Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- The line integral of conservative fields.
- Finding the potential of a conservative field.
- Comments on exact differential forms.

The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

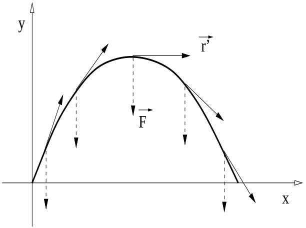
The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Example



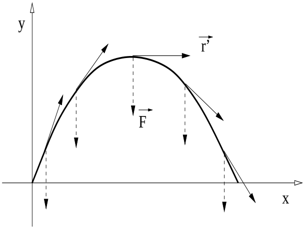
The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Example



Remark: An equivalent expression is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \hat{\mathbf{F}} \cdot \hat{\mathbf{u}}\, ds,$$

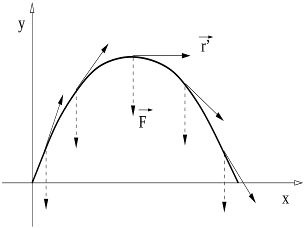
The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Example



Remark: An equivalent expression is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \hat{\mathbf{F}} \cdot \hat{\mathbf{u}}\, ds,$$

where $\hat{\mathbf{u}} = \dfrac{\mathbf{r}'(t(s))}{|\mathbf{r}'(t(s))|}$, and $\hat{\mathbf{F}} = \mathbf{F}(t(s))$.

Work done by a force on a particle.

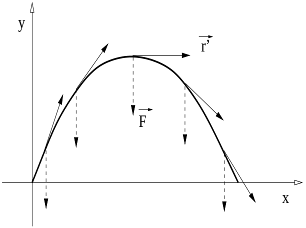
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

Work done by a force on a particle.

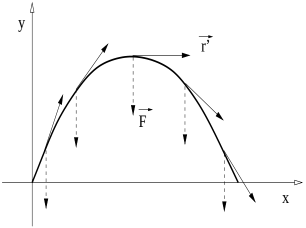
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.

Work done by a force on a particle.

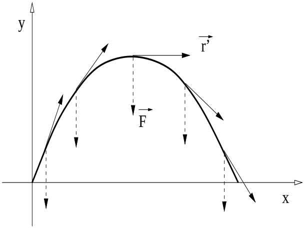
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- **Conservative fields.**
- The line integral of conservative fields.
- Finding the potential of a conservative field.
- Comments on exact differential forms.

Conservative fields.

Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called *conservative* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

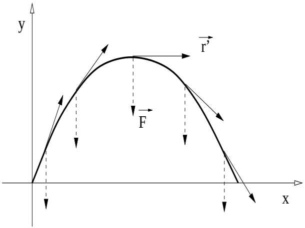
Conservative fields.

Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called *conservative* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

Example



A projectile of mass m moving on the surface of Earth.

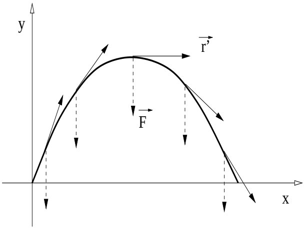
Conservative fields.

Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called *conservative* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.

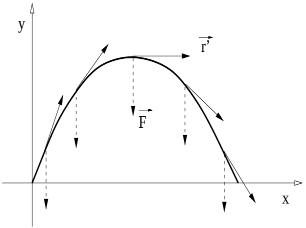
Conservative fields.

Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called *conservative* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $\mathbf{F} = \nabla f$, with $f = -mgy$.

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative iff there exists a potential function f such that $\mathbf{F} = \nabla f$,

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative iff there exists a potential function f such that $\mathbf{F} = \nabla f$, that is,

$$F_1 = \partial_{x_1} f, \qquad F_2 = \partial_{x_2} f, \qquad F_3 = \partial_{x_3} f.$$

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative iff there exists a potential function f such that $\mathbf{F} = \nabla f$, that is,

$$F_1 = \partial_{x_1} f, \qquad F_2 = \partial_{x_2} f, \qquad F_3 = \partial_{x_3} f.$$

Since

$$\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \Big[\big(x_1^2 + x_2^2 + x_3^2\big)^{-1/2} \Big], \quad i = 1, 2, 3,$$

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative iff there exists a potential function f such that $\mathbf{F} = \nabla f$, that is,

$$F_1 = \partial_{x_1} f, \qquad F_2 = \partial_{x_2} f, \qquad F_3 = \partial_{x_3} f.$$

Since

$$\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \Big[\big(x_1^2 + x_2^2 + x_3^2\big)^{-1/2} \Big], \quad i = 1, 2, 3,$$

then we conclude that $\mathbf{F} = \nabla f$, with $f = -\dfrac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$. $\triangleleft$

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- **The line integral of conservative fields.**
- Finding the potential of a conservative field.
- Comments on exact differential forms.

