Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- The line integral of conservative fields.
- Finding the potential of a conservative field.
- Comments on exact differential forms.

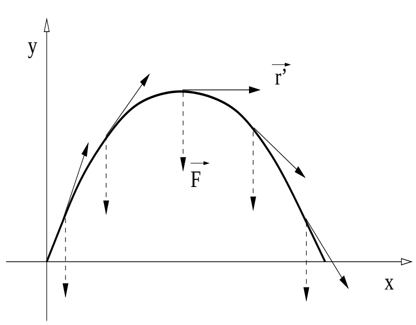
The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Example



Remark: An equivalent expression is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \hat{\mathbf{F}} \cdot \hat{\mathbf{u}}\, ds,$$

where $\hat{\mathbf{u}} = \dfrac{\mathbf{r}'(t(s))}{|\mathbf{r}'(t(s))|}$, and $\hat{\mathbf{F}} = \mathbf{F}(t(s))$.

Work done by a force on a particle.

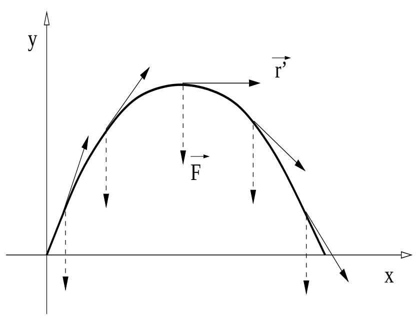
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- **Conservative fields.**
- The line integral of conservative fields.
- Finding the potential of a conservative field.
- Comments on exact differential forms.

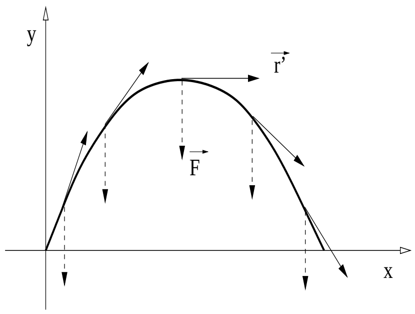
Conservative fields.

Definition

A vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is called *conservative* iff there exists a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, called *potential function*, such that

$$\mathbf{F} = \nabla f.$$

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $\mathbf{F} = \nabla f$, with $f = -mgy$.

Conservative fields.

Example

Show that the vector field $\mathbf{F} = \dfrac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \langle x_1, x_2, x_3 \rangle$ is conservative and find the potential function.

Solution: The field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative iff there exists a potential function f such that $\mathbf{F} = \nabla f$, that is,

$$F_1 = \partial_{x_1} f, \qquad F_2 = \partial_{x_2} f, \qquad F_3 = \partial_{x_3} f.$$

Since

$$\frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = -\partial_{x_i} \Big[\big(x_1^2 + x_2^2 + x_3^2 \big)^{-1/2} \Big], \quad i = 1, 2, 3,$$

then we conclude that $\mathbf{F} = \nabla f$, with $f = -\dfrac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$. $\triangleleft$

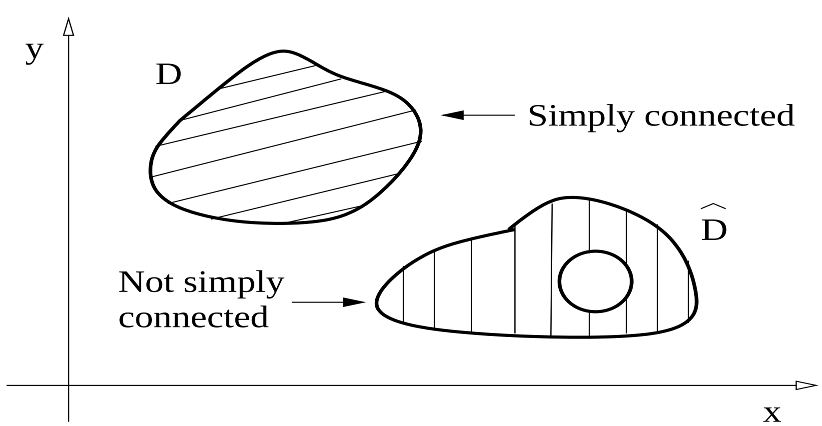
Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
- **The line integral of conservative fields.**
- Finding the potential of a conservative field.
- Comments on exact differential forms.

