $\partial_z f = xy \cos(z)$.
Integrating in x the first equation we get

$$f(x, y, z) = xy \sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x \sin(z) + \partial_y g = x \sin(z) \quad \Rightarrow \quad \partial_y g(y, z) = 0,$$

so $g(y, z) = h(z)$. That is, $f(x, y, z) = xy \sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy \cos(z) + h'(z) = xy \cos(z)$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative? If "yes", then find the potential function.

Solution: $\partial_x f = y \sin(z)$, $\partial_y f = x \sin(z)$, $\partial_z f = xy \cos(z)$.
Integrating in x the first equation we get

$$f(x, y, z) = xy \sin(z) + g(y, z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x \sin(z) + \partial_y g = x \sin(z) \quad \Rightarrow \quad \partial_y g(y, z) = 0,$$

so $g(y, z) = h(z)$. That is, $f(x, y, z) = xy \sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy \cos(z) + h'(z) = xy \cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

We conclude that $f(x,y,z) = xy\sin(z) + c$. $\triangleleft$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative,

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$,

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle\cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent,

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz \big]$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz \big]$$

$$I = f(1, 2, \pi/2) - f(1, 1, \pi/2)$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz \big]$$

$$I = f(1, 2, \pi/2) - f(1, 1, \pi/2) = 2\sin(\pi/2) - \sin(\pi/2)$$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz$, where C given by $\mathbf{r}(t) = \langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2 \rangle$ for $t \in [0, 1]$.

Solution: We know that the field $\mathbf{F} = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz.$$

We have computed f already, $f = xy \sin(z) + c$.
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$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y \sin(z)\, dx + x \sin(z)\, dy + xy \cos(z)\, dz \big]$$

$$I = f(1, 2, \pi/2) - f(1, 1, \pi/2) = 2 \sin(\pi/2) - \sin(\pi/2) \;\Rightarrow\; I = 1.$$