

Triple integrals in Cartesian coordinates (Sect. 15.4)

- ▶ Review: Triple integrals in arbitrary domains.
- ▶ Examples: Changing the order of integration.
- ▶ The average value of a function in a region in space.
- ▶ Triple integrals in arbitrary domains.

Review: Triple integrals in arbitrary domains.

Theorem

If $f : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous in the domain

$$D = \{x \in [x_0, x_1], y \in [h_0(x), h_1(x)], z \in [g_0(x, y), g_1(x, y)]\},$$

where $g_0, g_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f \, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

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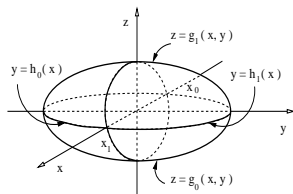
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Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0 .



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- ▶ **Examples: Changing the order of integration.**
- ▶ The average value of a function in a region in space.
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Changing the order of integration.

Example

Change the order of integration in the triple integral

$$V = \int_{-1}^1 \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz dy dx.$$

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Start from the outer integration limits to the inner limits.

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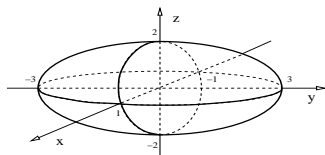
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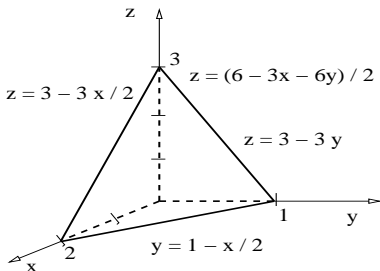
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Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz dy dx$.

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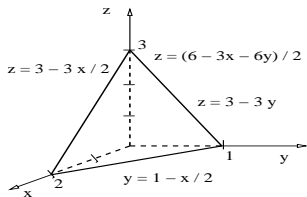
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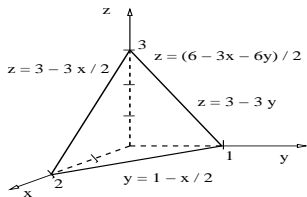
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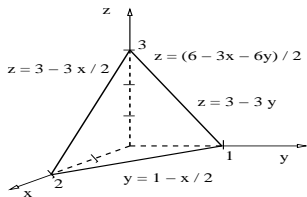
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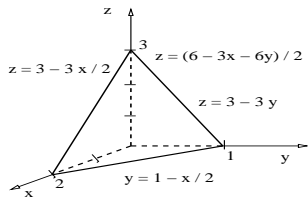
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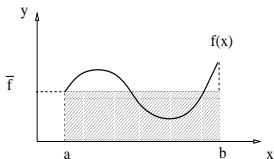
Average value of a function in a region in space.

Review: The average of a single variable function.

Definition

The *average* of a function $f : [a, b] \rightarrow \mathbb{R}$ on the interval $[a, b]$, denoted by \bar{f} , is given by

$$\bar{f} = \frac{1}{(b-a)} \int_a^b f(x) dx.$$



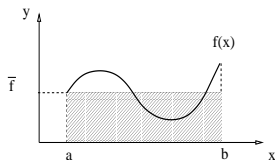
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Definition

The *average* of a function $f : R \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ on the region R with volume V , denoted by \bar{f} , is given by

$$\bar{f} = \frac{1}{V} \iiint_R f dv.$$

Average value of a function in a region in space.

Example

Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

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Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

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We conclude: $\bar{f} = 1/4$.



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Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

Solution: Sketch the integration region.