The line integral of conservative fields.

Definition
A set $D \subset \mathbb{R}^n$, with $n = 2, 3$, is called *simply connected* iff every two points in D can be connected by a smooth curve inside D and every loop in D can be smoothly contracted to a point without leaving D.

Remark: A set is simply connected iff it consists of one piece and it contains no holes.



The line integral of conservative fields.

Notation: If the path $C \in \mathbb{R}^n$, with $n = 2, 3$, has end points $\mathbf{r}_0$, $\mathbf{r}_1$, then denote the line integral of a field $\mathbf{F}$ along C as follows

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}.$$

(This notation emphasizes the end points, not the path.)

Theorem

A smooth vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, defined on a simply connected domain $D \subset \mathbb{R}^n$ is conservative with $\mathbf{F} = \nabla f$ iff for every two points $\mathbf{r}_0$, $\mathbf{r}_1 \in D$ the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining $\mathbf{r}_0$ to $\mathbf{r}_1$ and holds

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0).$$

Remark: A field $\mathbf{F}$ is conservative iff $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

The line integral of conservative fields.

Summary: $\mathbf{F} = \nabla f$ equivalent to $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$.

Proof: Only ($\Rightarrow$).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} (\nabla f)\Big|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{r}(t_1) = \mathbf{r}_1$. Therefore,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \frac{d}{dt}\big[f(\mathbf{r}(t)\big]\, dt = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)).$$

We conclude that $\displaystyle\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$. $\square$

(The statement ($\Leftarrow$) is more complicated to prove.)

The line integral of conservative fields.

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$.

Solution: I is a line integral for a field in $\mathbb{R}^3$, since

$$I = \int_{(0,0,0)}^{(1,2,3)} \langle 2x, 2y, 2z\rangle \cdot \langle dx, dy, dz\rangle.$$

Introduce $\mathbf{F} = \langle 2x, 2y, 2z\rangle$, $\mathbf{r}_0 = (0,0,0)$ and $\mathbf{r}_1 = (1,2,3)$, then $I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$. The field $\mathbf{F}$ is conservative, since $\mathbf{F} = \nabla f$ with potential $f(x,y,z) = x^2 + y^2 + z^2$. That is $f(\mathbf{r}) = |\mathbf{r}|^2$. Therefore,

$$I = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0) = |\mathbf{r}_1|^2 - |\mathbf{r}_0|^2 = (1+4+9).$$

We conclude that $I = 14$. $\lhd$

The line integral of conservative fields. (Along a path.)

Example

Evaluate $I = \int_{(0,0,0)}^{(1,2,3)} 2x\,dx + 2y\,dy + 2z\,dz$ along a straight line.

Solution: Consider the path C given by $\mathbf{r}(t) = \langle 1,2,3\rangle\, t$.
Then $\mathbf{r}(0) = \langle 0,0,0\rangle$, and $\mathbf{r}(1) = \langle 1,2,3\rangle$. We now evaluate $\mathbf{F} = \langle 2x, 2y, 2z\rangle$ along $\mathbf{r}(t)$, that is, $\mathbf{F}(t) = \langle 2t, 4t, 6t\rangle$. Therefore,

$$I = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle 2t, 4t, 6t\rangle \cdot \langle 1,2,3\rangle\,dt$$

$$I = \int_0^1 (2t + 8t + 18t)\,dt = \int_0^1 28t\,dt = 28\left(\frac{t^2}{2}\Big|_0^1\right).$$

We conclude that $I = 14$. $\lhd$

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
- Conservative fields.
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- **Finding the potential of a conservative field.**
- Comments on exact differential forms.

Finding the potential of a conservative field.

