Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- The area of a surface in space.

Next class:

- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
- Mass and center of mass thin shells.

Review: Arc length and line integrals.

- The integral of a function $f : [a, b] \to \mathbb{R}$ is
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- The double integral of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on a region $R \subset \mathbb{R}^2$, which is the volume under the graph of f and above the $z = 0$ plane, and is given by

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The area of a surface in space.

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, *the area of a level surface* $S = \{f(x, y, z) = 0\}$, *over a closed, bounded region* R *in the plane* $\{z = 0\}$, *is given by*

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

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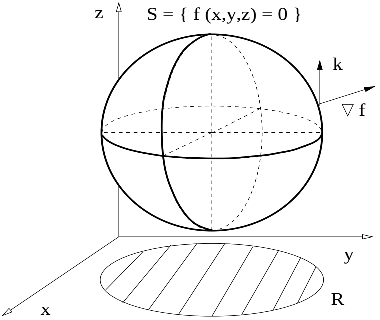

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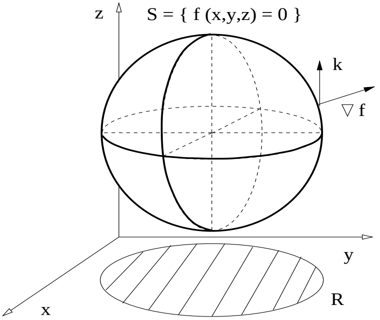

Remark: Eq. (3), page 1183, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

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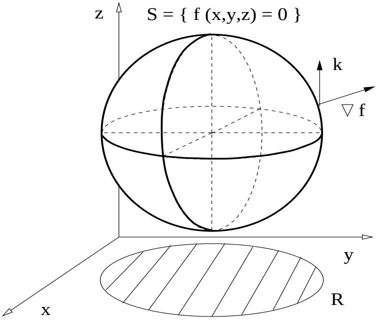

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The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

The area of a surface in space.

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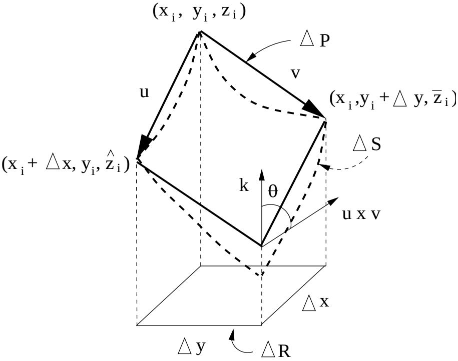

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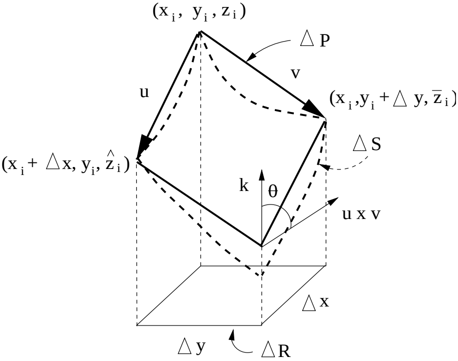

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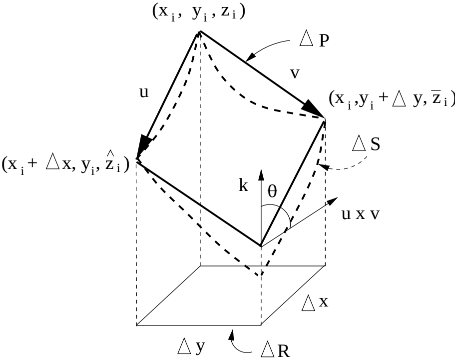

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$$\Delta P = |\mathbf{u} \times \mathbf{v}|,$$

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$$\mathbf{u} = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle,$$
$$\mathbf{v} = \langle 0, \Delta y, (z_i - \bar{z}_i) \rangle.$$

Therefore,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & (z_i - \hat{z}_i) \\ 0 & \Delta y & (z_i - \bar{z}_i) \end{vmatrix} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle.$$

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Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y(z_i - \hat{z}_i), -\Delta x(z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

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The area of a surface in space.

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Solution: The surface is the level surface of the function $f(x, y, z) = x^2 + y^2 - z$. The region R is the disk $z = x^2 + y^2 \leqslant 4$.

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy,$$

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Since R is a disk radius 2, it is convenient to use polar coordinates in $\mathbb{R}^2$.

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$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\,r\,dr\,d\theta.$$

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We conclude: $A(S) = \dfrac{\pi}{6} \big[(17)^{3/2} - 1 \big]$. $\triangleleft$

The area of a surface in space.

Remark: The formula for the area of a surface in space can be generalized as follows.

The area of a surface in space.