The line integral of conservative fields.

Definition

A set $D \subset \mathbb{R}^n$, with $n = 2, 3$, is called *simply connected* iff every two points in D can be connected by a smooth curve inside D and every loop in D can be smoothly contracted to a point without leaving D.

The line integral of conservative fields.

Definition

A set $D \subset \mathbb{R}^n$, with $n = 2, 3$, is called *simply connected* iff every two points in D can be connected by a smooth curve inside D and every loop in D can be smoothly contracted to a point without leaving D.

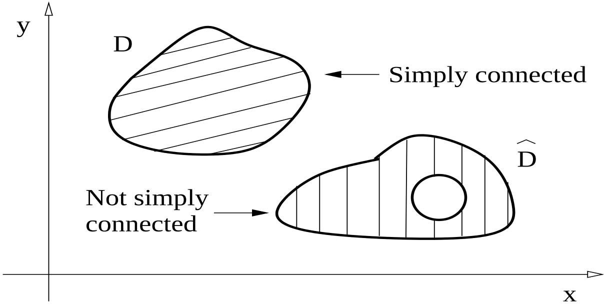
Remark: A set is simply connected iff it consists of one piece and it contains no holes.

The line integral of conservative fields.

Definition

A set $D \subset \mathbb{R}^n$, with $n = 2, 3$, is called *simply connected* iff every two points in D can be connected by a smooth curve inside D and every loop in D can be smoothly contracted to a point without leaving D.

Remark: A set is simply connected iff it consists of one piece and it contains no holes.



The line integral of conservative fields.

Notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a field $\mathbf{F}$ along C as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

The line integral of conservative fields.

Notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a field $\mathbf{F}$ along C as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

(This notation emphasizes the end points, not the path.)

The line integral of conservative fields.

Notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a field $\mathbf{F}$ along C as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

(This notation emphasizes the end points, not the path.)

Theorem

A smooth vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, *with* $n = 2, 3$, *defined on a simply connected domain* $D \subset \mathbb{R}^n$ *is conservative with* $\mathbf{F} = \nabla f$ *iff for every two points* $\mathbf{r}_0$, $\mathbf{r}_1 \in D$ *the line integral* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of the path* C *joining* $\mathbf{r}_0$ *to* $\mathbf{r}_1$ *and holds*

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$

The line integral of conservative fields.

Notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a field $\mathbf{F}$ along C as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

(This notation emphasizes the end points, not the path.)

Theorem

A smooth vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, defined on a simply connected domain $D \subset \mathbb{R}^n$ is conservative with $\mathbf{F} = \nabla f$ iff for every two points $\mathbf{r}_0$, $\mathbf{r}_1 \in D$ the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining $\mathbf{r}_0$ to $\mathbf{r}_1$ and holds

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$

Remark: A field $\mathbf{F}$ is conservative iff $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only $(\Rightarrow)$.

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r}$$

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only $(\Rightarrow)$.

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$.

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only $(\Rightarrow)$.

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt$$

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).$$

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).$$

We conclude that $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$. $\square$

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).$$

We conclude that $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$. $\square$

(The statement ($\Leftarrow$) is more complicated to prove.)

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$,

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$,

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative,

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$.

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$.

The line integral of conservative fields.

Example

Evaluate $I = \displaystyle\int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r}$$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz.$

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2$$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z \rangle \cdot \langle dx, dy, dz \rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z \rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2 = (1 + 4 + 9).$$

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2 = (1 + 4 + 9).$$

We conclude that $I = 14$. $\triangleleft$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$.

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate
$\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$,

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate
$\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$.

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \langle 2t, 4t, 6t \rangle \cdot \langle 1, 2, 3 \rangle\, dt$$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \langle 2t, 4t, 6t \rangle \cdot \langle 1, 2, 3 \rangle\, dt$$

$$I = \int_0^1 (2t + 8t + 18t)\, dt$$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1,2,3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0,0,0 \rangle$, and $\mathbf{r}(1) = \langle 1,2,3 \rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle 2t, 4t, 6t \rangle \cdot \langle 1,2,3 \rangle\,dt$$

$$I = \int_0^1 (2t + 8t + 18t)\,dt = \int_0^1 28t\,dt$$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \displaystyle\int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3\rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3\rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z\rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t\rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t)\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 2t, 4t, 6t\rangle\cdot\langle 1, 2, 3\rangle\,dt$$

$$I = \int_0^1 (2t + 8t + 18t)\,dt = \int_0^1 28t\,dt = 28\left(\frac{t^2}{2}\Big|_0^1\right).$$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1, 2, 3 \rangle\, t$.
Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(1) = \langle 1, 2, 3 \rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t \rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle 2t, 4t, 6t \rangle \cdot \langle 1, 2, 3 \rangle\,dt$$

$$I = \int_0^1 (2t + 8t + 18t)\,dt = \int_0^1 28t\,dt = 28\left(\frac{t^2}{2}\Big|_0^1\right).$$

