Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- The area of a surface in space.

Next class:

- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
- Mass and center of mass thin shells.

Review: Arc length and line integrals.

- The integral of a function $f : [a, b] \to \mathbb{R}$ is
  $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\,\Delta x.$
- The arc length of a curve $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ in space is
  $\displaystyle s_{t_1,t_0} = \int_{t_0}^{t_1} |\mathbf{r}'(t)|\,dt.$
- The integral of a function $f : \mathbb{R}^3 \to \mathbb{R}$ along a curve
  $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ is $\displaystyle\int_C f\,ds = \int_{t_0}^{t_1} f\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,dt.$
- The circulation of a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ along a curve
  $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^3$ is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{u}\,ds = \int_{t_0}^{t_1} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\,dt.$
- The flux of a function $\mathbf{F} : \{z = 0\} \cap \mathbb{R}^3 \to \{z = 0\} \cap \mathbb{R}^3$ along
  a loop $\mathbf{r} : [t_0, t_1] \to \{z = 0\} \cap \mathbb{R}^3$ is $\displaystyle\mathbb{F} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds.$

Surface area and surface integrals. (Sect. 16.5)

- Review: Arc length and line integrals.
- **Review: Double integral of a scalar function.**
- The area of a surface in space.

Review: Double integral of a scalar function.

- The double integral of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ on a region $R \subset \mathbb{R}^2$, which is the volume under the graph of f and above the $z = 0$ plane, and is given by

$$\iint_R f\, dA = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\, \Delta x\, \Delta y.$$

- The area of a plane surface $R \subset \mathbb{R}^2$ is the particular case $f = 1$, that is, $A(R) = \iint_R dA$.

We now show how to compute:

- The area of a surface in space.
- The integral of a scalar function on a surface is space.
- The flux of a vector-valued function on a surface in space.

Surface area and surface integrals. (Sect. 16.5)

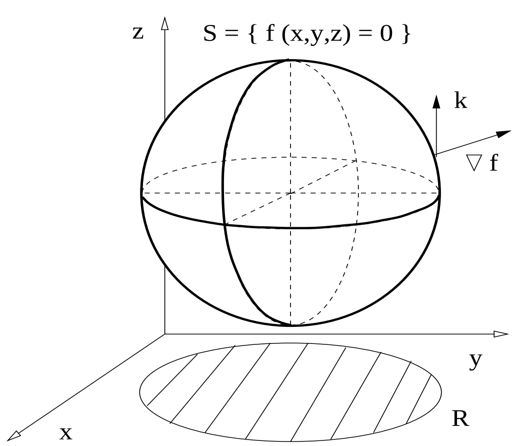
- Review: Arc length and line integrals.
- Review: Double integral of a scalar function.
- **The area of a surface in space.**

The area of a surface in space.

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, *the area of a level surface* $S = \{f(x, y, z) = 0\}$, *over a closed, bounded region* R *in the plane* $\{z = 0\}$, *is given by*

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

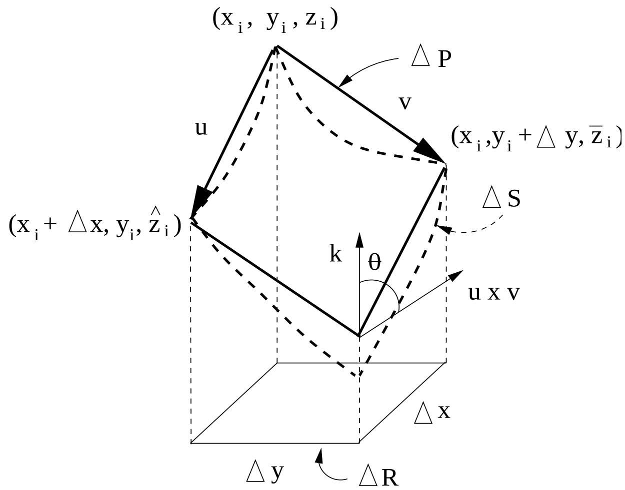


Remark: Eq. (3), page 1183, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

The vector $\mathbf{p}$ in the textbook is the vector normal to R. In our case $\mathbf{p} = \mathbf{k}$.

