Review for Exam 3.

- Sections 15.1-15.4, 15.6.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Triple integral in spherical coordinates (Sect. 15.6).

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

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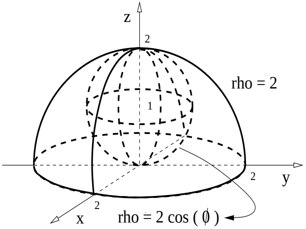

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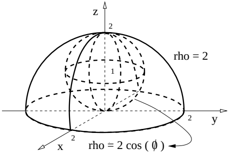

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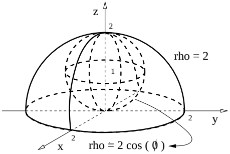

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

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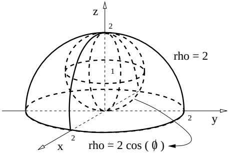

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$V = 2\pi \int_0^{\pi/2} \Big(\frac{\rho^3}{3}\Big|_{2\cos(\phi)}^{2} \Big) \sin(\phi)\, d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/2} \Big[8\sin(\phi) - 8\cos^3(\phi)\, \sin(\phi) \Big]\, d\phi.$$

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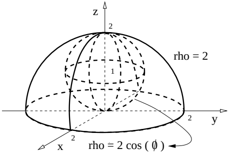

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$$V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi) \Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$$

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$$V = \frac{16\pi}{3}\Big[1 + \int_1^0 u^3\,du\Big]$$

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$$V = \frac{16\pi}{3}\Big[1 + \int_1^0 u^3\,du\Big] = \frac{16\pi}{3}\Big[1 + \Big(\frac{u^4}{4}\Big|_1^0\Big)\Big]$$

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$$V = \frac{16\pi}{3}\,\frac{3}{4} \quad \Rightarrow \quad V = 4\pi.$$

◁

Triple integral in cylindrical coordinates (Sect. 15.6).

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2+y^2=4$ by the planes $z=0$ and $z=-y$.

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- Since $0 \leqslant z \leqslant -y$, the integration region is on the $y \leqslant 0$ part of the $z = 0$ plane.

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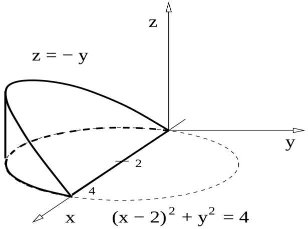

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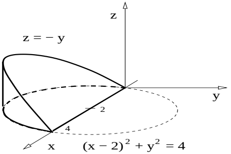

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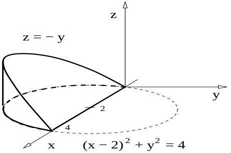

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r\,dz\,dr\,d\theta.$$

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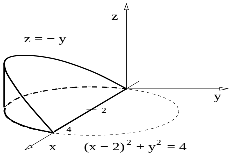

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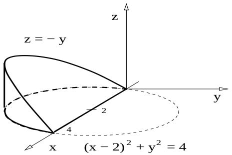

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We conclude: $V = \frac{16}{3}$. ◁

Triple integral in Cartesian coordinates (Sect. 15.4).

Example

Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 2$ and $0 \leqslant x \leqslant 1$, while the top side lies in the plane $x + y + z = 3$.

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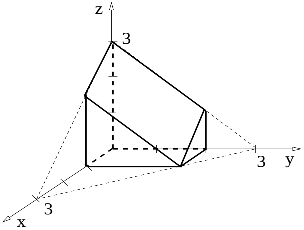

