The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- Stokes' Theorem in space.
- Idea of the proof of Stokes' Theorem.

The curl of a vector field in space.

Definition
The *curl* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ in $\mathbb{R}^3$ is the vector field

$$\operatorname{curl} \mathbf{F} = \big\langle (\partial_2 F_3 - \partial_3 F_2), (\partial_3 F_1 - \partial_1 F_3), (\partial_1 F_2 - \partial_2 F_1) \big\rangle.$$

Remark: Since the following formula holds,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl} \mathbf{F} = (\partial_2 F_3 - \partial_3 F_2)\,\mathbf{i} - (\partial_1 F_3 - \partial_3 F_1)\,\mathbf{j} + (\partial_1 F_2 - \partial_2 F_1)\,\mathbf{k},$$

then one also uses the notation

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}.$$

The curl of a vector field in space.

Example

Find the curl of the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$.

Solution: Since $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$, we get,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xyz & -y^2 \end{vmatrix} =$$

$$\big(\partial_y(-y^2) - \partial_z(xyz)\big)\,\mathbf{i} - \big(\partial_x(-y^2) - \partial_z(xz)\big)\,\mathbf{j} + \big(\partial_x(xyz) - \partial_y(xz)\big)\,\mathbf{k},$$

$$= \big(-2y - xy\big)\,\mathbf{i} - \big(0 - x\big)\,\mathbf{j} + \big(yz - 0\big)\,\mathbf{k},$$

We conclude that

$$\nabla \times \mathbf{F} = \langle -y(2+x), x, yz \rangle.$$

$\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- **The curl of conservative fields.**
- Stokes' Theorem in space.
- Idea of the proof of Stokes' Theorem.

The curl of conservative fields.

Recall: A vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ is *conservative* iff there exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

Theorem

If a vector field $\mathbf{F}$ *is conservative, then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Remark:

- This Theorem is usually written as $\nabla \times (\nabla f) = \mathbf{0}$.
- The converse is true only on simple connected sets. That is, if a vector field $\mathbf{F}$ satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ on a simple connected domain D, then there exists a scalar field $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

Proof of the Theorem:

$$\nabla \times \mathbf{F} = \big\langle (\partial_y \partial_z f - \partial_z \partial_y f), -(\partial_x \partial_z f - \partial_z \partial_x f), (\partial_x \partial_y f - \partial_y \partial_x f) \big\rangle$$

$\square$

The curl of conservative fields.

Example

Is the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$ conservative?

Solution: We have shown that $\nabla \times \mathbf{F} = \langle -y(2+x), x, yz \rangle$. Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not conservative. $\triangleleft$

Example

Is the vector field $\mathbf{F} = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$ conservative in $\mathbb{R}^3$?

Solution: Notice that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$$

$$= \big\langle (6xyz^2 - 6xyz^2), -(3y^2 z^2 - 3y^2 z^2), (2yz^3 - 2yz^3) \big\rangle = \mathbf{0}.$$

Since $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbb{R}^3$ is simple connected, then $\mathbf{F}$ is conservative, that is, there exists f in $\mathbb{R}^3$ such that $\mathbf{F} = \nabla f$. $\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- **Stokes' Theorem in space.**
- Idea of the proof of Stokes' Theorem.

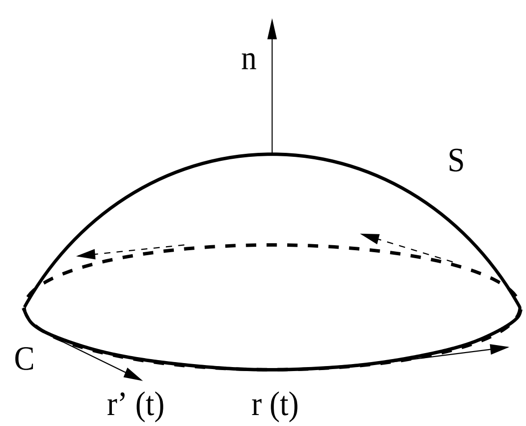
Stokes' Theorem in space.