The area of a surface in space.

Proof: Introduce a partition in $R \subset \mathbb{R}^2$, and consider an arbitrary rectangle ΔR in that partition. We compute the area ΔP.



It is simple to se that

$$\Delta P = |\mathbf{u} \times \mathbf{v}|,$$

and

$$\mathbf{u} = \langle \Delta x, 0, (z_i - \hat{z}_i) \rangle,$$
$$\mathbf{v} = \langle 0, \Delta y, (z_i - \bar{z}_i) \rangle.$$

Therefore,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & (z_i - \hat{z}_i) \\ 0 & \Delta y & (z_i - \bar{z}_i) \end{vmatrix} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \bar{z}_i), \Delta x \Delta y \rangle.$$

The area of a surface in space.

Proof: Recall: $\mathbf{u} \times \mathbf{v} = \langle -\Delta y (z_i - \hat{z}_i), -\Delta x (z_i - \bar{z}_i), \Delta x \Delta y \rangle$.

The linearization of $f(x, y, z)$ at (x_i, y_i, z_i) implies

$$f(x, y, z) \simeq f(x_i, y_i, z_i) + (\partial_x f)_i \Delta x + (\partial_y f)_i \Delta y + (\partial_z f)_i (z - z_i).$$

Since $f(x_i, y_i, z_i) = 0$, $f(x_i + \Delta x, y_i, \hat{z}_i) = 0$, $f(x_i, y_i + \Delta y, \bar{z}_i) = 0$,

$$0 = (\partial_x f)_i \Delta x + (\partial_z f)_i (z_i - \hat{z}_i) \quad \Rightarrow \quad (z_i - \hat{z}_i) = -\frac{(\partial_x f)_i}{(\partial_z f)_i} \Delta x,$$

$$0 = (\partial_y f)_i \Delta y + (\partial_z f)_i (z_i - \bar{z}_i) \quad \Rightarrow \quad (z_i - \bar{z}_i) = -\frac{(\partial_y f)_i}{(\partial_z f)_i} \Delta y.$$

$$\mathbf{u} \times \mathbf{v} = \langle (\partial_x f)_i, (\partial_y f)_i, (\partial_z f)_i \rangle \frac{\Delta x \Delta y}{(\partial_z f)_i} \Rightarrow \mathbf{u} \times \mathbf{v} = \frac{(\nabla f)_i}{(\nabla f \cdot \mathbf{k})_i} \Delta x \Delta y.$$

$$\Delta P = \frac{|(\nabla f)_i|}{|(\nabla f \cdot \mathbf{k})_i|} \Delta x \Delta y \quad \Rightarrow \quad A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA. \quad \square$$

The area of a surface in space.

Example

Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: The surface is the level surface of the function $f(x, y, z) = x^2 + y^2 - z$. The region R is the disk $z = x^2 + y^2 \leqslant 4$.

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy, \quad \nabla f = \langle 2x, 2y, -1\rangle, \quad \nabla f \cdot \mathbf{k} = -1,$$

$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy.$$

Since R is a disk radius 2, it is convenient to use polar coordinates in $\mathbb{R}^2$. We obtain

$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

The area of a surface in space.

Example

Find the area of the surface in space given by the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$.