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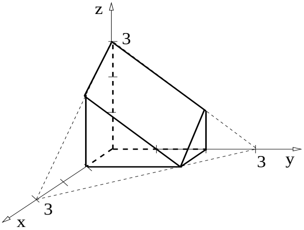

$$V = \int_0^1 \int_0^2 \int_0^{3-x-y} dz\, dy\, dx,$$

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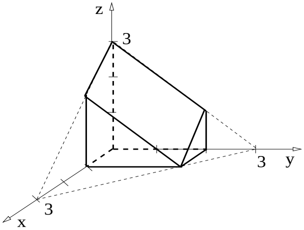

$$V = \int_0^1 \int_0^2 \int_0^{3-x-y} dz\, dy\, dx,$$

$$V = \int_0^1 \int_0^2 (3 - x - y)\, dy\, dx,$$

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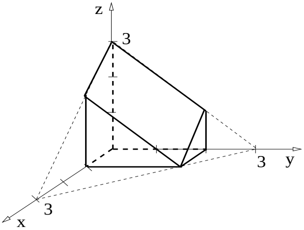

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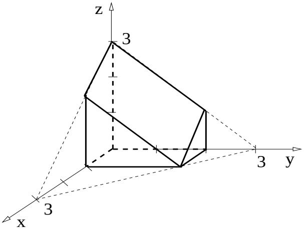

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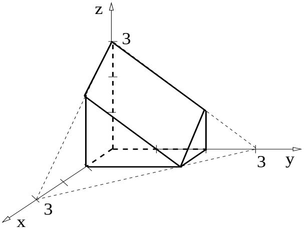

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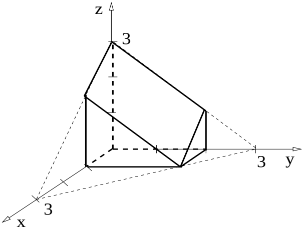
$$V = \int_0^1 (4 - 2x)\, dx = \Big[4\Big(x\Big|_0^1\Big) - \Big(x^2\Big|_0^1\Big)\Big]$$

Triple integral in Cartesian coordinates (Sect. 15.4).

Example

Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 2$ and $0 \leqslant x \leqslant 1$, while the top side lies in the plane $x + y + z = 3$.

Solution:



$$V = \int_0^1 \int_0^2 \int_0^{3-x-y} dz\, dy\, dx,$$

$$V = \int_0^1 \int_0^2 (3 - x - y)\, dy\, dx,$$

$$= \int_0^1 \Big[(3 - x)\Big(y\Big|_0^2\Big) - \frac{1}{2}\Big(y^2\Big|_0^2\Big)\Big]\, dx,$$

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$$V = \int_0^1 (4 - 2x)\, dx = \Big[4\Big(x\Big|_0^1\Big) - \Big(x^2\Big|_0^1\Big)\Big] = 4 - 1 \quad \Rightarrow \quad V = 3.$$

Double integrals in polar coordinates. (Sect. 15.3)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Double integrals in polar coordinates. (Sect. 15.3)

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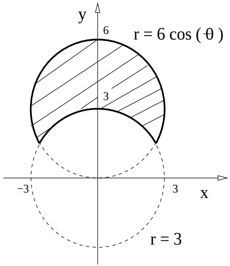

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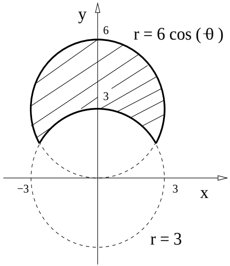

The condition $3 = r = 6\sin(\theta)$ determines the range in θ.

Double integrals in polar coordinates. (Sect. 15.3)

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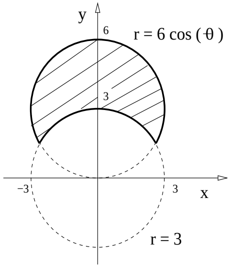

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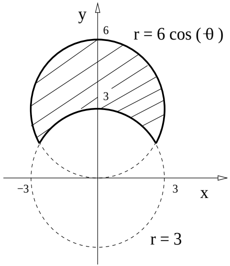

- The other curve is a circle $r = 3$ centered at the origin.

The condition $3 = r = 6\sin(\theta)$ determines the range in θ. Since $\sin(\theta) = 1/2$, we get $\theta_1 = 5\pi/6$ and $\theta_0 = \pi/6$.

Double integrals in polar coordinates. (Sect. 15.3)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Solution: Recall: $\theta \in [\pi/6, 5\pi/6]$.