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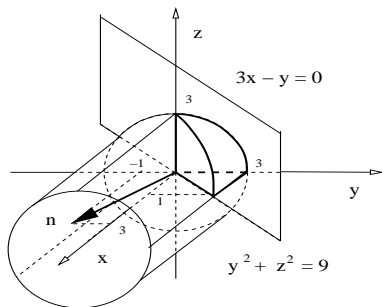
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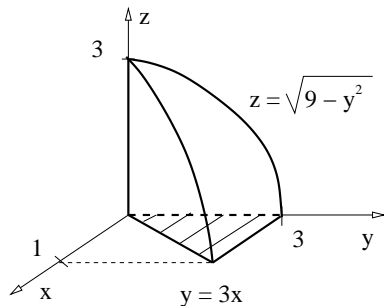


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Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

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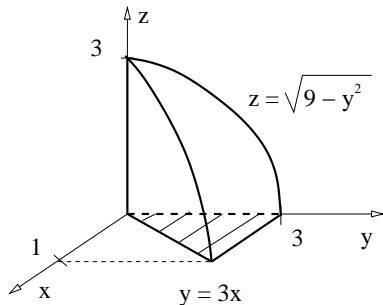


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The integration limits are:

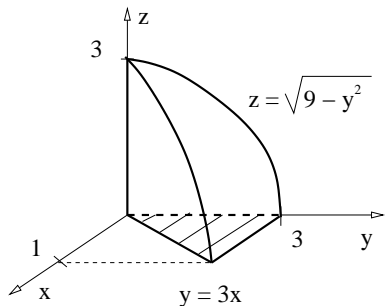
- ▶ Limits in z :
 $0 \leq z \leq \sqrt{9 - y^2}$.
- ▶ Limits in x : $0 \leq x \leq y/3$.
- ▶ Limits in y : $0 \leq y \leq 3$.

Triple integrals in arbitrary domains.

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 $0 \leq z \leq \sqrt{9 - y^2}$.
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We obtain $I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy$.

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

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Triple integrals in arbitrary domains.

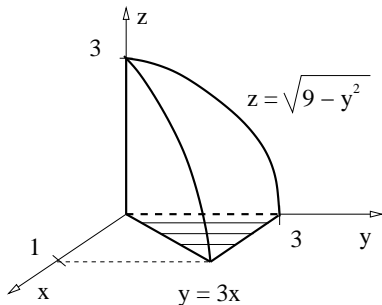
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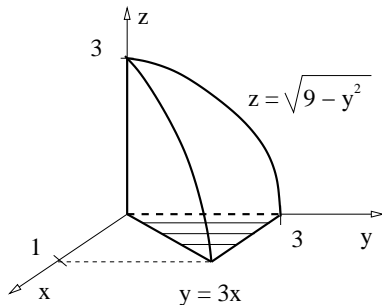
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The result is: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx.$



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Solution: Recall $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx$.

We now compute the integral:

$$\begin{aligned} \iiint_D f \, dv &= \int_0^1 \int_{3x}^3 \left(\frac{z^2}{2} \Big|_0^{\sqrt{9-y^2}} \right) dy \, dx, \\ &= \frac{1}{2} \int_0^1 \int_{3x}^3 (9 - y^2) dy \, dx, \\ &= \frac{1}{2} \int_0^1 \left[9 \left(y \Big|_{3x}^3 \right) - \left(\frac{y^3}{3} \Big|_{3x}^3 \right) \right] dx. \end{aligned}$$

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Therefore,

$$\begin{aligned} \iiint_D f \, dv &= \frac{1}{2} \int_0^1 \left[27(1-x) - 9(1-x)^3 \right] dx, \\ &= \frac{9}{2} \int_0^1 \left[3(1-x) - (1-x)^3 \right] dx. \end{aligned}$$

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Example

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Substitute $u = 1 - x$, then $du = -dx$, so,

$$\iiint_D f \, dv = \frac{9}{2} \int_0^1 (3u - u^3) du.$$

Triple integrals in arbitrary domains.

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Solution:

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We conclude $\iiint_D f \, dv = \frac{45}{8}$.



Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in cylindrical coordinates.
 - ▶ Review: Polar coordinates in a plane.
 - ▶ Cylindrical coordinates in space.
 - ▶ Triple integral in cylindrical coordinates.