Theorem

The circulation of a differentiable vector field $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ *around the boundary* C *of the oriented surface* $S \subset D$ *satisfies the equation*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma,$$

where $d\mathbf{r}$ *points counterclockwise when the unit vector* $\mathbf{n}$ *normal to* S *points in the direction to the viewer (right-hand rule).*

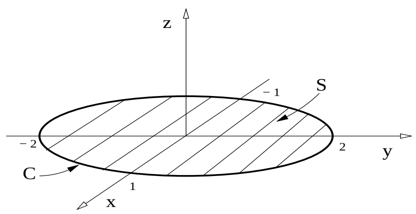


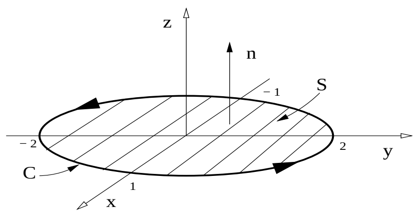
Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \; : \; 4x^2 + y^2 \leqslant 4, \; z = 0\}$.

Solution: We compute both sides in $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$.





We start computing the circulation integral on the ellipse $x^2 + \frac{y^2}{2^2} = 1$.
We need to choose a counterclockwise parametrization, hence the normal to S points upwards.
We choose, for $t \in [0, 2\pi]$,

$$\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle.$$

Therefore, the right-hand rule normal $\mathbf{n}$ to S is $\mathbf{n} = \langle 0, 0, 1 \rangle$.

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \; : \; 4x^2 + y^2 \leqslant 4, \; z = 0\}$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$, with $\mathbf{r}(t) = \langle \cos(t), 2\sin(t), 0 \rangle$, $t \in [0, 2\pi]$ and $\mathbf{n} = \langle 0, 0, 1 \rangle$.
The circulation integral is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{2\pi} \langle \cos^2(t), 2\cos(t), 0 \rangle \cdot \langle -\sin(t), 2\cos(t), 0 \rangle \, dt.$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left[-\cos^2(t)\sin(t) + 4\cos^2(t) \right] dt.$$

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x,y,z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \big[-\cos^2(t)\sin(t) + 4\cos^2(t)\big]\,dt.$

The substitution on the first term $u = \cos(t)$ and $du = -\sin(t)\,dt$, implies $\displaystyle\int_0^{2\pi} -\cos^2(t)\sin(t)\,dt = \int_1^1 u^2\,du = 0.$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 4\cos^2(t)\,dt = \int_0^{2\pi} 2\big[1 + \cos(2t)\big]\,dt.$$

Since $\displaystyle\int_0^{2\pi} \cos(2t)\,dt = 0$, we conclude that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi.$

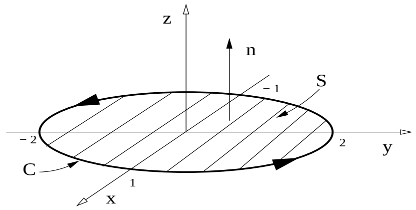
Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x,y,z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ and $\mathbf{n} = \langle 0,0,1 \rangle$.

We now compute the right-hand side in Stokes' Theorem.



$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2x & z^2 \end{vmatrix} \quad \Rightarrow \quad \nabla \times \mathbf{F} = \langle 0,0,2 \rangle.$$

S is the flat surface $\{x^2 + \frac{y^2}{2^2} \leqslant 1, \ z = 0\}$, so $d\sigma = dx\,dy$.

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \{(x, y, z) \ : \ 4x^2 + y^2 \leqslant 4, \ z = 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\mathbf{n} = \langle 0, 0, 1 \rangle$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, and $d\sigma = dx\, dy$.

Then, $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \langle 0, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle \, dy\, dx.$

The right-hand side above is twice the area of the ellipse. Since we know that an ellipse $x^2/a^2 + y^2/b^2 = 1$ has area πab, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 4\pi.$$

This verifies Stokes' Theorem. $\triangleleft$

Stokes' Theorem in space.

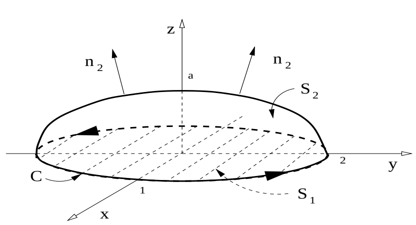
Remark: Stokes' Theorem implies that for any smooth field $\mathbf{F}$ and any two surfaces S_1, S_2 having the same boundary curve C holds,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, d\sigma_1 = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, d\sigma_2.$$

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \ : \ x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1, \ z \geqslant 0\}$.