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Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

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$$A = \int_{\pi/6}^{5\pi/6} \int_{3}^{6\sin(\theta)} r dr \, d\theta$$

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$$A = 3^2\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big) - \frac{3^2}{2}\Big(\sin(2\theta)\Big|_{\pi/6}^{5\pi/6}\Big) - \frac{3^2}{2}\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big).$$

$A = 6\pi - 3\pi - \dfrac{3^2}{2}\Big(-\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2}\Big)$, hence $A = 3\pi + 9\sqrt{3}/2$. $\triangleleft$

Double integrals in Cartesian coordinates. (Sect. 15.2)

Example

Find the y-component of the centroid vector in Cartesian coordinates in the plane of the region given by the disk $x^2 + y^2 \leqslant 9$ minus the first quadrant.

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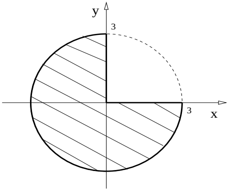

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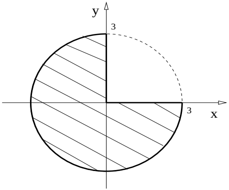

$\overline{y} = \dfrac{1}{A} \iint_R y\, dA$, where $A = \pi R^2 (3/4)$, with $R = 3$.

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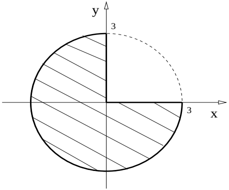

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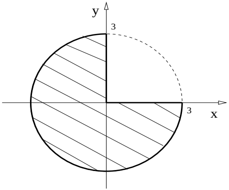

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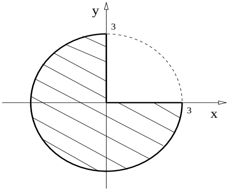

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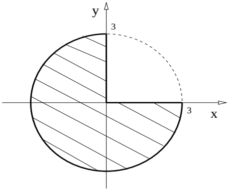

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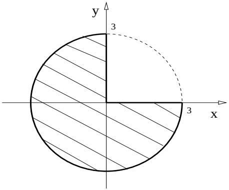

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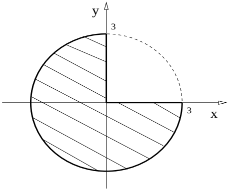

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Double integrals in polar coordinates. (Sect. 15.2)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{-\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx.$$

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- For $x \in [-2, -\sqrt{2}]$, we have
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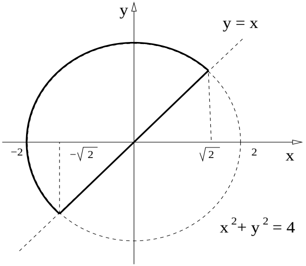

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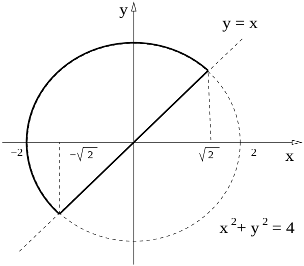

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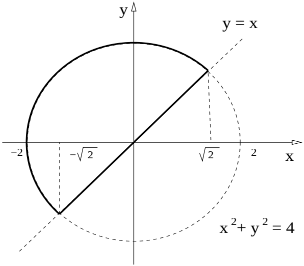

$$I = \int_{\pi/4}^{5\pi/4} \int_0^2 r^2\, r dr\, d\theta$$

$$I = \Big(\frac{5\pi}{4} - \frac{\pi}{4}\Big) \int_0^2 r^3\, dr$$

$$I = \pi \Big(\frac{r^4}{4}\Big|_0^2 \Big)$$

We conclude: $I = 4\pi$. ◁

Double integrals in polar coordinates. (Sect. 15.2)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$$

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Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$$

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Double integrals in polar coordinates. (Sect. 15.2)

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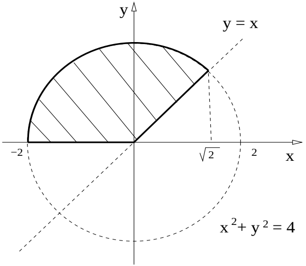