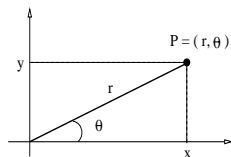
Next class:

- ▶ Integration in spherical coordinates.
 - ▶ Review: Cylindrical coordinates.
 - ▶ Spherical coordinates in space.
 - ▶ Triple integral in spherical coordinates.

Review: Polar coordinates in plane.

Definition

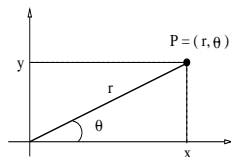
The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



Review: Polar coordinates in plane.

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Theorem (Cartesian-polar transformations)

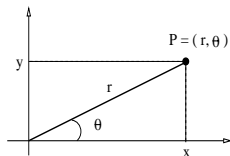
The Cartesian coordinates of a point $P = (r, \theta)$ in the first quadrant are given by

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

Review: Polar coordinates in plane.

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The polar coordinates of a point $P = (x, y)$ in the first quadrant are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Recall: Polar coordinates in a plane.

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Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

Recall: Polar coordinates in a plane.

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Then, for every $y \in [0, 2]$ the x coordinate satisfies $x \in [0, y]$.

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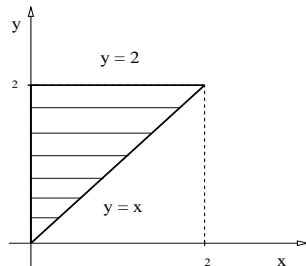
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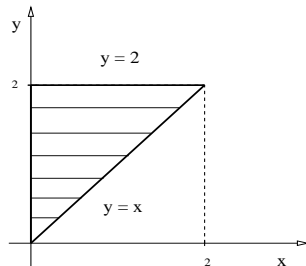
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Now is simple to describe this domain in polar coordinates:

Recall: Polar coordinates in a plane.

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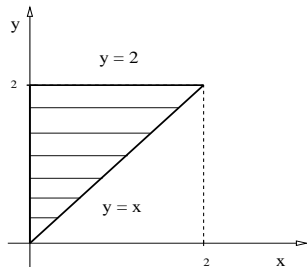
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The upper limit for x is the curve $y = x$.



Now is simple to describe this domain in polar coordinates:

The line $y = x$ is $\theta_0 = \pi/4$; the line $x = 0$ is $\theta_1 = \pi/2$.

Recall: Polar coordinates in a plane.

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Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

Solution: Recall: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$.

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The lower integration limit in r is $r = 0$.

Recall: Polar coordinates in a plane.

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Solution: Recall: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$.

The lower integration limit in r is $r = 0$.

The upper integration limit is $y = 2$,
that is, $2 = y = r \sin(\theta)$.

Recall: Polar coordinates in a plane.

Example

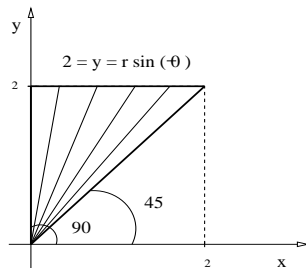
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Solution: Recall: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$.

The lower integration limit in r is $r = 0$.

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Hence $r = 2/\sin(\theta)$.



Recall: Polar coordinates in a plane.

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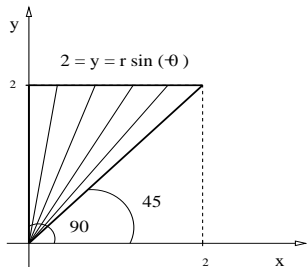
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Hence $r = 2/\sin(\theta)$.



We conclude:
$$\int_0^2 \int_0^y x \, dx \, dy = \int_{\pi/4}^{\pi/2} \int_0^{2/\sin(\theta)} r \cos(\theta) (r \, dr) \, d\theta. \triangleleft$$

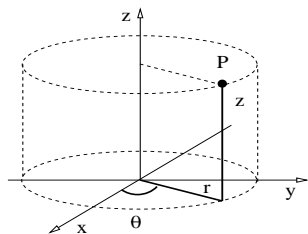
Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in cylindrical coordinates.
 - ▶ Review: Polar coordinates in a plane.
 - ▶ **Cylindrical coordinates in space.**
 - ▶ Triple integral in cylindrical coordinates.