Solution: Recall: $A(S) = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$

$$A(S) = 2\pi \int_0^2 \sqrt{1 + 4r^2}\, r\, dr, \qquad u = 1 + 4r^2, \; du = 8r\, dr.$$

$$A(S) = \frac{2\pi}{8} \int_1^{17} u^{1/2}\, du = \frac{2\pi}{8}\,\frac{2}{3}\left(u^{3/2}\Big|_1^{17}\right).$$

We conclude: $A(S) = \frac{\pi}{6}\big[(17)^{3/2} - 1\big].$ $\triangleleft$

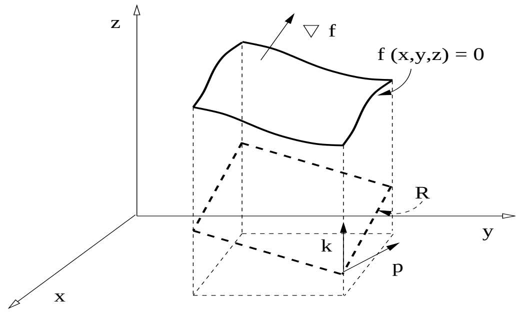
The area of a surface in space.

Remark: The formula for the area of a surface in space can be generalized as follows.

Theorem

The area of a surface S given by $f(x,y,z) = 0$ over a closed and bounded plane region R in space is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA,$$



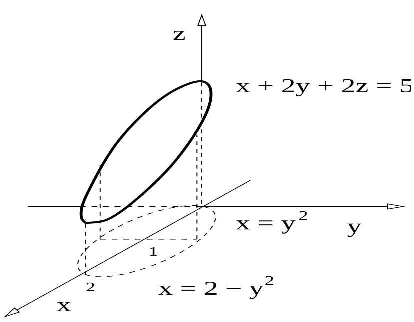
where $\mathbf{p}$ is a unit vector normal to the region R and $\nabla f \cdot \mathbf{p} \neq 0$.

The area of a surface in space.

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution:



The surface is given by $f = 0$ with

$$f(x,y,z) = x + 2y + 2z - 5.$$

The region R is in the plane $z = 0$,

$$R = \left\{ \begin{array}{c} (x,y,z) \;:\; z = 0,\; y \in [-1,1] \\ x \in [y^2, (2-y^2)] \end{array} \right\}.$$

Recall: $A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$. Here $\mathbf{p} = \mathbf{k}$, $\nabla f = \langle 1, 2, 2 \rangle$.

The area of a surface in space.

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$. Here $\mathbf{p} = \mathbf{k}$, $\nabla f = \langle 1, 2, 2 \rangle$.

Therefore: $|\nabla f| = \sqrt{1 + 4 + 4} = 3$, and $|\nabla f \cdot \mathbf{k}| = 2$.

And the region $R = \{(x, y) \ : \ y \in [-1, 1],\ x \in [y^2, (2 - y^2)]\}$.

So we can write down the expression for $A(S)$ as follows,

$$A(S) = \iint_R \frac{3}{2}\, dx\, dy = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx\, dy.$$

The area of a surface in space.

Example

Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder with walls $x = y^2$ and $x = 2 - y^2$.

Solution: $A(S) = \frac{3}{2} \int_{-1}^{1} \int_{y^2}^{2-y^2} dx\, dy$.

$$A(S) = \frac{3}{2} \int_{-1}^{1} (2 - y^2 - y^2)\, dy = \frac{3}{2} \int_{-1}^{1} (2 - 2y^2)\, dy$$

$$A(S) = 3 \int_{-1}^{1} (1 - y^2)\, dy = 3\Big(y - \frac{y^3}{3}\Big)\Big|_{-1}^{1} = 3\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big)$$

$$A(S) = 3\Big(2 - \frac{2}{3}\Big) = 3\,\frac{4}{3} \quad \Rightarrow \quad A(S) = 4. \qquad \triangleleft$$

Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
- Surface integrals of a scalar field.
- The flux of a vector field on a surface.
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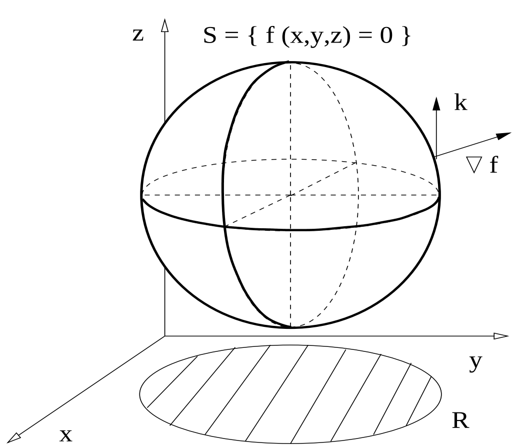
Review: The area of a surface in space.