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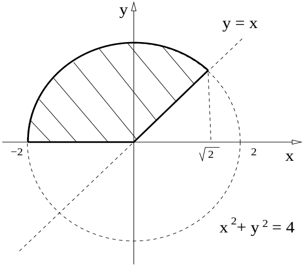

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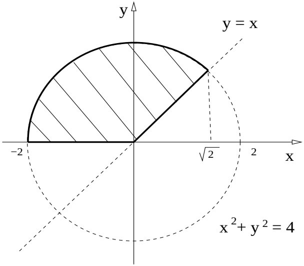

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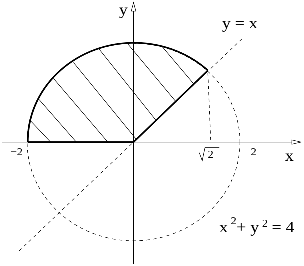

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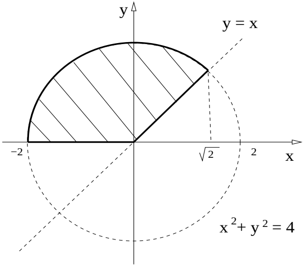

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We conclude: $I = 3\pi$. $\triangleleft$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- Mass and center of mass of wires.

Line integrals in space.

Definition

The *line integral* of a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along a curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C f \, ds = \int_{s_0}^{s_1} f\big(\hat{\mathbf{r}}(s)\big) \, ds,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points t_0, t_1, respectively.

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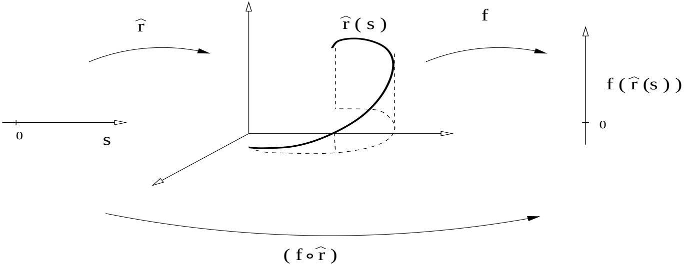

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Remarks:

- A line integral is an integral of a function along a curved path.

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  This Δx_i generalizes to Δs_i on a curved path. This is why the arc length parametrization is needed in the line integral.

Line integrals in space.

Theorem (Arbitrary parametrization.)

The line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ *along a differentiable curve* $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ *is given by*

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Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- **The addition of line integrals.**
- Mass and center of mass of wires.

The addition of line integrals.

Theorem

If a curve $C \subset D$ in space is the union of the differentiable curves $C_1, \cdots, C_n$, then the line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along C satisfies

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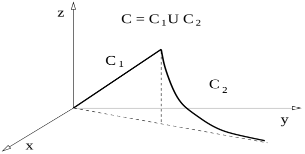

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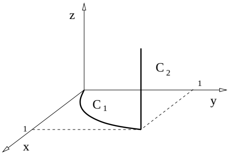

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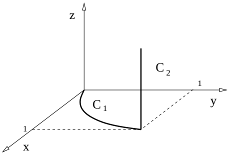

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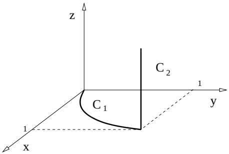

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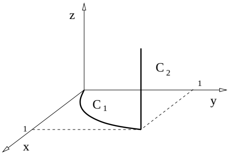

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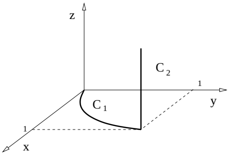

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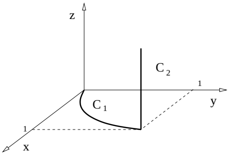

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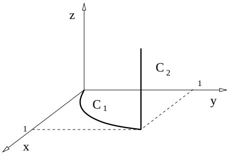

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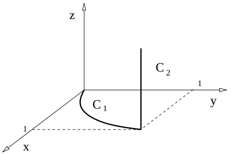