Theorem (Characterization of potential fields)

A smooth field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *on a simply connected domain* $D \subset \mathbb{R}^3$ *is a conservative field iff hold*

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

Proof: Only ($\Rightarrow$).

Since the vector field $\mathbf{F}$ is conservative, there exists a scalar field f such that $\mathbf{F} = \nabla f$. Then the equations above are satisfied, since for $i, j = 1, 2, 3$ hold

$$F_i = \partial_i f \quad \Rightarrow \quad \partial_i F_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j F_i.$$

□

(The statement ($\Leftarrow$) is more complicated to prove.)

Finding the potential of a conservative field.

Example

Show that the field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$ is conservative.

Solution: We need to show that the equations in the Theorem above hold, that is

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. This is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$

$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$

$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

$\triangleleft$

Finding the potential of a conservative field.

Example

Find the potential function of the conservative field $\mathbf{F} = \langle 2xy, (x^2 - z^2), -2yz \rangle$.

Solution: We know there exists a scalar function f solution of

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \partial_x f = 2xy, \quad \partial_y f = x^2 - z^2, \quad \partial_z f = -2yz.$$

$$f = \int 2xy\, dx + g(y,z) \quad \Rightarrow \quad f = x^2 y + g(y,z).$$

$$\partial_y f = x^2 + \partial_y g(y,z) = x^2 - z^2 \quad \Rightarrow \quad \partial_y g(y,z) = -z^2.$$

$$g(y,z) = -\int z^2\, dy + h(z) = -z^2 y + h(z) \Rightarrow f = x^2 y - z^2 y + h(z).$$

$$\partial_z f = -2zy + \partial_z h(z) = -2yz \Rightarrow \partial_z h(z) = 0 \Rightarrow f = (x^2 - z^2) y + c_0.$$

$\triangleleft$

Conservative fields and potential functions. (Sect. 16.3)

- Review: Line integral of a vector field.
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- Finding the potential of a conservative field.
- **Comments on exact differential forms.**

Comments on exact differential forms.

Notation: We call a *differential form* to the integrand in a line integral for a smooth field **F**, that is,

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = F_x dx + F_y dy + F_z dz.$$

Remark: A differential form is a quantity that can be integrated along a path.

Definition

A differential form $\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$ is called *exact* iff there exists a scalar function f such that

$$F_x dx + F_y dy + F_z dz = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$

Remarks:

- A differential form $\mathbf{F} \cdot d\mathbf{r}$ is exact iff $\mathbf{F} = \nabla f$.
- An exact differential form is nothing else than another name for a conservative field.

Comments on exact differential forms.

Example

Show that the differential form given below is exact, where
$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$.

Solution: We need to do the same calculation we did above:
Writing $\mathbf{F} \cdot d\mathbf{r} = F_1\,dx_1 + F_2\,dx_2 + F_3\,dx_3$, show that

$$\partial_2 F_3 = \partial_3 F_2, \qquad \partial_3 F_1 = \partial_1 F_3, \qquad \partial_1 F_2 = \partial_2 F_1.$$

with $x_1 = x$, $x_2 = y$, and $x_3 = z$. We showed that this is the case, since

$$\partial_1 F_2 = 2x, \qquad \partial_2 F_1 = 2x,$$
$$\partial_2 F_3 = -2z, \qquad \partial_3 F_2 = -2z,$$
$$\partial_3 F_1 = 0, \qquad \partial_1 F_3 = 0.$$

So, there exists f such that $\mathbf{F} \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r}$. $\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green's Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- Tangential and normal forms equivalence.

Review: The line integral of a vector field along a curve.

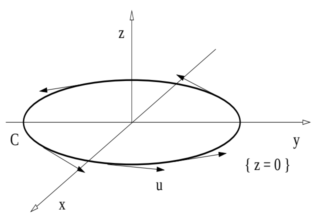
Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, with arc length function s, is given by

$$\int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{u} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$, and $s_0 = s(t_0)$, $s_1 = s(t_1)$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, in components,

$$\int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt.$$

Review: The line integral of a vector field along a curve.