Solution: (The previous example was the case $a \to 0$.)



We must verify Stokes' Theorem on S_2,

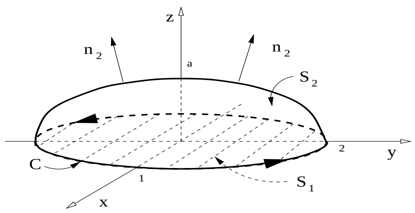
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, d\sigma_2.$$

Stokes' Theorem in space.

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \;:\; x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1,\; z \geqslant 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$, $\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$, $\displaystyle I = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2$.



S_2 is the level surface $\mathbb{F} = 0$ of

$$\mathbb{F}(x, y, z) = x^2 + \frac{y^2}{2^2} + \frac{z^2}{a^2} - 1.$$

$$\mathbf{n}_2 = \frac{\nabla \mathbb{F}}{|\nabla \mathbb{F}|}, \quad \nabla \mathbb{F} = \Big\langle 2x, \frac{y}{2}, \frac{2z}{a^2} \Big\rangle, \quad (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 = 2\,\frac{2z/a^2}{|\nabla \mathbb{F}|}.$$

$$d\sigma_2 = \frac{|\nabla \mathbb{F}|}{|\nabla \mathbb{F} \cdot \mathbf{k}|} = \frac{|\nabla \mathbb{F}|}{2z/a^2} \quad \Rightarrow \quad (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 2.$$

Example

Verify Stokes' Theorem for the field $\mathbf{F} = \langle x^2, 2x, z^2 \rangle$ on any half-ellipsoid $S_2 = \{(x, y, z) \;:\; x^2 + \dfrac{y^2}{2^2} + \dfrac{z^2}{a^2} = 1,\; z \geqslant 0\}$.

Solution: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 2$.

Therefore,

$$\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = \iint_{S_1} 2 \, dx \, dy = 2(2\pi).$$

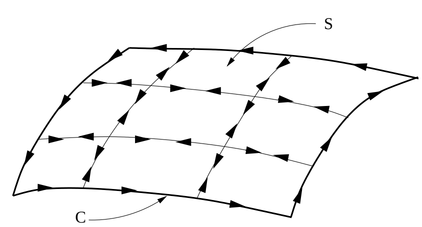
We conclude that $\displaystyle\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma_2 = 4\pi$, no matter what is the value of $a > 0$. $\triangleleft$

The Stokes Theorem. (Sect. 16.7)

- The curl of a vector field in space.
- The curl of conservative fields.
- Stokes' Theorem in space.
- **Idea of the proof of Stokes' Theorem.**

Idea of the proof of Stokes' Theorem.

Split the surface S into n surfaces S_i, for $i = 1, \cdots, n$, as it is done in the figure for $n = 9$.



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \oint_{C_i} \mathbf{F} \cdot d\mathbf{r}_i$$

$$\simeq \sum_{i=1}^{n} \oint_{\tilde{C}_i} \mathbf{F} \cdot d\tilde{\mathbf{r}}_i \quad (\tilde{C}_i \text{ the border of small rectangles});$$

$$= \sum_{i=1}^{n} \iint_{\tilde{R}_i} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_i \, dA \text{ (Green's Theorem on a plane)};$$

$$\simeq \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma. \qquad \square$$

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- The meaning of Curls and Divergences.
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The divergence of a vector field in space.

Definition

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y, F_z \rangle$ is the scalar field

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z.$$

Remarks:

- It is also used the notation $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.
- The divergence of a vector field measures the expansion (positive divergence) or contraction (negative divergence) of the vector field.
- A heated gas expands, so the divergence of its velocity field is positive.
- A cooled gas contracts, so the divergence of its velocity field is negative.

The divergence of a vector field in space.

Example

Find the divergence and the curl of $\mathbf{F} = \langle 2xyz, -xy, -z^2 \rangle$.

Solution: Recall: $\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$.

$$\partial_x F_x = 2yz, \quad \partial_y F_y = -x, \quad \partial_z F_z = -2z.$$

Therefore $\nabla \cdot \mathbf{F} = 2yz - x - 2z$, that is $\nabla \cdot \mathbf{F} = 2z(y-1) - x$.