Cylindrical coordinates in space.

Definition

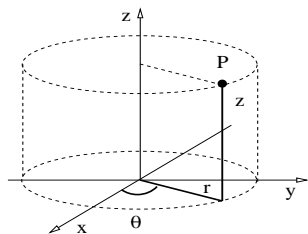
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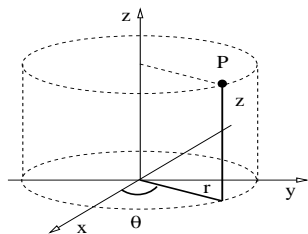


Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z .

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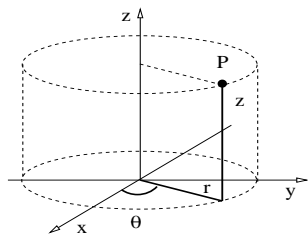
Theorem (Cartesian-cylindrical transformations)

The Cartesian coordinates of a point $P = (r, \theta, z)$ in the first quadrant are given by $x = r \cos(\theta)$, $y = r \sin(\theta)$, and $z = z$.

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Cylindrical coordinates in space.

Example

Use cylindrical coordinates to describe the region

$$R = \{(x, y, z) : x^2 + (y - 1)^2 \leq 1, 0 \leq z \leq x^2 + y^2\}.$$

Cylindrical coordinates in space.

Example

Use cylindrical coordinates to describe the region

$$R = \{(x, y, z) : x^2 + (y - 1)^2 \leq 1, 0 \leq z \leq x^2 + y^2\}.$$

Solution: We first sketch the region.

Cylindrical coordinates in space.

Example

Use cylindrical coordinates to describe the region

$$R = \{(x, y, z) : x^2 + (y - 1)^2 \leq 1, 0 \leq z \leq x^2 + y^2\}.$$

Solution: We first sketch the region.

The base of the region is at $z = 0$,
given by the disk $x^2 + (y - 1)^2 \leq 1$.

Cylindrical coordinates in space.

Example

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$$R = \{(x, y, z) : x^2 + (y - 1)^2 \leq 1, 0 \leq z \leq x^2 + y^2\}.$$

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The base of the region is at $z = 0$,
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The top of the region is the paraboloid
 $z = x^2 + y^2$.

Cylindrical coordinates in space.

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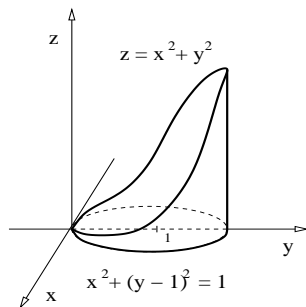
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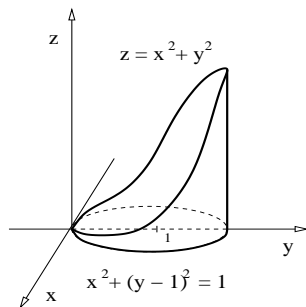
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The top of the region is the paraboloid $z = x^2 + y^2$.



In cylindrical coordinates: $z = x^2 + y^2 \Leftrightarrow z = r^2$, and

Cylindrical coordinates in space.