We conclude that $I = 14$. $\triangleleft$

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- The line integral of conservative fields.
- **Finding the potential of a conservative field.**
- Comments on exact differential forms.

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative,

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only $(\Rightarrow)$.

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$.

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied,

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f$$

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f$$

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ on a simply connected domain $D \subset \mathbb{R}^3$ is a conservative field iff hold

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f$$

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).
Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i.$$

□

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ on a simply connected domain $D \subset \mathbb{R}^3$ is a conservative field iff hold

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i.$$

□

(The statement ($\Leftarrow$) is more complicated to prove.)

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold,

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x,$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z,$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0,$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0, \qquad \partial_1 F_3$$

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

◁

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y, z)$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z)$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy \, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z)$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z)$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z)$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z) = -2yz$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0$$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field
$\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0 \Rightarrow f = (x^2 - z^2) y + c_0.$$

◁

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- The line integral of conservative fields.
- Finding the potential of a conservative field.
- **Comments on exact differential forms.**

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$,

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r}$$

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field **F**, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle$$

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Definition

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function f such that

$$F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Definition

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function f such that

$$F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$

Remarks:

- A differential form $\mathbf{F} \cdot d\mathbf{r}$ is exact iff $\mathbf{F} = \nabla f$.

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field $\mathbf{F}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Definition

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function f such that

$$F_x dx + F_y dy + F_z dz = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz.$$

Remarks:

- A differential form $\mathbf{F} \cdot d\mathbf{r}$ is exact iff $\mathbf{F} = \nabla f$.
- An exact differential form is nothing else than another name for a conservative field.

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$.

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$.

Solution: We need to do the same calculation we did above:

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\, dx + (x^2 - z^2)\, dy - 2yz\, dz$.

Solution: We need to do the same calculation we did above:
Writing $\mathbf{F} \cdot d\mathbf{r} = F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$, show that

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$.

Solution: We need to do the same calculation we did above: Writing $\mathbf{F} \cdot d\mathbf{r} = F_1\,dx_1 + F_2\,dx_2 + F_3\,dx_3$, show that

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. We showed that this is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$
$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$
$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$.

Solution: We need to do the same calculation we did above:
Writing $\mathbf{F} \cdot d\mathbf{r} = F_1\,dx_1 + F_2\,dx_2 + F_3\,dx_3$, show that

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. We showed that this is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$
$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$
$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

So, there exists f such that $\mathbf{F} \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r}$. $\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green's Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.

Review: The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, with arc length function s, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{u} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$, and $s_0 = s(t_0)$, $s_1 = s(t_1)$.

Review: The line integral of a vector field along a curve.

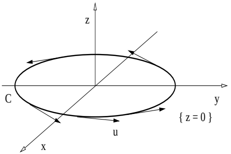
Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, with arc length function s, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{u} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$, and $s_0 = s(t_0)$, $s_1 = s(t_1)$.

Example



Review: The line integral of a vector field along a curve.

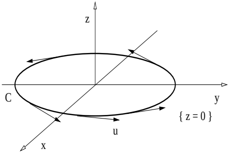
Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, with arc length function s, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{u} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$, and $s_0 = s(t_0)$, $s_1 = s(t_1)$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, in components,

$$\int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt.$$

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big] \, dt,$$

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \big[F_x(t)x'(t) + F_y(t)y'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \big]\, dt,$$

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[\sin^2(t) + \cos^2(t) \big]\, dt$$

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[\sin^2(t) + \cos^2(t) \big]\, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi.$$

Review: The flux across a plane loop.

Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Review: The flux across a plane loop.

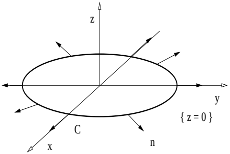
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Review: The flux across a plane loop.

Definition

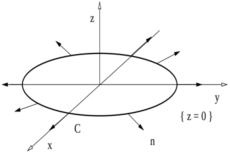
The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$,

Example



Review: The flux across a plane loop.

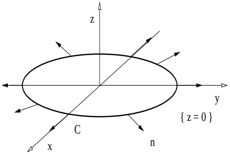
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$,
$\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$,

Review: The flux across a plane loop.