Example

Evaluate the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t) \rangle$.
Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.
Then evaluate the line integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_{t_0}^{t_1} \big[F_x(t) x'(t) + F_y(t) y'(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t) \big]\, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \int_0^{2\pi} \big[\sin^2(t) + \cos^2(t) \big]\, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi.$$

Review: The flux across a plane loop.

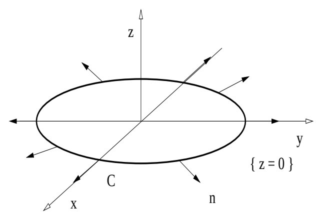
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remark: Since $\mathbf{F} = \langle F_x, F_y, 0 \rangle$, $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$, $ds = |\mathbf{r}'(t)| \, dt$, and $\mathbf{n} = \frac{1}{|\mathbf{r}'|} \langle y'(t), -x'(t), 0 \rangle$, in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big] \, dt.$$

Review: The flux across a plane loop.

Example

Evaluate the flux of $\mathbf{F} = \langle -y, x, 0 \rangle$ along the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ for $t \in [0, 2\pi]$.

Solution: Evaluate $\mathbf{F}$ along the curve: $\mathbf{F}(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the derivative vector $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Now compute the normal vector $\mathbf{n}(t) = \langle y'(t), -x'(t), 0 \rangle$, that is, $\mathbf{n}(t) = \langle \cos(t), \sin(t), 0 \rangle$. Evaluate the flux integral in components,

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t_0}^{t_1} \big[F_x(t) y'(t) - F_y(t) x'(t) \big] \, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} \big[-\sin(t) \cos(t) - \cos(t)(-\sin(t)) \big] \, dt,$$

$$\oint_C \mathbf{F} \cdot \mathbf{u} \, ds = \int_0^{2\pi} 0 \, dt \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot \mathbf{u} \, ds = 0.$$

Green's Theorem on a plane. (Sect. 16.4)

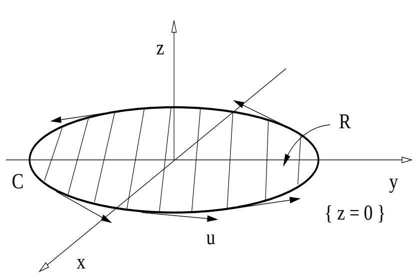
- Review: Line integrals and flux integrals.
- **Green's Theorem on a plane.**
  - **Circulation-tangential form.**
  - Flux-normal form.
- Tangential and normal forms equivalence.

Green's Theorem on a plane.

Theorem (Circulation-tangential form)

The counterclockwise line integral $\oint_C \mathbf{F}\cdot\mathbf{u}\,ds$ *of the field* $\mathbf{F} = \langle F_x, F_y\rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ *for* $t \in [t_0, t_1]$ *and with unit tangent vector* $\mathbf{u}$*, satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\,x'(t) + F_y(t)\,y'(t)\big]\,dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$



Equivalently,

$$\oint_C \mathbf{F}\cdot\mathbf{u}\,ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\,dx\,dy.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem tangential form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = 2\pi$.

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy$ and we verify that we get the same result, 2π.

$$I = \iint_R \big[1 - (-1)\big]\, dx\, dy = 2 \iint_R dx\, dy = 2 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$I = 2(2\pi)\Big(\frac{r^2}{2}\Big|_0^1\Big) \quad \Rightarrow \quad I = 2\pi.$$

We verified that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy = 2\pi.$ ◁

Green's Theorem on a plane. (Sect. 16.4)

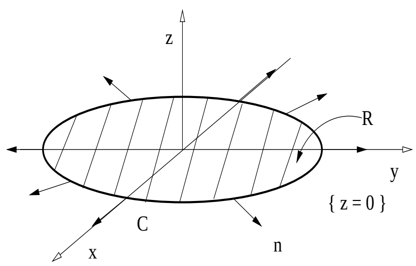
- Review: Line integrals and flux integrals.
- **Green's Theorem on a plane.**
  - Circulation-tangential form.
  - **Flux-normal form.**
- Tangential and normal forms equivalence.