Recall: $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & -xy & -z^2 \end{vmatrix} = \langle (0-0), -(0-2xy), (-y-2xz) \rangle$$

We conclude: $\nabla \times \mathbf{F} = \langle 0, 2xy, -(2xz+y) \rangle$. $\triangleleft$

The divergence of a vector field in space.

Example

Find the divergence of $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. (Notice: $|\mathbf{F}| = 1/\rho^2$.)

Solution: The field components are $F_x = \dfrac{x}{\rho^3}$, $F_y = \dfrac{y}{\rho^3}$, $F_z = \dfrac{z}{\rho^3}$.

$$\partial_x F_x = \partial_x \big[x (x^2+y^2+z^2)^{-3/2} \big]$$

$$\partial_x F_x = (x^2+y^2+z^2)^{-3/2} - \frac{3}{2} x (x^2+y^2+z^2)^{-5/2} (2x)$$

$$\partial_x F_x = \frac{1}{\rho^3} - 3\frac{x^2}{\rho^5} \;\Rightarrow\; \partial_y F_y = \frac{1}{\rho^3} - 3\frac{y^2}{\rho^5}, \quad \partial_z F_z = \frac{1}{\rho^3} - 3\frac{z^2}{\rho^5}.$$

$$\nabla \cdot \mathbf{F} = \frac{3}{\rho^3} - 3\frac{(x^2+y^2+z^2)}{\rho^5} = \frac{3}{\rho^3} - 3\frac{\rho^2}{\rho^5} = \frac{3}{\rho^3} - \frac{3}{\rho^3}.$$

We conclude: $\nabla \cdot \mathbf{F} = 0$. $\triangleleft$

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- **The Divergence Theorem in space.**
- The meaning of Curls and Divergences.
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The Divergence Theorem in space.

Theorem

The flux of a differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *across a closed oriented surface* $S \subset \mathbb{R}^3$ *in the direction of the surface outward unit normal vector* $\mathbf{n}$ *satisfies the equation*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV,$$

where $V \subset \mathbb{R}^3$ *is the region enclosed by the surface* S*.*

Remarks:

- The volume integral of the divergence of a field $\mathbf{F}$ in a volume V in space equals the outward flux (normal flow) of $\mathbf{F}$ across the boundary S of V.
- The expansion part of the field $\mathbf{F}$ in V minus the contraction part of the field $\mathbf{F}$ in V equals the net normal flow of $\mathbf{F}$ across S out of the region V.

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV$.

We start with the flux integral across S. The surface S is the level surface $f = 0$ of the function $f(x, y, z) = x^2 + y^2 + z^2 - R^2$. Its outward unit normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \langle 2x, 2y, 2z \rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2R,$$

We conclude that $\mathbf{n} = \frac{1}{R} \langle x, y, z \rangle$, where $z = z(x, y)$.

Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy$, then $d\sigma = \dfrac{R}{z} \, dx \, dy$, with $z = z(x, y)$.

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: Recall: $\mathbf{n} = \frac{1}{R} \langle x, y, z \rangle$, $d\sigma = \dfrac{R}{z} \, dx \, dy$, with $z = z(x, y)$.

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \Big(\langle x, y, z \rangle \cdot \frac{1}{R} \langle x, y, z \rangle \Big) \, d\sigma.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \frac{1}{R} \iint_S (x^2 + y^2 + z^2) \, d\sigma = R \iint_S d\sigma.$$

The integral on the sphere S can be written as the sum of the integral on the upper half plus the integral on the lower half, both integrated on the disk $R = \{x^2 + y^2 \leqslant R^2, \; z = 0\}$, that is,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 2R \iint_R \frac{R}{z} \, dx \, dy.$$

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2R \iint_R \frac{R}{z}\, dx\, dy$.

Using polar coordinates on $\{z = 0\}$, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \int_0^{2\pi} \int_0^R \frac{R^2}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta.$$

The substitution $u = R^2 - r^2$ implies $du = -2r\, dr$, so,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^2 \int_{R^2}^0 u^{-1/2} \frac{(-du)}{2} = 2\pi R^2 \int_0^{R^2} u^{-1/2} du$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2\pi R^2 \Big(2u^{1/2} \Big|_0^{R^2} \Big) \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3.$$

The Divergence Theorem in space.