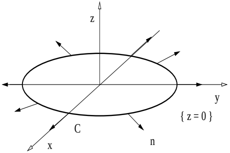
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0\rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$, $ds = |\mathbf{r}'(t)|\, dt$,

Review: The flux across a plane loop.

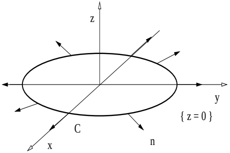
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0\rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$, $ds = |\mathbf{r}'(t)|\, dt$, and

$$\mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0\rangle,$$

Review: The flux across a plane loop.

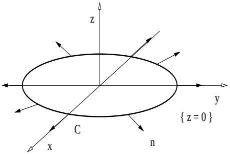
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0\rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$, $ds = |\mathbf{r}'(t)|\, dt$, and $\mathbf{n} = \dfrac{1}{|\mathbf{r}'|}\langle y'(t), -x'(t), 0\rangle$, in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t)y'(t) - F_y(t)x'(t)\big]\, dt.$$

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$.

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$.

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$,

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle$.

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0\rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0\rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle$.
Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0\rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0\rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t)y'(t) - F_y(t)x'(t)\big]\, dt,$$

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$.
Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[-\sin(t)\cos(t) - \cos(t)(-\sin(t)) \big]\, dt,$$

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[-\sin(t)\cos(t) - \cos(t)(-\sin(t)) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} 0\, dt$$

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[-\sin(t)\cos(t) - \cos(t)(-\sin(t)) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} 0\, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 0.$$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- **Green's Theorem on a plane.**
  - **Circulation-tangential form.**
  - Flux-normal form.
- Tangential and normal forms equivalence.

Green's Theorem on a plane.

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit tangent vector* $\mathbf{u}$, *satisfies that*

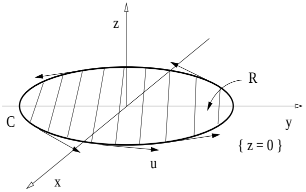
$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t) \big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x \big)\, dx\, dy.$$

Green's Theorem on a plane.

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit tangent vector* $\mathbf{u}$*, satisfies that*

$$\int_{t_0}^{t_1} \left[F_x(t)\, x'(t) + F_y(t)\, y'(t) \right] dt = \iint_R \left(\partial_x F_y - \partial_y F_x \right) dx\, dy.$$

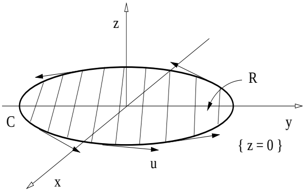


Green's Theorem on a plane.

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit tangent vector* $\mathbf{u}$*, satisfies that*

$$\int_{t_0}^{t_1} \left[F_x(t)\, x'(t) + F_y(t)\, y'(t) \right] dt = \iint_R \left(\partial_x F_y - \partial_y F_x \right) dx\, dy.$$



Equivalently,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \left(\partial_x F_y - \partial_y F_x \right) dx\, dy.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy = 2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy = 2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$I = 2(2\pi) \Big(\frac{r^2}{2} \Big|_0^1 \Big)$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy = 2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$I = 2(2\pi)\Big(\frac{r^2}{2}\Big|_0^1\Big) \quad \Rightarrow \quad I = 2\pi.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy = 2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$I = 2(2\pi)\left(\frac{r^2}{2}\Big|_0^1\right) \quad \Rightarrow \quad I = 2\pi.$$

We verified that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy = 2\pi$. $\quad\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- **Green's Theorem on a plane.**
  - Circulation-tangential form.
  - **Flux-normal form.**
- Tangential and normal forms equivalence.

Green's Theorem on a plane.

Theorem (Flux-normal form)

The counterclockwise flux integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit normal vector* $\mathbf{n}$*, satisfies that*

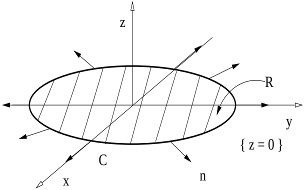
$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t) \big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y \big)\, dx\, dy.$$

Green's Theorem on a plane.

Theorem (Flux-normal form)

The counterclockwise flux integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit normal vector* $\mathbf{n}$, *satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t) \big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y \big)\, dx\, dy.$$

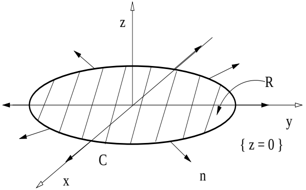


Green's Theorem on a plane.