Theorem

Given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, the area of a level surface $S = \{f(x, y, z) = 0\}$, over a closed, bounded region R in the plane $\{z = 0\}$, is given by

$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$



Remark: Eq. (3), page 1183, in the textbook is more general than the equation above, since the region R can be located on any plane, not only the plane $\{z = 0\}$ considered here.

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Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
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Surface integrals of a scalar field.

Theorem

The integral of a continuous scalar function $g : \mathbb{R}^3 \to \mathbb{R}$ over a surface S defined as the level set of $f(x, y, z) = 0$ over the bounded plane R is given by

$$\iint_S g \, d\sigma = \iint_R g \, \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA,$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Remark: In the particular case $g = 1$, we recover the formula for the area $A(S) = \iint_S d\sigma$ of the surface S, that is,

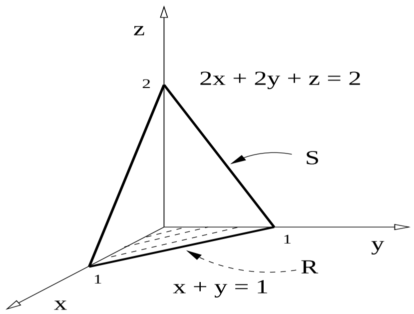
$$A(S) = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA.$$

Surface integrals of a scalar field.

Example

Integrate the function $g(x,y,z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution: Recall: $\displaystyle\iint_S g\, d\sigma = \iint_R g\, \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA.$



Here $f = 2x + 2y + z - 2$, so the surface S is given by $f = 0$ in the first octant. Hence, the region R is on the $z = 0$ plane, (therefore $\mathbf{p} = \mathbf{k}$) given by the triangle with sides $x = 0$, $y = 0$ and $x + y = 1$.

So, $\nabla f = \langle 2, 2, 1\rangle$, hence $|\nabla f| = 3$, and $|\nabla f \cdot \mathbf{k}| = 1$. Therefore

$$\iint_S g\, d\sigma = \iint_R g(x,y,z)\, 3\, dA.$$

Surface integrals of a scalar field.

Example

Integrate the function $g(x,y,z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution: Recall: $\displaystyle\iint_S g\, d\sigma = \iint_R g(x,y,z)\, 3\, dA.$

Now, function g must be evaluated on the surface S. That means

$$g(x,y,z(x,y)) = x + y + z(x,y) = x + y + (2 - 2x - 2y).$$

$$g(x,y,z(z,y)) = 2 - x - y.$$

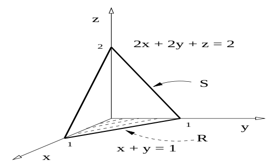
$$\iint_S g\, d\sigma = 3\iint_R (2 - x - y)\, dA.$$

Surface integrals of a scalar field.

Example

Integrate the function $g(x, y, z) = x + y + z$ over the surface given by the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution: $\displaystyle\iint_S g\, d\sigma = 3 \iint_R (2 - x - y)\, dA.$



The region R is the triangle in the plane $z = 0$ given by the lines $x = 0$, $y = 0$, and $x + y = 1$. Therefore,

$$3 \int_0^1 \int_0^{1-y} (2-x-y)\, dx\, dy = 3 \int_0^1 \Big[(2-y)\Big(x\Big|_0^{1-y}\Big) - \Big(\frac{x^2}{2}\Big|_0^{1-y}\Big)\Big]\, dy$$

$$\iint_S g\, d\sigma = 3 \int_0^1 \Big[(2-y)(1-y) - \frac{1}{2}(1-y)^2\Big]\, dy$$

$$\iint_S g\, d\sigma = 3 \int_0^1 \Big(\frac{3}{2} - 2y + \frac{y^2}{2}\Big)\, dy \quad \Rightarrow \quad \iint_S g\, d\sigma = 2. \quad \triangleleft$$