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Theorem

The area of a surface S given by $f(x, y, z) = 0$ over a closed and bounded plane region R in space is given by

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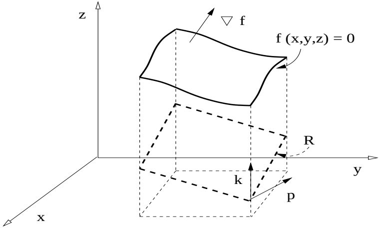

where $\mathbf{p}$ is a unit vector normal to the region R and $\nabla f \cdot \mathbf{p} \neq 0$.

The area of a surface in space.

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

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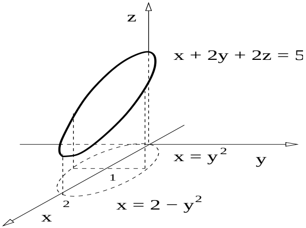

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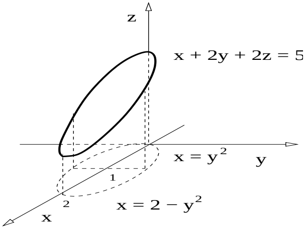

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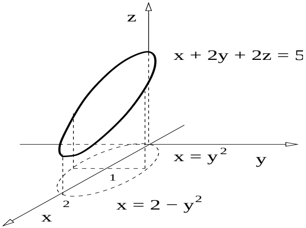

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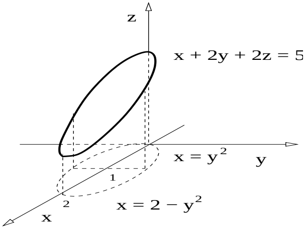

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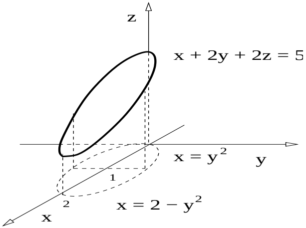

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$$A(S) = 3\Big(2 - \frac{2}{3}\Big) = 3\,\frac{4}{3} \quad \Rightarrow \quad A(S) = 4. \qquad \triangleleft$$

Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
- Mass and center of mass thin shells.

Review: The area of a surface in space.

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$*, the area of a level surface* $S = \{f(x, y, z) = 0\}$*, over a closed, bounded region* R *in the plane* $\{z = 0\}$*, is given by*

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

Review: The area of a surface in space.

Theorem

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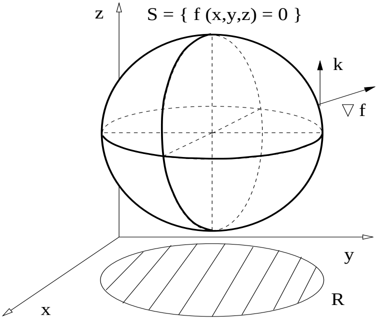

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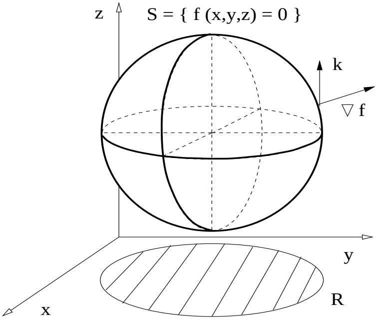

Remark: Eq. (3), page 1183, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

Review: The area of a surface in space.

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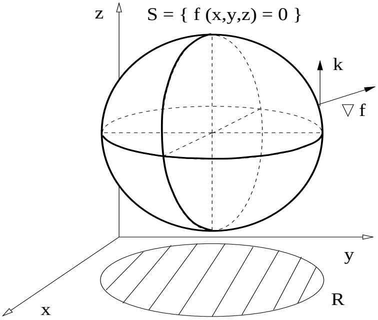

Remark: Eq. (3), page 1183, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
- **Surface integrals of a scalar field.**
- The flux of a vector field on a surface.
- Mass and center of mass thin shells.

Surface integrals of a scalar field.

Theorem

The integral of a continuous scalar function $g : \mathbb{R}^3 \to \mathbb{R}$ *over a surface* S *defined as the level set of* $f(x, y, z) = 0$ *over the bounded plane* R *is given by*

$$\iint_S g\, d\sigma = \iint_R g\, \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA,$$

where $\mathbf{p}$ *is a unit vector normal to* R *and* $\nabla f \cdot \mathbf{p} \neq 0$*.*

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where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Remark: In the particular case $g = 1$, we recover the formula for the area $A(S) = \iint_S d\sigma$ of the surface S, that is,

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA.$$

Surface integrals of a scalar field.