Theorem (Flux-normal form)

The counterclockwise flux integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit normal vector* $\mathbf{n}$, *satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$



Equivalently,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$ and we verify that we get the same result, 0.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \big[\partial_x(-y) + \partial_y(x)\big]\, dx\, dy$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y \big)\, dx\, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \big[\partial_x(-y) + \partial_y(x) \big]\, dx\, dy = \iint_R 0\, dx\, dy = 0.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y \big) \, dx \, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \big[\partial_x(-y) + \partial_y(x) \big] \, dx \, dy = \iint_R 0 \, dx \, dy = 0.$$

We verified that $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \big(\partial_x F_x + \partial_y F_y \big) \, dx \, dy = 0.$ $\triangleleft$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$,

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$,
and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$,
and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Therefore, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[2a^2\cos^2(t) - 3a^2\sin^2(t) \big]\, dt,$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$, and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Therefore, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[2a^2\cos^2(t) - 3a^2\sin^2(t) \big]\, dt,$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \Big[2a^2 \frac{1}{2}\big(1 + \cos(2t)\big) - 3a^2 \frac{1}{2}\big(1 - \cos(2t)\big) \Big]\, dt.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big]\, dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$, and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Therefore, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \big[2a^2\cos^2(t) - 3a^2\sin^2(t) \big]\, dt,$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \Big[2a^2 \frac{1}{2}\big(1 + \cos(2t)\big) - 3a^2 \frac{1}{2}\big(1 - \cos(2t)\big) \Big]\, dt.$$

Since $\displaystyle\int_0^{2\pi} \cos(2t)\, dt = 0,$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \left[F_x(t) y'(t) - F_y(t) x'(t) \right] dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t) \rangle$,
and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t) \rangle$.

Therefore, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[2a^2 \cos^2(t) - 3a^2 \sin^2(t) \right] dt,$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left[2a^2 \frac{1}{2}\big(1 + \cos(2t)\big) - 3a^2 \frac{1}{2}\big(1 - \cos(2t)\big) \right] dt.$$

Since $\displaystyle\int_0^{2\pi} \cos(2t)\, dt = 0$, we conclude $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2 - 3)\, dx\, dy.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy = -\int_0^{2\pi} \int_0^a r\, dr\, d\theta$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy = -\int_0^{2\pi} \int_0^a r\, dr\, d\theta = -2\pi \left(\frac{r^2}{2} \Big|_0^a \right)$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy = -\int_0^{2\pi}\int_0^a r\, dr\, d\theta = -2\pi\Big(\frac{r^2}{2}\Big|_0^a\Big) = -\pi a^2.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y \rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = -\pi a^2$.

Now we compute the double integral $I = \displaystyle\iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$.

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\, dx\, dy = \iint_R (2-3)\, dx\, dy.$$

$$I = -\iint_R dx\, dy = -\int_0^{2\pi}\int_0^a r\, dr\, d\theta = -2\pi\Big(\frac{r^2}{2}\Big|_0^a\Big) = -\pi a^2.$$

Hence, $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy = -\pi a^2.$ $\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green's Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- **Tangential and normal forms equivalence.**

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t) \big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x \big)\, dx\, dy.$$

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$,

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t) \big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x \big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$.

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$. We obtain

$$\int_{t_0}^{t_1} \big[-F_y(t)\, x'(t) + F_x(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_x - \partial_y(-F_y)\big)\, dx\, dy,$$

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$. We obtain

$$\int_{t_0}^{t_1} \big[-F_y(t)\, x'(t) + F_x(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_x - \partial_y(-F_y)\big)\, dx\, dy,$$

so, $$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy,$$

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$. We obtain

$$\int_{t_0}^{t_1} \big[-F_y(t)\, x'(t) + F_x(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_x - \partial_y(-F_y)\big)\, dx\, dy,$$

so, $$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy,$$

which is Green's Theorem in normal form.

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$. We obtain

$$\int_{t_0}^{t_1} \big[-F_y(t)\, x'(t) + F_x(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_x - \partial_y(-F_y)\big)\, dx\, dy,$$

so, $\displaystyle\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy,$

which is Green's Theorem in normal form. The converse implication is proved in the same way. □

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \Big[2x \Big(y \Big|_0^x \Big) - \Big(y^2 \Big|_0^x \Big) \Big]\, dx$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[2x\left(y\Big|_0^x \right) - \left(y^2\Big|_0^x \right) \right] dx = \int_0^3 \left(2x^2 - x^2 \right) dx,$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[2x \left(y \Big|_0^x \right) - \left(y^2 \Big|_0^x \right) \right] dx = \int_0^3 \left(2x^2 - x^2 \right) dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2\, dx$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[2x \left(y \Big|_0^x \right) - \left(y^2 \Big|_0^x \right) \right] dx = \int_0^3 \left(2x^2 - x^2 \right) dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2\, dx = \frac{x^3}{3} \Big|_0^3$$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[2x \left(y \Big|_0^x \right) - \left(y^2 \Big|_0^x \right) \right] dx = \int_0^3 \left(2x^2 - x^2 \right) dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 9. \qquad \triangleleft$$

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

Review: Green's Theorem on a plane.