Green's Theorem on a plane.

Theorem (Flux-normal form)

The counterclockwise flux integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *of the field* $\mathbf{F} = \langle F_x, F_y \rangle$ *along a loop* C *enclosing a region* $R \in \mathbb{R}^2$ *and given by the function* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ *for* $t \in [t_0, t_1]$ *and with unit normal vector* $\mathbf{n}$, *satisfies that*

$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$



Equivalently,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle -y, x \rangle$ and the loop $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Solution: Recall: We found that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$.

Now we compute the double integral $I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy$ and we verify that we get the same result, 0.

$$I = \iint_R \big[\partial_x(-y) + \partial_y(x)\big]\, dx\, dy = \iint_R 0\, dx\, dy = 0.$$

We verified that $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy = 0.$ $\triangleleft$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y\rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: We start with the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[F_x(t)y'(t) - F_y(t)x'(t)\big]\,dt.$$

It is simple to see that $\mathbf{F}(t) = \langle 2a\cos(t), -3a\sin(t)\rangle$, and also that $\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t)\rangle$.

Therefore, $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \big[2a^2\cos^2(t) - 3a^2\sin^2(t)\big]\,dt,$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \Big[2a^2\frac{1}{2}\big(1+\cos(2t)\big) - 3a^2\frac{1}{2}\big(1-\cos(2t)\big)\Big]\,dt.$$

Since $\displaystyle\int_0^{2\pi}\cos(2t)\,dt = 0$, we conclude $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = -\pi a^2.$

Green's Theorem on a plane.

Example

Verify Green's Theorem normal form for the field $\mathbf{F} = \langle 2x, -3y\rangle$ and the loop $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$, $a > 0$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = -\pi a^2.$

Now we compute the double integral $\displaystyle I = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\,dx\,dy.$

$$I = \iint_R \big[\partial_x(2x) + \partial_y(-3y)\big]\,dx\,dy = \iint_R (2-3)\,dx\,dy.$$

$$I = -\iint_R dx\,dy = -\int_0^{2\pi}\int_0^a r\,dr\,d\theta = -2\pi\Big(\frac{r^2}{2}\Big|_0^a\Big) = -\pi a^2.$$

Hence, $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\,dx\,dy = -\pi a^2.$ $\triangleleft$

Green's Theorem on a plane. (Sect. 16.4)

- Review: Line integrals and flux integrals.
- Green's Theorem on a plane.
  - Circulation-tangential form.
  - Flux-normal form.
- **Tangential and normal forms equivalence.**

Tangential and normal forms equivalence.

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Proof: Green's Theorem in tangential form for $\mathbf{F} = \langle F_x, F_y \rangle$ says

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t) \big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x \big)\, dx\, dy.$$

Apply this Theorem for $\hat{\mathbf{F}} = \langle -F_y, F_x \rangle$, that is, $\hat{F}_x = -F_y$ and $\hat{F}_y = F_x$. We obtain

$$\int_{t_0}^{t_1} \big[-F_y(t)\, x'(t) + F_x(t)\, y'(t) \big]\, dt = \iint_R \big(\partial_x F_x - \partial_y(-F_y) \big)\, dx\, dy,$$

so, $\displaystyle\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t) \big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y \big)\, dx\, dy,$

which is Green's Theorem in normal form. The converse implication is proved in the same way. $\square$

Using Green's Theorem

Example

Use Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F} = \langle (y^2 - x^2), (x^2 + y^2) \rangle$ along the curve C that is the triangle bounded by $y = 0$, $x = 3$ and $y = x$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\partial_x F_y - \partial_y F_x)\, dx\, dy.$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (2x - 2y)\, dx\, dy = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \Big[2x \Big(y\Big|_0^x \Big) - \Big(y^2\Big|_0^x \Big) \Big]\, dx = \int_0^3 \big(2x^2 - x^2 \big)\, dx,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 9. \qquad \triangleleft$$

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

Review: Green's Theorem on a plane.