Example

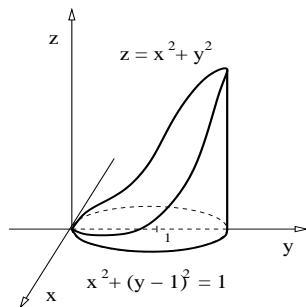
Use cylindrical coordinates to describe the region

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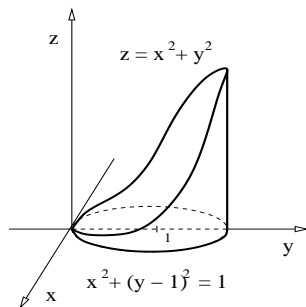
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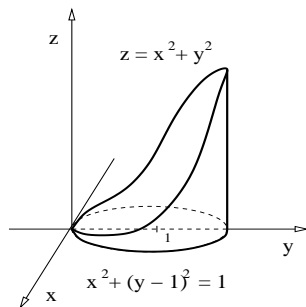
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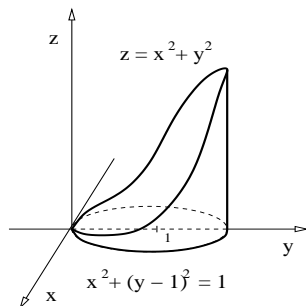
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Hence: $R = \{(r, \theta, z) : \theta \in [0, \pi], r \in [0, 2 \sin(\theta)], z \in [0, r^2]\}.$ ◁

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in cylindrical coordinates.
 - ▶ Review: Polar coordinates in a plane.
 - ▶ Cylindrical coordinates in space.
 - ▶ **Triple integral in cylindrical coordinates.**

Triple integrals using cylindrical coordinates.

Theorem

If the function $f : R \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in cylindrical coordinates as follows,

$$\iiint_R f \, dv = \iiint_R f(r, \theta, z) r \, dr \, d\theta \, dz.$$

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Remark:

- ▶ Cylindrical coordinates are useful when the integration region R is described in a simple way using cylindrical coordinates.
- ▶ Notice the extra factor r on the right-hand side.

Triple integrals using cylindrical coordinates.

Example

Find the volume of a cylinder of radius R and height h .

Triple integrals using cylindrical coordinates.

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Find the volume of a cylinder of radius R and height h .

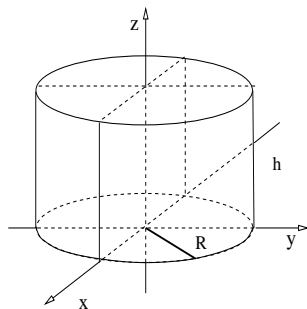
Solution: $R = \{(r, \theta, z) : \theta \in [0, 2\pi], r \in [0, R], z \in [0, h]\}$.

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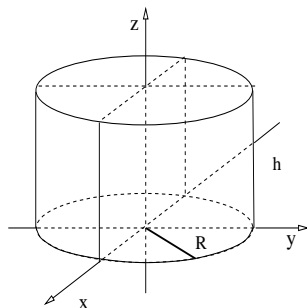


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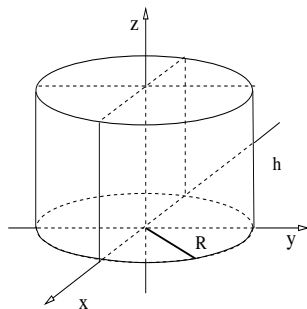
$$\begin{aligned} V &= \int_0^{2\pi} \int_0^R \int_0^h dz (r dr) d\theta, \\ &= h \int_0^{2\pi} \int_0^R r dr d\theta, \\ &= h \frac{R^2}{2} \int_0^{2\pi} d\theta, \\ &= h \frac{R^2}{2} 2\pi, \end{aligned}$$

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We conclude: $V = \pi R^2 h$.



Triple integrals using cylindrical coordinates.

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Find the volume of a cone of base radius R and height h .

Triple integrals using cylindrical coordinates.

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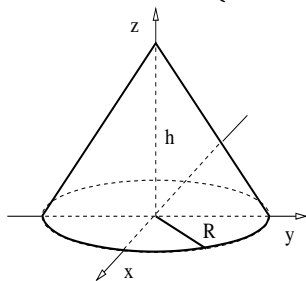
Solution: $R = \left\{ \theta \in [0, 2\pi], r \in [0, R], z \in \left[0, -\frac{h}{R}r + h \right] \right\}$.