Theorem

Given a field $\mathbf{F} = \langle F_x, F_y \rangle$ *and a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *described by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$*, with unit tangent vector* $\mathbf{u}$ *and exterior normal vector* $\mathbf{n}$*, then holds:*

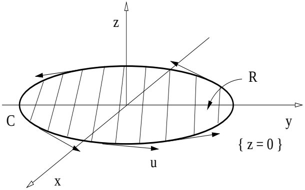
- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *satisfies:*

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t) \big]\, dt = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy.$$

- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *satisfies:*

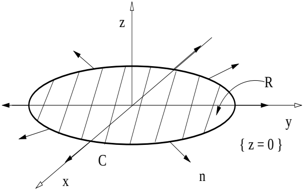
$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t) \big]\, dt = \iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy.$$

Review: Green's Theorem on a plane.



Circulation-tangential form:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$



Flux-normal form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- **Sketch of the proof of Green's Theorem.**
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

Sketch of the proof of Green's Theorem.

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

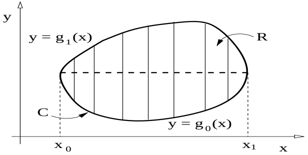
$$\int_C \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

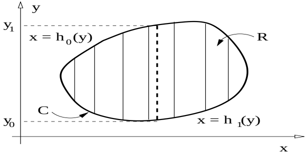
Sketch of the proof of Green's Theorem.

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \left[F_x(t)\, x'(t) + F_y(t)\, y'(t) \right] dt = \iint_R \left(\partial_x F_y - \partial_y F_x \right) dx\, dy.$$

We only consider a simple domain like the one in the pictures.



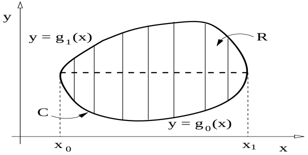


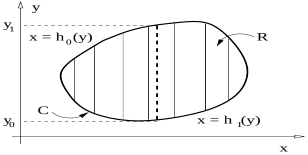
Sketch of the proof of Green's Theorem.

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \big[F_x(t)\,x'(t) + F_y(t)\,y'(t)\big]\,dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$

We only consider a simple domain like the one in the pictures.





Using the picture on the left we show that

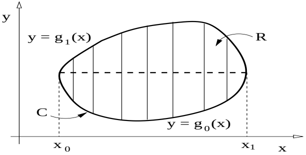
$$\int_C F_x(t)\,x'(t)\,dt = \iint_R \big(-\partial_y F_x\big)\,dx\,dy;$$

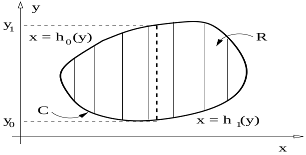
Sketch of the proof of Green's Theorem.

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \big[F_x(t)\,x'(t) + F_y(t)\,y'(t)\big]\,dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$

We only consider a simple domain like the one in the pictures.





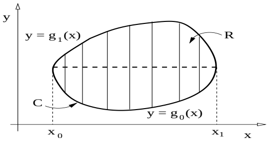
Using the picture on the left we show that

$$\int_C F_x(t)\,x'(t)\,dt = \iint_R \big(-\partial_y F_x\big)\,dx\,dy;$$

and using the picture on the right we show that

$$\int_C F_y(t)\,y'(t)\,dt = \iint_R \big(\partial_x F_y\big)\,dx\,dy.$$

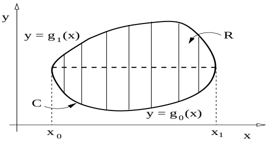
Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

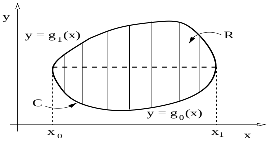
$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t) \rangle, \qquad t \in [x_0, x_1]$$

$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t) \rangle \qquad t \in [x_0, x_1].$$

Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

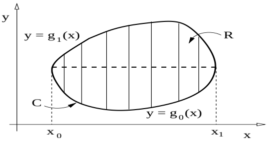
The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

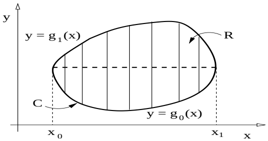
$$\mathbf{r}_0(t) = \langle t, g_0(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,

Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,
and $F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))$ on $\mathbf{r}_1$.