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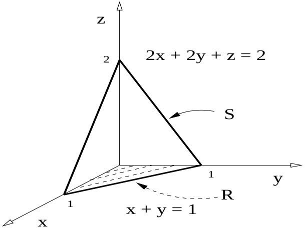

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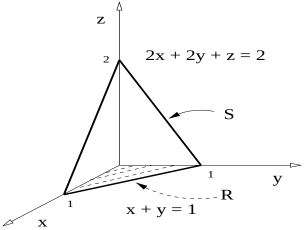

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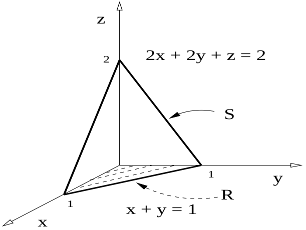

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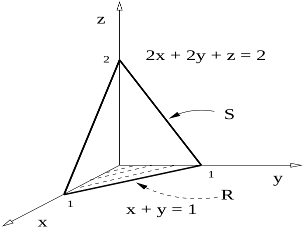

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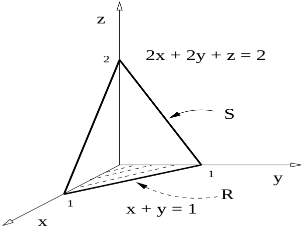

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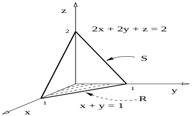

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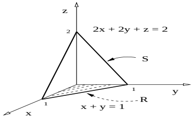

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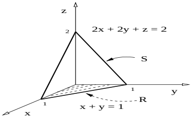

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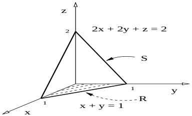

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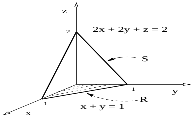

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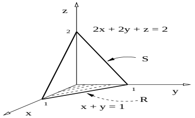

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Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
- Surface integrals of a scalar field.
- **The flux of a vector field on a surface.**
- Mass and center of mass thin shells.

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A surface $S \subset \mathbb{R}^3$ is called *orientable* if it is possible to define on S a continuous, unit vector field $\mathbf{n}$ normal to S.

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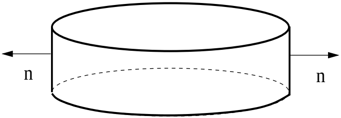

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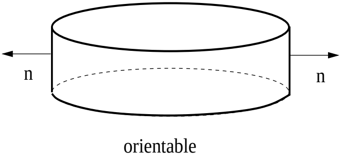

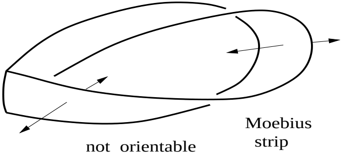

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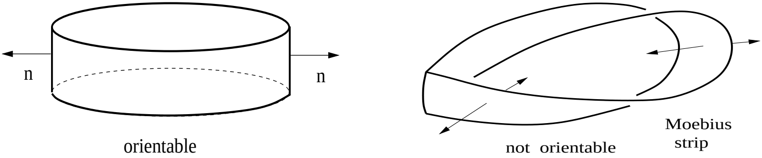

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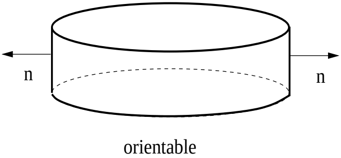

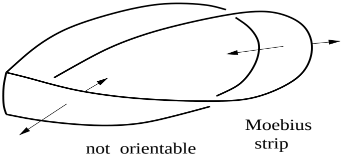

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Remark: $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$, where S is the level surface $f = 0$.

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Find the flux of the field $\mathbf{F} = \langle 0, 0, z \rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

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Solution: Recall: $\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$ and $\mathbf{n} = \frac{1}{a} \langle x, y, z \rangle$ on S.

Since $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} \, dx \, dy$, and $\nabla f = 2\langle x, y, z \rangle$, which on S says

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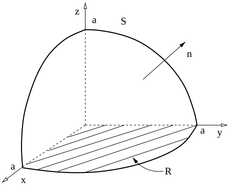

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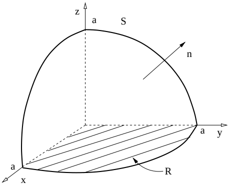

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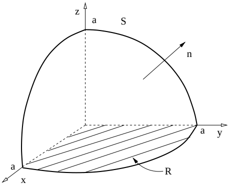

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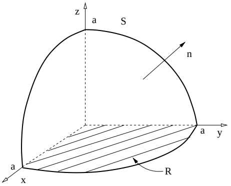

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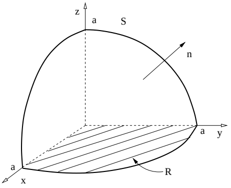

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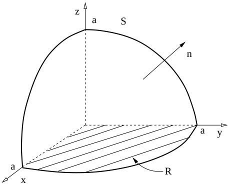

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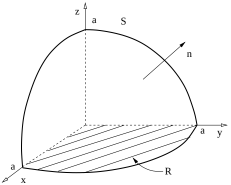

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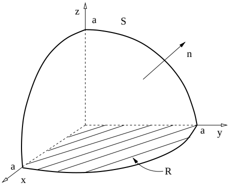

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Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
- **Mass and center of mass of thin shells.**

Mass and center of mass of thin shells.