Triple integrals using cylindrical coordinates.

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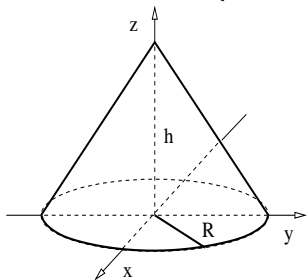


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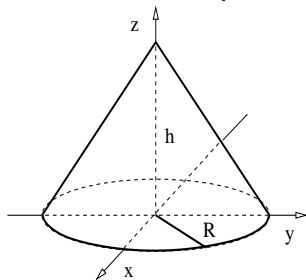
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We conclude: $V = \frac{1}{3}\pi R^2 h$.



Triple integrals using cylindrical coordinates.

Example

Sketch the region with volume $V = \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{9-r^2}} rdz dr d\theta$.

Triple integrals using cylindrical coordinates.

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Solution: The integration region is

$$R = \{(r, \theta, z) : \theta \in [0, \pi/2], r \in [0, 2], z \in [0, \sqrt{9-r^2}]\}.$$

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We upper boundary is a sphere, since

$$z^2 = 9 - r^2 \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 3^2.$$

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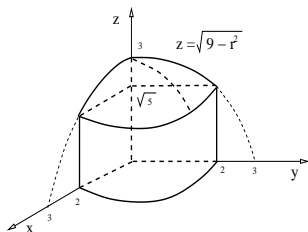
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The upper limit for r is $r = 2$, so

$$z = \sqrt{9 - 2^2} \Rightarrow z = \sqrt{5}.$$



Triple integrals using cylindrical coordinates.

Example

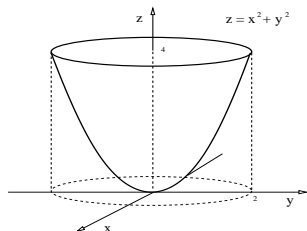
Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) : x^2 + y^2 \leq 2^2, x^2 + y^2 \leq z \leq 4\}$.

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Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) : x^2 + y^2 \leq 2^2, x^2 + y^2 \leq z \leq 4\}$.

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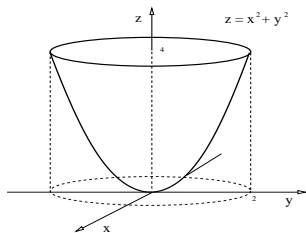
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Solution:

The symmetry of the region implies $\bar{x} = 0$ and $\bar{y} = 0$. (We verify this result later on.) We only need to compute \bar{z} .

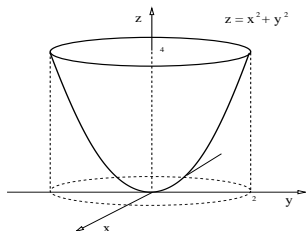


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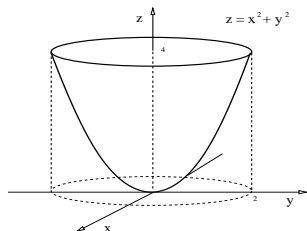
Since $\bar{z} = \frac{1}{V} \iiint_R z \, dv$, we start computing the total volume V .

Triple integrals using cylindrical coordinates.

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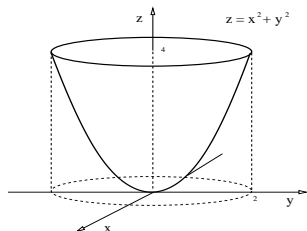
We use cylindrical coordinates.