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau)$$

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau) = -\int_{x_0}^{x_1} F_x(\tau, g_1(\tau))\,d\tau.$$

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau) = -\int_{x_0}^{x_1} F_x(\tau, g_1(\tau))\,d\tau.$$

Therefore, $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \big[F_x(t, g_0(t)) - F_x(t, g_1(t))\big]\,dt.$

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\,dt$$

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\,dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau) = -\int_{x_0}^{x_1} F_x(\tau, g_1(\tau))\,d\tau.$$

Therefore, $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \big[F_x(t, g_0(t)) - F_x(t, g_1(t))\big]\,dt.$

We obtain: $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t, y)\big]\,dy\,dt.$

Sketch of the proof of Green's Theorem.

Recall: $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t,y)\big]\,dy\,dt.$

Sketch of the proof of Green's Theorem.

Recall: $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t,y)\big]\,dy\,dt.$

This result is precisely what we wanted to prove:

$$\int_C F_x(t)x'(t)\,dt = \iint_R \big(-\partial_y F_x\big)\,dy\,dx.$$

Sketch of the proof of Green's Theorem.

Recall: $\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t,y)\big]\,dy\,dt.$

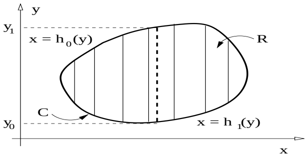
This result is precisely what we wanted to prove:

$$\int_C F_x(t)x'(t)\,dt = \iint_R \big(-\partial_y F_x\big)\,dy\,dx.$$

We just mention that the result

$$\int_C F_y(t)\,y'(t)\,dt = \iint_R \big(\partial_x F_y\big)\,dx\,dy.$$

is proven in a similar way using the parametrization of the C given in the picture.



□

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- **Divergence and curl of a function on a plane.**
- Area computed with a line integral.

Divergence and curl of a function on a plane.

Definition

The *curl* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\left(\operatorname{curl} \mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x.$$

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y.$$

Divergence and curl of a function on a plane.

Definition

The *curl* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\left(\operatorname{curl} \mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x.$$

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y.$$

Remark: Both forms of Green's Theorem can be written as:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \left(\operatorname{curl} \mathbf{F}\right)_z dx\, dy.$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dx\, dy.$$

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $(\operatorname{curl} \mathbf{F})_z$?

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $(\operatorname{curl} \mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle$$

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $(\mathrm{curl}\,\mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

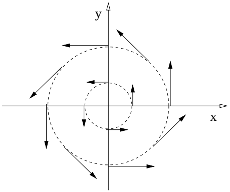
$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\mathrm{curl}\,\mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $(\operatorname{curl}\mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\operatorname{curl}\mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$

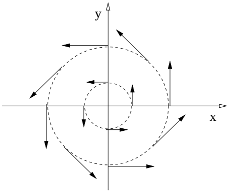


Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $(\operatorname{curl} \mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\operatorname{curl} \mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$



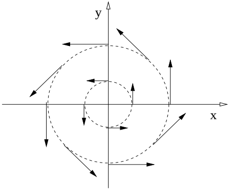
If we place a small ball at $(0, 0)$, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl} \mathbf{F})_z$.

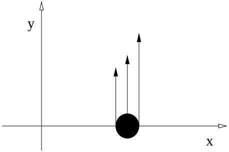
Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\left(\operatorname{curl}\mathbf{F}\right)_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad \left(\operatorname{curl}\mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x = 2.$$





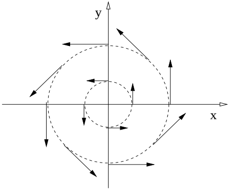
If we place a small ball at $(0, 0)$, the ball will spin around the z-axis with speed proportional to $\left(\operatorname{curl}\mathbf{F}\right)_z$.

Divergence and curl of a function on a plane.

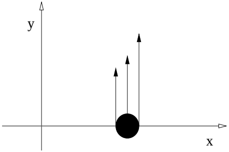
Remark: What type of information about $\mathbf{F}$ is given in $(\operatorname{curl}\mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\operatorname{curl}\mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$



If we place a small ball at $(0,0)$, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl}\mathbf{F})_z$.



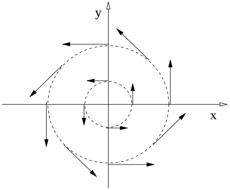
If we place a small ball at everywhere in the plane, the ball will spin around the z-axis with speed proportional to $(\operatorname{curl}\mathbf{F})_z$.