Theorem

Given a field $\mathbf{F} = \langle F_x, F_y \rangle$ and a loop C enclosing a region $R \in \mathbb{R}^2$ described by the function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [t_0, t_1]$, with unit tangent vector $\mathbf{u}$ and exterior normal vector $\mathbf{n}$, then holds:

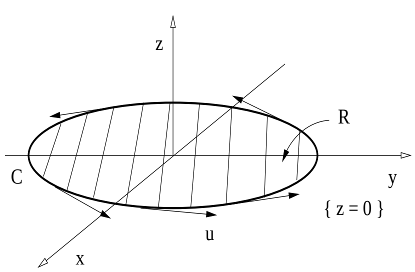
- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{u}\, ds$ *satisfies:*

$$\int_{t_0}^{t_1} \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

- *The counterclockwise line integral* $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ *satisfies:*

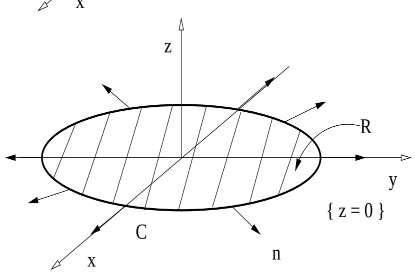
$$\int_{t_0}^{t_1} \big[F_x(t)\, y'(t) - F_y(t)\, x'(t)\big]\, dt = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Review: Green's Theorem on a plane.



Circulation-tangential form:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$



Flux-normal form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \big(\partial_x F_x + \partial_y F_y\big)\, dx\, dy.$$

Lemma

The Green Theorem in tangential form is equivalent to the Green Theorem in normal form.

Green's Theorem on a plane. (Sect. 16.4)

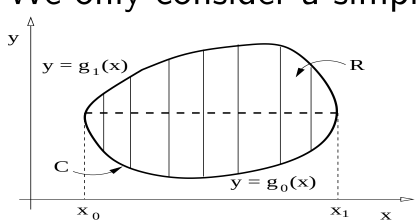
- Review of Green's Theorem on a plane.
- **Sketch of the proof of Green's Theorem.**
- Divergence and curl of a function on a plane.
- Area computed with a line integral.

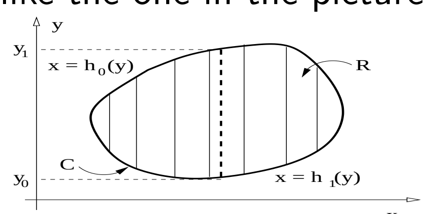
Sketch of the proof of Green's Theorem.

We want to prove that for every differentiable vector field $\mathbf{F} = \langle F_x, F_y \rangle$ the Green Theorem in tangential form holds,

$$\int_C \big[F_x(t)\, x'(t) + F_y(t)\, y'(t)\big]\, dt = \iint_R \big(\partial_x F_y - \partial_y F_x\big)\, dx\, dy.$$

We only consider a simple domain like the one in the pictures.





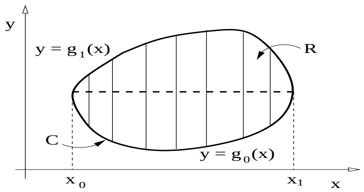
Using the picture on the left we show that

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R \big(-\partial_y F_x\big)\, dx\, dy;$$

and using the picture on the right we show that

$$\int_C F_y(t)\, y'(t)\, dt = \iint_R \big(\partial_x F_y\big)\, dx\, dy.$$

Sketch of the proof of Green's Theorem.