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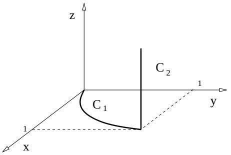

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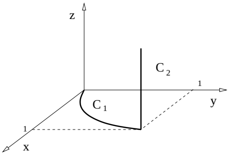

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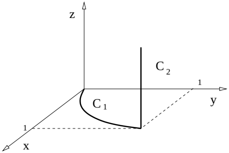

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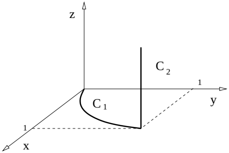

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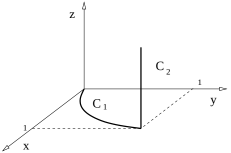

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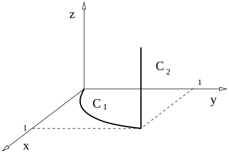

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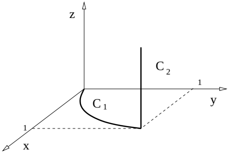

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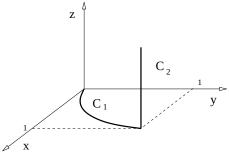

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$$\int_{C_2} f\, ds = \int_0^1 (2 - t^2)\, dt$$

The addition of line integrals.

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Evaluate the line integral of $f(x, y, z) = x + \sqrt{y} - z^2$ along the path $C = C_1 \cup C_2$, where C_1 is the image of $\mathbf{r}_1(t) = \langle t, t^2, 0\rangle$ for $t \in [0, 1]$, and C_2 is the image of $\mathbf{r}_2(t) = \langle 1, 1, t\rangle$ for $t \in [0, 1]$.

Solution:

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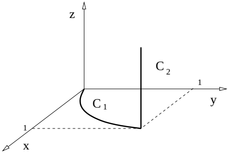

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$$f(\mathbf{r}_2(t)) = 1 + 1 - t^2 = 2 - t^2.$$

$$\int_{C_2} f\, ds = \int_0^1 (2 - t^2)\, dt = 2\Big(t\Big|_0^1\Big) - \Big(\frac{t^3}{3}\Big|_0^1\Big)$$

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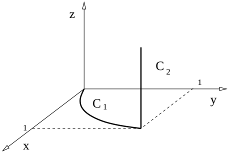

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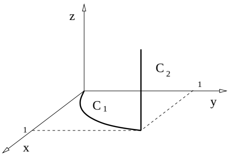

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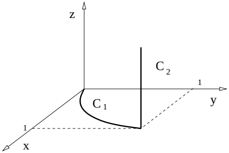

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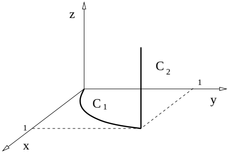

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Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- **Mass and center of mass of wires.**

Mass and center of mass of wires.

Remark:
The total mass, the center of mass, and the moments of inertia of wires with arbitrary shapes in space, given by a curve C and having a density function ρ, can be computed using line integrals.

- $M = \int_C \rho \, ds$;
- $\overline{x} = \dfrac{1}{M} \displaystyle\int_C x\rho \, ds, \quad \overline{y} = \dfrac{1}{M} \displaystyle\int_C y\rho \, ds, \quad \overline{z} = \dfrac{1}{M} \displaystyle\int_C z\rho \, ds$;
- $I_x = \dfrac{1}{M} \displaystyle\int_C (y^2 + z^2)\rho \, ds$,
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◁

Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
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Vector fields on a plane and in space.

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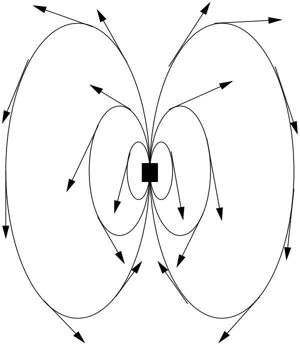

Magnetic field of a small magnet

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The gradient field of a scalar-valued function.