Surface area and surface integrals. (Sect. 16.5)

- Review: The area of a surface in space.
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- **The flux of a vector field on a surface.**
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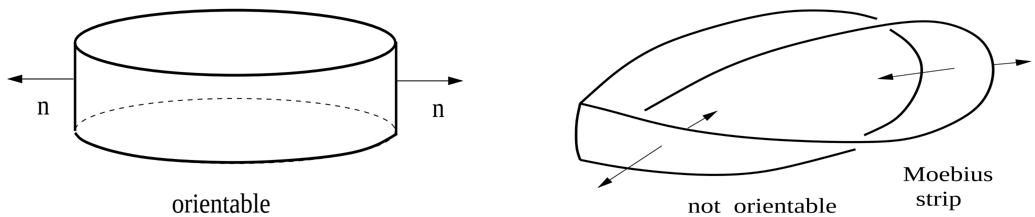
The flux of a vector field on a surface.

Definition

A surface $S \subset \mathbb{R}^3$ is called *orientable* if it is possible to define on S a continuous, unit vector field $\mathbf{n}$ normal to S.



Definition

The *flux* of a continuous vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ over an orientable surface S in the direction of a unit normal $\mathbf{n}$ is given by

$$\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

Remark: $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA$, where S is the level surface $f = 0$.

The flux of a vector field on a surface.

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z \rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

In this case S is the level surface $f = 0$, for $f = x^2 + y^2 + z^2 - a^2$. The unit normal vector $\mathbf{n}$ is proportional to ∇f.

$$\nabla f = \langle 2x, 2y, 2z \rangle, \quad |\nabla f| = 2\sqrt{x^2 + y^2 + z^2}.$$

On the surface S we have that $x^2 + y^2 + z^2 = a^2$, therefore, $|\nabla f| = 2a$ on this surface. We obtain that on S the appropriate normal vector is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} \quad \Rightarrow \quad \mathbf{n} = \frac{1}{a} \langle x, y, z \rangle, \quad z|_S = z(x, y).$$

The flux of a vector field on a surface.

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z\rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ and $\mathbf{n} = \frac{1}{a}\langle x, y, z\rangle$ on S.

Since $d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dx\, dy$, and $\nabla f = 2\langle x, y, z\rangle$, which on S says $|\nabla f| = 2a$, we conclude, $d\sigma = \frac{2a}{2z}\, dx\, dy$, hence $d\sigma = \frac{a}{z}\, dx\, dy$.

$$\mathbb{F} = \iint_R \Big(\langle 0, 0, z\rangle \cdot \frac{1}{a}\langle x, y, z\rangle\Big)\, \frac{a}{z}\, dx\, dy.$$

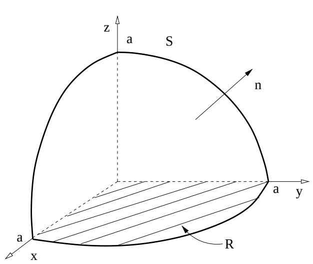
$$\mathbb{F} = \iint_R \frac{z^2}{a}\,\frac{a}{z}\, dx\, dy \quad \Rightarrow \quad \mathbb{F} = \iint_R z\, dx\, dy, \quad z|_S = z(x, y).$$

The flux of a vector field on a surface.

Example

Find the flux of the field $\mathbf{F} = \langle 0, 0, z\rangle$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

Solution: Recall: $\mathbb{F} = \iint_R z\, dx\, dy$, and z must be evaluated on S.