Divergence and curl of a function on a plane.

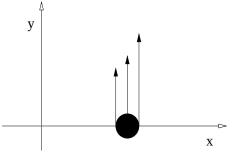
Remark: What type of information about $\mathbf{F}$ is given in $(\mathrm{curl}\,\mathbf{F})_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad (\mathrm{curl}\,\mathbf{F})_z = \partial_x F_y - \partial_y F_x = 2.$$



If we place a small ball at $(0,0)$, the ball will spin around the z-axis with speed proportional to $(\mathrm{curl}\,\mathbf{F})_z$.



If we place a small ball at everywhere in the plane, the ball will spin around the z-axis with speed proportional to $(\mathrm{curl}\,\mathbf{F})_z$.

Remark: The curl of a field measures its rotation.

Divergence and curl of a function on a plane.

Remark: What type of information about **F** is given in $\operatorname{div} \mathbf{F}$?

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle$$

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

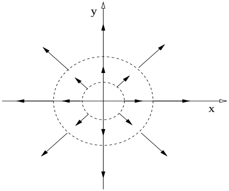
$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$

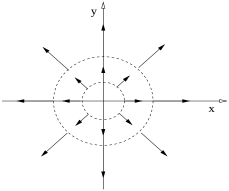


Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div}\mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



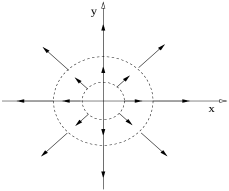
The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out form $(0,0)$. The $\operatorname{div}\mathbf{F}$ measures that expansion.

Divergence and curl of a function on a plane.

Remark: What type of information about **F** is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose **F** is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



The field **F** represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out form $(0,0)$. The $\operatorname{div} \mathbf{F}$ measures that expansion.

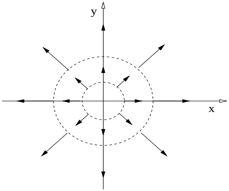
Remark: The divergence of a field measures its expansion.

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out form $(0,0)$. The $\operatorname{div} \mathbf{F}$ measures that expansion.

Remark: The divergence of a field measures its expansion.

Remarks:

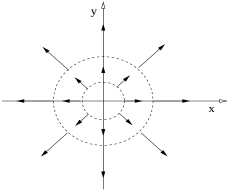
- Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $\big(\operatorname{curl} \mathbf{F}\big)_z = 0$.

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div} \mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0, 0)$. The heated gas expands in all directions, radially out form $(0, 0)$. The $\operatorname{div} \mathbf{F}$ measures that expansion.

Remark: The divergence of a field measures its expansion.

Remarks:

- Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $\big(\operatorname{curl} \mathbf{F}\big)_z = 0$.
- Notice that for $\mathbf{F} = \langle -y, x \rangle$ we have $\operatorname{div} \mathbf{F} = 0$.

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- **Area computed with a line integral.**

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral.

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If **F** is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R.

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0 \rangle$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R \left(\partial_x F_x + \partial_y F_y\right) dx\, dy = \oint_C \left(F_x\, dy - F_y\, dx\right)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0 \rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y \rangle$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0 \rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y \rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C -y\, dx.$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R \left(\partial_x F_x + \partial_y F_y\right) dx\, dy = \oint_C \left(F_x\, dy - F_y\, dx\right)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C -y\, dx.$$

$$\mathbf{F} = \frac{1}{2}\langle x, y\rangle$$

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R \left(\partial_x F_x + \partial_y F_y\right) dx\, dy = \oint_C \left(F_x\, dy - F_y\, dx\right)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C -y\, dx.$$

$$\mathbf{F} = \frac{1}{2}\langle x, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \frac{1}{2}\oint_C \left(x\, dy - y\, dx\right).$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t)$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$.

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1}{2}\big[1 + \cos(2t)\big]\,dt.$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1}{2}\big[1 + \cos(2t)\big]\,dt.$$

Since $\int_0^{2\pi} \cos(2t)\,dt = 0,$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1}{2}\big[1+\cos(2t)\big]\,dt.$$

Since $\int_0^{2\pi} \cos(2t)\,dt = 0$, we obtain $A(R) = \frac{ab}{2}\,2\pi$,

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\,dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\,y'(t)\,dt = \int_0^{2\pi} a\cos(t)\,b\cos(t)\,dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1}{2}\big[1 + \cos(2t)\big]\,dt.$$

Since $\int_0^{2\pi} \cos(2t)\,dt = 0$, we obtain $A(R) = \frac{ab}{2}\,2\pi$, that is,

$$A(R) = \pi ab.$$

$\triangleleft$