Remark:

- Given a scalar-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, its gradient vector, $\nabla f = \langle \partial_x f, \partial_y f \rangle$ or $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$, respectively, is a vector field in a plane or in space.

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Solution: We know the graph of f is a paraboloid. The gradient field is $\nabla f = \langle 2x, 2y \rangle$.

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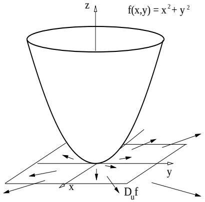

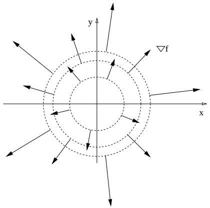

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$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

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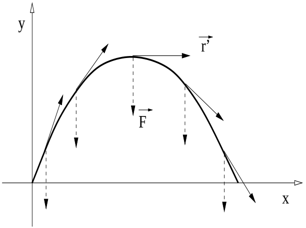

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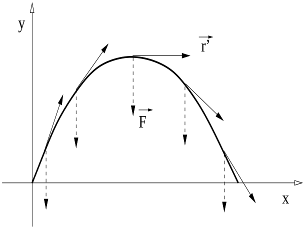

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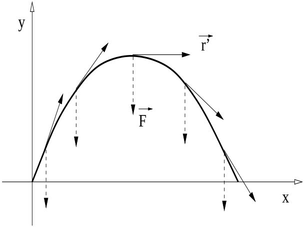

Remark: An equivalent expression is:

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$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \hat{\mathbf{F}} \cdot \hat{\mathbf{u}}\, ds,$$

where $\hat{\mathbf{u}} = \dfrac{\mathbf{r}'(t(s))}{|\mathbf{r}'(t(s))|}$, and $\hat{\mathbf{F}} = \mathbf{F}(t(s))$.

Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - **Work done by a force on a particle.**
  - The flow of a fluid along a curve.
- The flux across a plane curve.

Work done by a force on a particle.

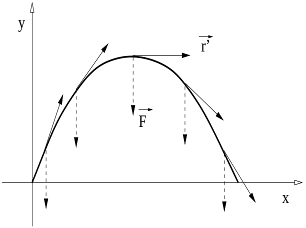
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

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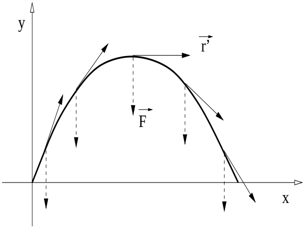

A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.

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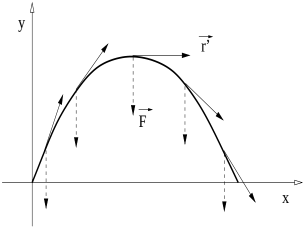

A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

Work done by a force on a particle.

Example

Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

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First: Evaluate $\mathbf{F}$ along $\mathbf{r}$.

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Solution:

First: Evaluate **F** along **r**. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1 \rangle$.

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Find the work done by the force $\mathbf{F}(x,y,z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0,1]$.

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Second: Compute $\mathbf{r}'(t)$.

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$$W = \int_0^1 \left[(3t^2 - 3t) + (6t^5) + (4t^3) \right] dt$$

$$= \left(t^3 - \frac{3}{2} t^2 + t^6 + t^4 \right) \Big|_0^1 = 1 - \frac{3}{2} + 1 + 1.$$

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So, $W = 3 - \dfrac{3}{2}$.

Work done by a force on a particle.

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Find the work done by the force $\mathbf{F}(x, y, z) = \langle (3x^2 - 3x), 3z, 1 \rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4 \rangle$, $t \in [0, 1]$.

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So, $W = 3 - \frac{3}{2}$. We conclude: The work done is $W = \frac{3}{2}$. $\triangleleft$

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  - **The flow of a fluid along a curve.**
- The flux across a plane curve.

The flow of a fluid along a curve.

Definition

In the case that the vector field $\mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is the velocity field of a flow and $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is any smooth curve, then the line integral

$$F = \int_C \mathbf{v} \cdot d\mathbf{r},$$

is called a *flow integral*. If the curve is a closed loop, the flow integral is called the *circulation* of the fluid around the loop.