Definition

The *mass* M of a thin shell described by the surface S in space with mass per unit area function $\rho : S \to \mathbb{R}$ is given by

$$M = \iint_S \rho \, d\sigma.$$

The *center of mass* $\bar{\mathbf{r}} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle$ of the thin shell above is

$$\overline{x}_i = \frac{1}{M} \iint_S x_i \, \rho \, d\sigma, \qquad i = 1, 2, 3.$$

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- See in the textbook the definitions of moments of inertia I_{x_i}, with $i = 1, 2, 3$, for thin shells.

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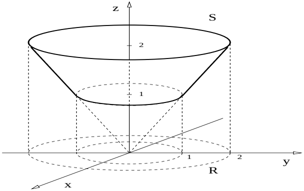

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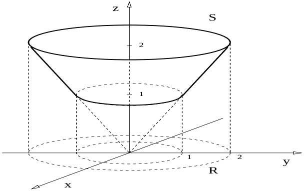

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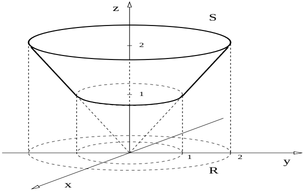

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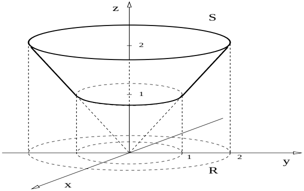

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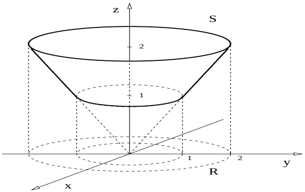

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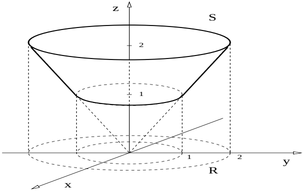

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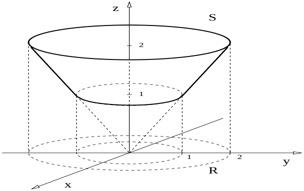

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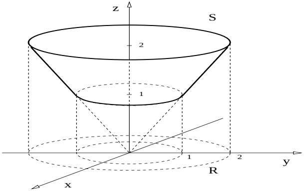

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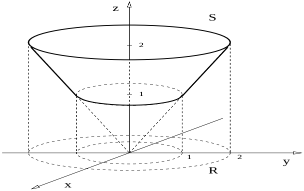

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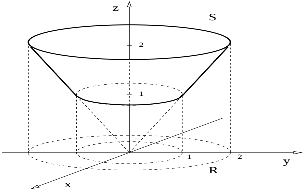

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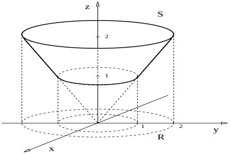

Mass and center of mass of thin shells.

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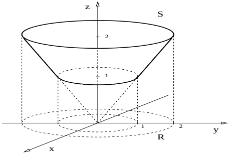

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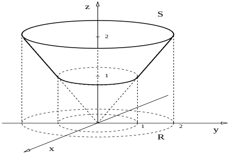

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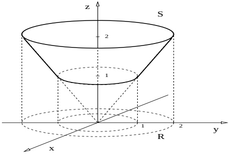

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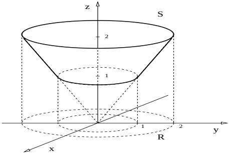

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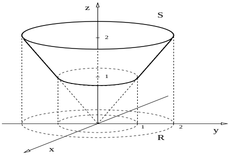

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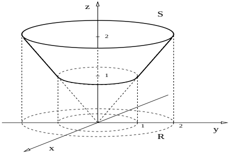

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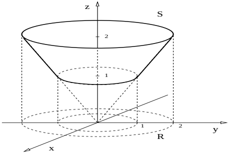

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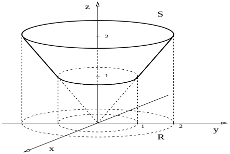

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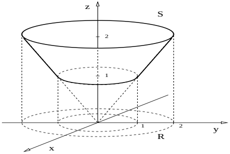

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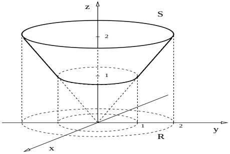

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