Example

Verify the Divergence Theorem for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = R^2$.

Solution: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi R^3$.

We now compute the volume integral $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV$. The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1$, that is, $\nabla \cdot \mathbf{F} = 3$. Therefore

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = 3\Big(\frac{4}{3}\pi R^3\Big)$$

We obtain $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV = 4\pi R^3$.

We have verified the Divergence Theorem in this case. $\triangleleft$

The Divergence Theorem in space.

Example

Find the flux of the field $\mathbf{F} = \dfrac{\mathbf{r}}{\rho^3}$ across the boundary of the region between the spheres of radius $R_1 > R_0 > 0$, where $\mathbf{r} = \langle x, y, z \rangle$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Solution: We use the Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$$

Since $\nabla \cdot \mathbf{F} = 0$, then $\displaystyle\iiint_V (\nabla \cdot \mathbf{F})\, dV = 0.$ Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

The flux along any surface S vanishes as long as $\mathbf{0}$ is not included in the region surrounded by S. ($\mathbf{F}$ is not differentiable at $\mathbf{0}$.) ◁

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- **The meaning of Curls and Divergences.**
- Applications in electromagnetism:
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

The meaning of Curls and Divergences.

Remarks: The meaning of the Curl and the Divergence of a vector field **F** is best given through the Stokes and Divergence Theorems.

- $\nabla \times \mathbf{F} = \lim_{S \to \{P\}} \frac{1}{A(S)} \oint_C \mathbf{F} \cdot d\mathbf{r}$,

  where S is a surface containing the point P with boundary given by the loop C and $A(S)$ is the area of that surface.

- $\nabla \cdot \mathbf{F} = \lim_{R \to \{P\}} \frac{1}{V(R)} \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$,

  where R is a region in space containing the point P with boundary given by the closed orientable surface S and $V(R)$ is the volume of that region.

The Divergence Theorem. (Sect. 16.8)

- The divergence of a vector field in space.
- The Divergence Theorem in space.
- The meaning of Curls and Divergences.
- **Applications in electromagnetism:**
  - Gauss' law. (Divergence Theorem.)
  - Faraday's law. (Stokes Theorem.)

Applications in electromagnetism: Gauss' Law.

Gauss' law: Let $q : \mathbb{R}^3 \to \mathbb{R}$ be the charge density in space, and $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ be the electric field generated by that charge. Then

$$\iiint_R q\, dV = k \iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma,$$

that is, the total charge in a region R in space with closed orientable surface S is proportional to the integral of the electric field $\mathbf{E}$ on this surface S.

The Divergence Theorem relates surface integrals with volume integrals, that is, $\iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \iiint_R (\nabla \cdot \mathbf{E})\, dV$.

Using the Divergence Theorem we obtain the differential form of Gauss' law,

$$\nabla \cdot \mathbf{E} = \frac{1}{k}\, q.$$

Applications in electromagnetism: Faraday's Law.

Faraday's law: Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be the magnetic field across an orientable surface S with boundary given by the loop C, and let $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ measured on that loop. Then

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = - \oint_C \mathbf{E} \cdot d\mathbf{r},$$

that is, the time variation of the magnetic flux across S is the negative of the electromotive force on the loop.

The Stokes Theorem relates line integrals with surface integrals, that is, $\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma$.

Using the Stokes Theorem we obtain the differential form of Faraday's law,

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}.$$

Review for Exam 4.

- Sections 16.1-16.5, 16.7, 16.8.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Line integrals (16.1).

Example

Integrate the function $f(x, y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0, 2]$, from the point $(0, 0)$ to $(2, 2)$.

Solution: We have to compute $I = \int_C f\, ds$, by that we mean

$$\int_C f\, ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\, |\mathbf{r}'(t)|\, dt,$$

where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is a parametrization of the path C. In this case the path is given by the parabola $y = x^2/2$, so a simple parametrization is to use $x = t$, that is,

$$\mathbf{r}(t) = \Big\langle t, \frac{t^2}{2}\Big\rangle, \quad t \in [0, 2] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, t\rangle.$$

Line integrals (16.1).