The flow of a fluid along a curve.

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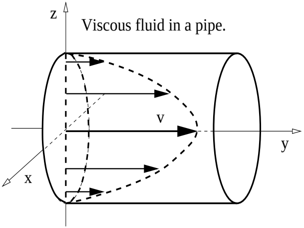

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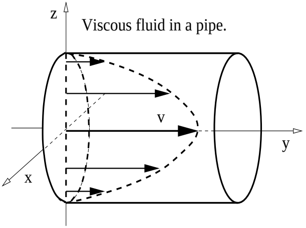

- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.

The flow of a fluid along a curve.

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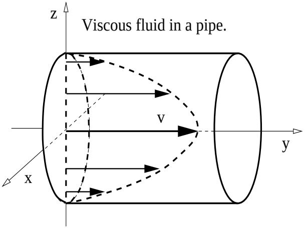

- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
- The flow vanishes on any curve perpendicular to the section of the pipe.

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

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Solution: The circulation is: $F = \displaystyle\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$.

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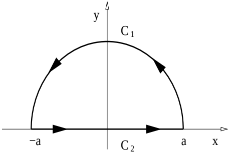

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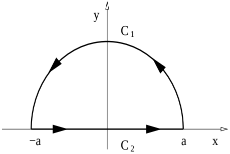

The first term is given by:

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} \mathbf{v}(t) \cdot \mathbf{r}_1'(t)\, dt.$$

The flow of a fluid along a curve.

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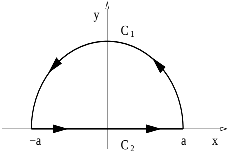

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$$\mathbf{v}(t) = \langle -a\sin(t), a\cos(t) \rangle,$$

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

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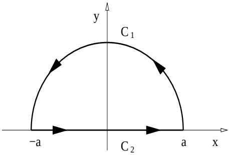

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The flow of a fluid along a curve.

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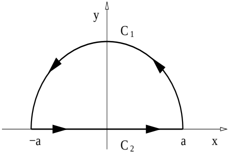

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The flow of a fluid along a curve.

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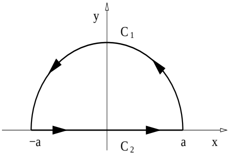

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$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} a^2 \big[\sin^2(t) + \cos^2(t)\big]\, dt \quad \Rightarrow \quad \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2.$$

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

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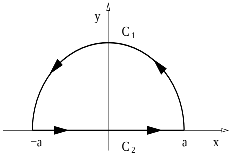

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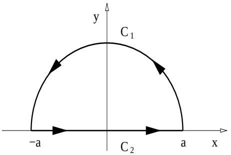

The second term is given by:

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}_2'(t)\, dt,$$

The flow of a fluid along a curve.

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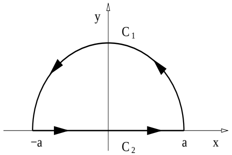

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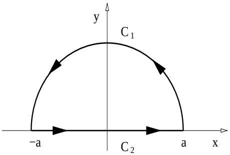

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Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

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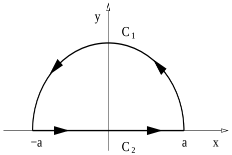

The second term is given by:

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t) \cdot \mathbf{r}_2'(t)\, dt,$$

$$\mathbf{v}(t) = \langle 0, t \rangle, \quad \mathbf{r}_2'(t) = \langle 1, 0 \rangle.$$

$$\mathbf{v}(t) \cdot \mathbf{r}_2'(t) = 0$$

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

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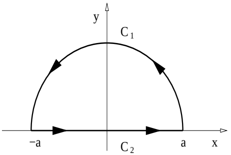

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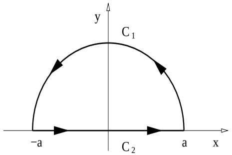

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Since $\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2$,

The flow of a fluid along a curve.