Show that for $F_x(t) = F_x(x(t), y(t))$ holds

$$\int_C F_x(t)\, x'(t)\, dt = \iint_R (-\partial_y F_x)\, dx\, dy;$$

The path C can be described by the curves $\mathbf{r}_0$ and $\mathbf{r}_1$ given by

$$\mathbf{r}_0(t) = \langle t, g_0(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1(t) = \langle (x_1 + x_0 - t), g_1(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Therefore,

$$\mathbf{r}_0'(t) = \langle 1, g_0'(t)\rangle, \qquad t \in [x_0, x_1]$$
$$\mathbf{r}_1'(t) = \langle -1, -g_1'(x_1 + x_0 - t)\rangle \qquad t \in [x_0, x_1].$$

Recall: $F_x(t) = F_x(t, g_0(t))$ on $\mathbf{r}_0$,
and $F_x(t) = F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))$ on $\mathbf{r}_1$.

Sketch of the proof of Green's Theorem.

$$\int_C F_x(t) x'(t)\, dt = \int_{x_0}^{x_1} F_x(t, g_0(t))\, dt$$
$$- \int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\, dt$$

Substitution in the second term: $\tau = x_1 + x_0 - t$, so $d\tau = -dt$.

$$-\int_{x_0}^{x_1} F_x((x_1 + x_0 - t), g_1(x_1 + x_0 - t))\, dt =$$

$$-\int_{x_1}^{x_0} F_x(\tau, g_1(\tau))\,(-d\tau) = -\int_{x_0}^{x_1} F_x(\tau, g_1(\tau))\, d\tau.$$

Therefore, $\displaystyle\int_C F_x(t) x'(t)\, dt = \int_{x_0}^{x_1} \big[F_x(t, g_0(t)) - F_x(t, g_1(t))\big]\, dt.$

We obtain: $\displaystyle\int_C F_x(t) x'(t)\, dt = \int_{x_0}^{x_1} \int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t, y)\big]\, dy\, dt.$

Sketch of the proof of Green's Theorem.

Recall: $\displaystyle\int_C F_x(t)x'(t)\,dt = \int_{x_0}^{x_1}\int_{g_0(t)}^{g_1(t)} \big[-\partial_y F_x(t,y)\big]\,dy\,dt.$

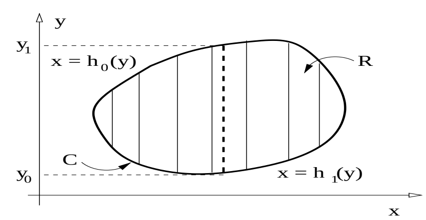
This result is precisely what we wanted to prove:

$$\int_C F_x(t)x'(t)\,dt = \iint_R \big(-\partial_y F_x\big)\,dy\,dx.$$

We just mention that the result

$$\int_C F_y(t)\,y'(t)\,dt = \iint_R \big(\partial_x F_y\big)\,dx\,dy.$$

is proven in a similar way using the parametrization of the C given in the picture.



□

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- **Divergence and curl of a function on a plane.**
- Area computed with a line integral.

Divergence and curl of a function on a plane.

Definition

The *curl* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\left(\operatorname{curl} \mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x.$$

The *divergence* of a vector field $\mathbf{F} = \langle F_x, F_y \rangle$ in $\mathbb{R}^2$ is the scalar

$$\operatorname{div} \mathbf{F} = \partial_x F_x + \partial_y F_y.$$

Remark: Both forms of Green's Theorem can be written as:

$$\oint_C \mathbf{F} \cdot \mathbf{u}\, ds = \iint_R \left(\operatorname{curl} \mathbf{F}\right)_z\, dx\, dy.$$

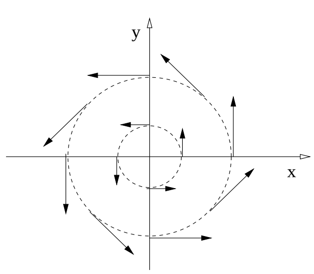
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dx\, dy.$$

Divergence and curl of a function on a plane.

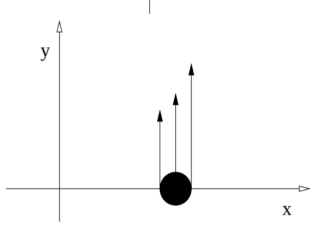
Remark: What type of information about $\mathbf{F}$ is given in $\left(\operatorname{curl} \mathbf{F}\right)_z$?