Example

Integrate the function $f(x, y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0, 2]$, from the point $(0, 0)$ to $(2, 2)$.

Solution: $\mathbf{r}(t) = \Big\langle t, \frac{t^2}{2}\Big\rangle$ for $t \in [0, 2]$, and $\mathbf{r}'(t) = \langle 1, t\rangle$.

$$\int_C f\, ds = \int_{t_0}^{t_1} f\big(x(t), y(t)\big)\, |\mathbf{r}'(t)|\, dt = \int_0^2 \frac{t^3}{t^2/2}\sqrt{1 + t^2}\, dt,$$

$$\int_C f\, ds = \int_0^2 2t\sqrt{1 + t^2}\, dt, \quad u = 1 + t^2, \quad du = 2t\, dt.$$

$$\int_C f\, ds = \int_1^5 u^{1/2}\, du = \frac{2}{3}\, u^{3/2}\Big|_1^5 = \frac{2}{3}\big(5^{3/2} - 1\big).$$

We conclude that $\int_C f\, ds = \frac{2}{3}\big(5\sqrt{5} - 1\big)$. ◁

Review for Exam 4.

- (16.1) Line integrals.
- **(16.2) Vector fields, work, circulation, flux (plane).**
- (16.3) Conservative fields, potential functions.
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Solution: The formula for the work done by a force on a particle moving along C given by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

In this case $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$. We now need to evaluate $\mathbf{F}$ along the curve, that is,

$$\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t), z(t)\big) = \langle (t^2)t, t(t^3), -(t^3)t^2 \rangle$$

We obtain $\mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle$.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy \rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ for $t \in [0, 2]$.

Solution: $\mathbf{F}(t) = \langle t^3, t^4, -t^5 \rangle$ and $\mathbf{r}'(t) = \langle 3t^2, 2t, 1 \rangle$ for $t \in [0, 2]$. The Work done by the force on the particle is

$$W = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^2 \langle t^3, t^4, -t^5 \rangle \cdot \langle 3t^2, 2t, 1 \rangle\, dt$$

$$W = \int_0^2 \left(3t^5 + 2t^5 - t^5\right) dt = \int_0^2 4t^5\, dt = \frac{4}{6} t^6 \Big|_0^2 = \frac{2}{3}\, 2^6.$$

We conclude that $W = 2^7/3$.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz \rangle$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: The flow (also called circulation) of the field $\mathbf{F}$ along a curve C parametrized by $\mathbf{r}(t)$ for $t \in [t_0, t_1]$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt.$$

We use $t = x$ as the parameter of the curve $\mathbf{r}$, so we obtain

$$\mathbf{r}(t) = \langle t, t^2, t \rangle, \quad t \in [0, 1] \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 1 \rangle.$$

$$\mathbf{F}(t) = \langle t(t^2), (t^2)^2, -t^2(t) \rangle \quad \Rightarrow \quad \mathbf{F}(t) = \langle t^3, t^4, -t^3 \rangle.$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz\rangle$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: $\mathbf{r}'(t) = \langle 1, 2t, 1\rangle$ for $t \in [0,1]$ and $\mathbf{F}(t) = \langle t^3, t^4, -t^3\rangle$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t)\cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle t^3, t^4, -t^3\rangle \cdot \langle 1, 2t, 1\rangle\,dt,$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(t^3 + 2t^5 - t^3\right)dt = \int_0^1 2t^5\,dt = \frac{2}{6}\,t^6\Big|_0^1.$$

We conclude that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{3}$. ◁

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y)\rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$.

Solution: The flux (also normal flow) of the field $\mathbf{F} = \langle F_x, F_y\rangle$ across a loop C parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $t \in [t_0, t_1]$ is given by

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \left[\mathbf{F}_x y'(t) - F_y x'(t)\right]dt.$$

Recall that $\mathbf{n} = \dfrac{1}{|\mathbf{r}'(t)|}\,\langle y'(y), -x'(t)\rangle$ and $ds = |\mathbf{r}'(t)|\,dt$, therefore

$$\mathbf{F}\cdot\mathbf{n}\,ds = \left(\langle F_x, F_y\rangle \cdot \frac{1}{|\mathbf{r}'(t)|}\,\langle y'(y), -x'(t)\rangle\right)|\mathbf{r}'(t)|\,dt,$$

so we obtain $\mathbf{F}\cdot\mathbf{n}\,ds = \left[\mathbf{F}_x y'(t) - F_y x'(t)\right]dt$.