The integral is only on the first octant.

$$\mathbb{F} = \iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy.$$

We use polar coordinates on $R \subset \{z = 0\}$.

$$\mathbb{F} = \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta. \quad u = a^2 - r^2, \quad du = -2r\, dr.$$

$$\mathbb{F} = \frac{\pi}{2} \int_{a^2}^0 u^{1/2} \frac{(-du)}{2} = \frac{\pi}{4} \int_0^{a^2} u^{1/2}\, du = \frac{\pi}{4}\frac{2}{3}\,(a^2)^{3/2} \Rightarrow \mathbb{F} = \frac{\pi a^3}{6}.$$

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Mass and center of mass of thin shells.

Definition
The *mass* M of a thin shell described by the surface S in space with mass per unit area function $\rho : S \to \mathbb{R}$ is given by

$$M = \iint_S \rho \, d\sigma.$$

The *center of mass* $\bar{\mathbf{r}} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle$ of the thin shell above is

$$\overline{x}_i = \frac{1}{M} \iint_S x_i \, \rho \, d\sigma, \qquad i = 1, 2, 3.$$

Remark:

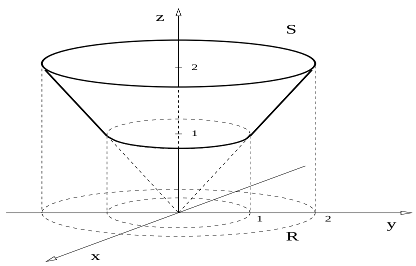
- The *centroid vector* is the particular case of the center of mass vector for an object with constant density.
- See in the textbook the definitions of moments of inertia I_{x_i}, with $i = 1, 2, 3$, for thin shells.

Mass and center of mass of thin shells.

Example

Find the centroid of the surface S given by $x^2 + y^2 = z^2$ between the planes $z = 1$ and $z = 2$.

Solution: The surface S is a cone section, given in the figure.



We first compute the area, M, of S,

$$M = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA.$$

Here $f = x^2 + y^2 - z^2$, therefore,

$$\nabla f = \langle 2x, 2y, -2z \rangle.$$

Hence $|\nabla f| = 2\sqrt{x^2 + y^2 + z^2}$, evaluated on S. Since $z^2 = x^2 + y^2$, we get $|\nabla f| = 2\sqrt{2}\, z$. Also $\nabla f \cdot \mathbf{k} = -2z$. So,

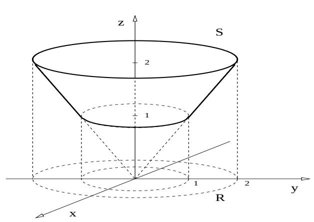
$$\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \frac{2\sqrt{2}\, z}{2z} = \sqrt{2} \quad \Rightarrow \quad M = \iint_R \sqrt{2}\, dA.$$

Mass and center of mass of thin shells.

Example

Find the centroid of the surface S given by $x^2 + y^2 = z^2$ between the planes $z = 1$ and $z = 2$.

Solution: Recall: $\dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} = \sqrt{2}$ and $M = \iint_R \sqrt{2}\, dA$.



We use polar coordinates in $\{z = 0\}$,

$$M = \sqrt{2} \int_0^{2\pi} \int_1^2 r\, dr\, d\theta = 2\pi\sqrt{2} \Big(\frac{r^2}{2} \Big|_1^2 \Big)$$

We conclude $M = 3\sqrt{2}\,\pi$.

By symmetry, the only non-zero component of the centroid is $\bar{z}$.

$$\bar{z} = \frac{1}{M} \iint_R z\, \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dA = \frac{\sqrt{2}}{3\sqrt{2}\pi} \iint_R \sqrt{x^2 + y^2}\, dx\, dy.$$

$$\bar{z} = \frac{1}{3\pi} \int_0^{2\pi} \int_1^2 r^2 dr\, d\theta = \frac{2\pi}{3\pi} \Big(\frac{r^3}{3} \Big|_1^3 \Big) = \frac{2}{9}\,(8 - 1) \quad \Rightarrow \quad \bar{z} = \frac{14}{9}.$$