Example: Suppose $\mathbf{F}$ is the velocity field of a viscous fluid and

$$\mathbf{F} = \langle -y, x \rangle \quad \Rightarrow \quad \left(\operatorname{curl} \mathbf{F}\right)_z = \partial_x F_y - \partial_y F_x = 2.$$



If we place a small ball at $(0, 0)$, the ball will spin around the z-axis with speed proportional to $\left(\operatorname{curl} \mathbf{F}\right)_z$.



If we place a small ball at everywhere in the plane, the ball will spin around the z-axis with speed proportional to $\left(\operatorname{curl} \mathbf{F}\right)_z$.

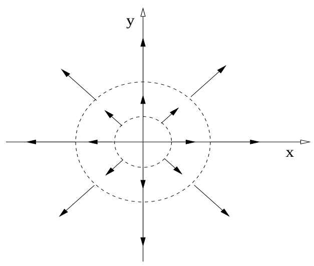
Remark: The curl of a field measures its rotation.

Divergence and curl of a function on a plane.

Remark: What type of information about $\mathbf{F}$ is given in $\operatorname{div}\mathbf{F}$?

Example: Suppose $\mathbf{F}$ is the velocity field of a gas and

$$\mathbf{F} = \langle x, y \rangle \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = \partial_x F_x + \partial_y F_y = 2.$$



The field $\mathbf{F}$ represents the gas as is heated with a heat source at $(0,0)$. The heated gas expands in all directions, radially out form $(0,0)$. The $\operatorname{div}\mathbf{F}$ measures that expansion.

Remark: The divergence of a field measures its expansion.

Remarks:

- Notice that for $\mathbf{F} = \langle x, y \rangle$ we have $\big(\operatorname{curl}\mathbf{F}\big)_z = 0$.
- Notice that for $\mathbf{F} = \langle -y, x \rangle$ we have $\operatorname{div}\mathbf{F} = 0$.

Green's Theorem on a plane. (Sect. 16.4)

- Review of Green's Theorem on a plane.
- Sketch of the proof of Green's Theorem.
- Divergence and curl of a function on a plane.
- **Area computed with a line integral.**

Area computed with a line integral.

Remark: Any of the two versions of Green's Theorem can be used to compute areas using a line integral. For example:

$$\iint_R (\partial_x F_x + \partial_y F_y)\, dx\, dy = \oint_C (F_x\, dy - F_y\, dx)$$

If $\mathbf{F}$ is such that the left-hand side above has integrand 1, then that integral is the area $A(R)$ of the region R. Indeed:

$$\mathbf{F} = \langle x, 0\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C x\, dy.$$

$$\mathbf{F} = \langle 0, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \oint_C -y\, dx.$$

$$\mathbf{F} = \frac{1}{2}\langle x, y\rangle \quad \Rightarrow \quad \iint_R dx\, dy = A(R) = \frac{1}{2}\oint_C (x\, dy - y\, dx).$$

Area computed with a line integral.

Example

Use Green's Theorem to find the area of the region enclosed by the ellipse $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, with $t \in [0, 2\pi]$ and a, b positive.

Solution: We use: $A(R) = \oint_C x\, dy$.

We need to compute $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t)\rangle$. Then,

$$A(R) = \int_0^{2\pi} x(t)\, y'(t)\, dt = \int_0^{2\pi} a\cos(t)\, b\cos(t)\, dt.$$

$$A(R) = ab\int_0^{2\pi} \cos^2(t)\, dt = ab\int_0^{2\pi} \frac{1}{2}\big[1 + \cos(2t)\big]\, dt.$$

Since $\int_0^{2\pi} \cos(2t)\, dt = 0$, we obtain $A(R) = \frac{ab}{2}\, 2\pi$, that is,

$$A(R) = \pi ab.$$

$\triangleleft$