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[\mathbf{F}_x y'(t) - F_y x'(t)\big]\,dt.$

We evaluate $\mathbf{F}$ along the loop,

$$\mathbf{F}(t) = \langle -a\cos(t), a\big[\cos(t) - \sin(t)\big] \rangle,$$

and compute $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$. Therefore,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a\cos(t)a\cos(t) - a\big(\cos(t) - \sin(t)\big)(-a)\sin(t)\big]\,dt$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\,dt$$

Vector fields, work, circulation, flux (plane) (16.2).

Example

Find the flux of the field $\mathbf{F} = \langle -x, (x-y) \rangle$ across loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[-a^2\cos^2(t) + a^2\sin(t)\cos(t) - a^2\sin^2(t)\big]\,dt.$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = a^2\int_0^{2\pi} \big[-1 + \sin(t)\cos(t)\big]\,dt,$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = a^2\int_0^{2\pi} \Big[-1 + \frac{1}{2}\sin(2t)\Big]\,dt.$$

Since $\displaystyle\int_0^{2\pi} \sin(2t)\,dt = 0$, we obtain $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = -2\pi a^2$. $\lhd$

Review for Exam 4.

- (16.1) Line integrals.
- (16.2) Vector fields, work, circulation, flux (plane).
- **(16.3) Conservative fields, potential functions.**
- (16.4) The Green Theorem in a plane.
- (16.5) Surface area, surface integrals.
- (16.7) The Stokes Theorem.
- (16.8) The Divergence Theorem.

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: We need to check the equations

$$\partial_y F_z = \partial_z F_y, \quad \partial_x F_z = \partial_z F_x, \quad \partial_x F_y = \partial_y F_x.$$

$$\partial_y F_z = x\cos(z) = \partial_z F_y,$$

$$\partial_x F_z = y\cos(z) = \partial_z F_x,$$

$$\partial_x F_y = \sin(z) = \partial_y F_x.$$

Therefore, $\mathbf{F}$ is a conservative field, that means there exists a scalar field f such that $\mathbf{F} = \nabla f$. The equations for f are

$$\partial_x f = y\sin(z), \quad \partial_y f = x\sin(z), \quad \partial_z f = xy\cos(z).$$

Conservative fields, potential functions (16.3).

Example

Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
If "yes", then find the potential function.

Solution: $\partial_x f = y\sin(z)$, $\partial_y f = x\sin(z)$, $\partial_z f = xy\cos(z)$.
Integrating in x the first equation we get

$$f(x,y,z) = xy\sin(z) + g(y,z).$$

Introduce this expression in the second equation above,

$$\partial_y f = x\sin(z) + \partial_y g = x\sin(z) \quad \Rightarrow \quad \partial_y g(y,z) = 0,$$

so $g(y,z) = h(z)$. That is, $f(x,y,z) = xy\sin(z) + h(z)$.
Introduce this expression into the last equation above,

$$\partial_z f = xy\cos(z) + h'(z) = xy\cos(z) \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = c.$$

We conclude that $f(x,y,z) = xy\sin(z) + c$. $\triangleleft$

Conservative fields, potential functions (16.3).

Example

Compute $I = \int_C y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz$, where C given by $\mathbf{r}(t) = \langle\cos(2\pi t), 1+t^5, \cos^2(2\pi t)\pi/2\rangle$ for $t \in [0,1]$.

Solution: We know that the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative, so there exists f such that $\mathbf{F} = \nabla f$, or equivalently

$$df = y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz.$$

We have computed f already, $f = xy\sin(z) + c$.
Since $\mathbf{F}$ is conservative, the integral I is path independent, and

$$I = \int_{(1,1,\pi/2)}^{(1,2,\pi/2)} \big[y\sin(z)\,dx + x\sin(z)\,dy + xy\cos(z)\,dz\big]$$

$$I = f(1,2,\pi/2) - f(1,1,\pi/2) = 2\sin(\pi/2) - \sin(\pi/2) \;\Rightarrow\; I = 1.$$