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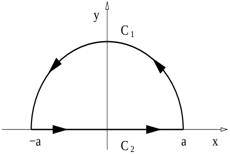

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Since $\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2$, we conclude: $F = \pi a^2$. $\triangleleft$

Integrals of vector fields. (Sect. 16.2)

- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- **The flux across a plane curve.**

The flux across a plane curve.

Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

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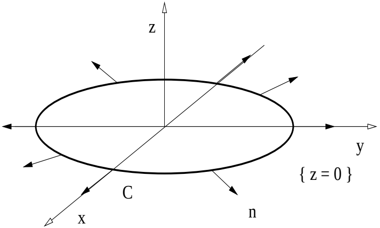

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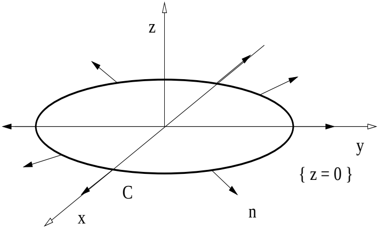

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- $\mathbf{F}$ is defined on $\{z = 0\}$.

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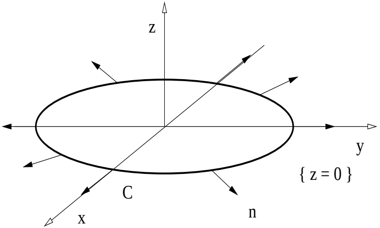

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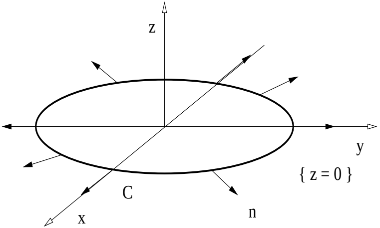

Remarks:

- $\mathbf{F}$ is defined on $\{z = 0\}$.
- The loop C lies on $\{z = 0\}$.
- Simple formula for $\mathbf{n}$?

The flux across a plane curve.

Theorem (Counterclockwise loops.)

The flux of a vector field $\mathbf{F} = \langle F_x(x,y), F_y(x,y), 0\rangle$ *along a closed, counterclockwise plane loop* $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$ *for* $t \in [t_0, t_1]$ *is given by*

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x\, y'(t) - F_y\, x'(t)\big]\, dt.$$

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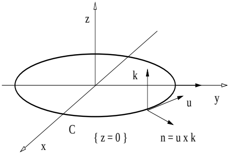

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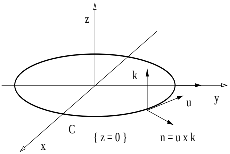

Remarks: Since C is counterclockwise traversed, $\mathbf{n} = \mathbf{u}\times\mathbf{k}$, where $\mathbf{u} = \mathbf{r}'/|\mathbf{r}'|$.

The flux across a plane curve.

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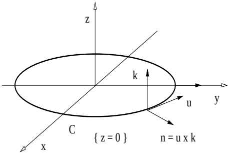

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$$\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|}\langle x'(t), y'(t), 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle.$$

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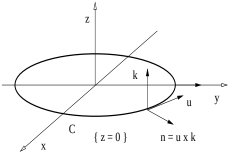

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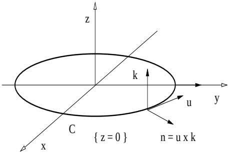

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Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

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Along C_1 we have: $\mathbf{F}_1(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$

The flux across a plane curve.

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The flux across a plane curve.

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Therefore,

$F_{1x}(t)\, y'(t) - F_{1y}(t)\, x'(t)$

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$$F_{1x}(t)\,y'(t) - F_{1y}(t)\,x'(t) = -a^2\sin(t)\cos(t) + a^2\sin(t)\cos(t) = 0.$$

Hence: $\displaystyle\int_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$

The flux across a plane curve.

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Along C_2 we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$

The flux across a plane curve.

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1\, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2\, ds$

Along C_2 we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$ and $x'(t) = 1$, $y'(t) = 0$.

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We conclude: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0.$ $\triangleleft$