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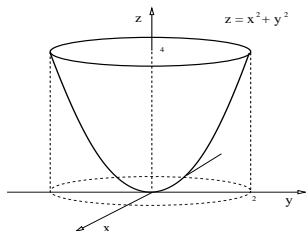
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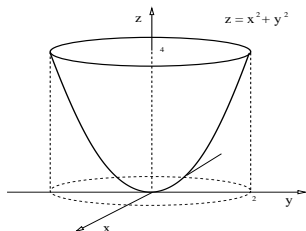
$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 dz \, r \, dr \, d\theta = 2\pi \int_0^2 \left(z \Big|_{r^2}^4 \right) r \, dr$$

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$$\bar{z} = \frac{1}{8} \int_0^2 (16r - r^5) dr = \frac{1}{8} \left[16 \left(\frac{r^2}{2} \Big|_0^2 \right) - \left(\frac{r^6}{6} \Big|_0^2 \right) \right];$$

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$$\bar{z} = \frac{1}{8} \left(32 - \frac{64}{6} \right)$$

Triple integrals using cylindrical coordinates.

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) : x^2 + y^2 \leq 2^2, x^2 + y^2 \leq z \leq 4\}$.

Solution: $V = 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[4 \left(\frac{r^2}{2} \Big|_0^2 \right) - \left(\frac{r^4}{4} \Big|_0^2 \right) \right].$

Hence $V = 2\pi(8 - 4)$, so $V = 8\pi$. Then, \bar{z} is given by,

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Triple integrals using cylindrical coordinates.

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Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) : x^2 + y^2 \leq 2^2, x^2 + y^2 \leq z \leq 4\}$.

Solution: We obtained $\bar{z} = \frac{8}{3}$.

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It is simple to see that $\bar{x} = 0$ and $\bar{y} = 0$.

Triple integrals using cylindrical coordinates.

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It is simple to see that $\bar{x} = 0$ and $\bar{y} = 0$. For example,

$$\begin{aligned}\bar{x} &= \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 [r \cos(\theta)] dz r dr d\theta \\ &= \frac{1}{8\pi} \left[\int_0^{2\pi} \cos(\theta) d\theta \right] \left[\int_0^2 \int_{r^2}^4 dz r^2 dr \right].\end{aligned}$$

Triple integrals using cylindrical coordinates.

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space $R = \{(x, y, z) : x^2 + y^2 \leq 2^2, x^2 + y^2 \leq z \leq 4\}$.

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But $\int_0^{2\pi} \cos(\theta) d\theta = \sin(2\pi) - \sin(0) = 0$, so $\bar{x} = 0$.

Triple integrals using cylindrical coordinates.

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But $\int_0^{2\pi} \cos(\theta) d\theta = \sin(2\pi) - \sin(0) = 0$, so $\bar{x} = 0$.

A similar calculation shows $\bar{y} = 0$. Hence $\bar{\mathbf{r}} = \langle 0, 0, 8/3 \rangle$. \triangleleft

Triple integrals using cylindrical coordinates.

Example

Change the integration order and compute the integral

$$I = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr dz d\theta.$$

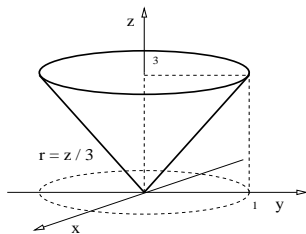
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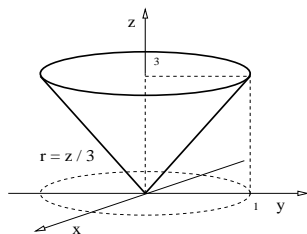
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Example

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Solution:



$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_{3r}^3 dz r^3 dr d\theta \\ &= 2\pi \int_0^1 \left(z \Big|_{3r}^3 \right) r^3 dr \\ &= 2\pi \int_0^1 3(r^3 - r^4) dr \\ &= 6\pi \left(\frac{r^4}{4} - \frac{r^5}{5} \right) \Big|_0^1. \end{aligned}$$

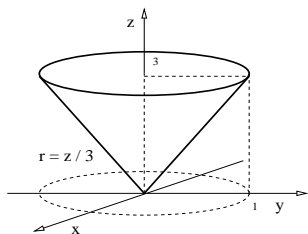
Triple integrals using cylindrical coordinates.

Example

Change the integration order and compute the integral

$$I = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr dz d\theta.$$

Solution:



So, $I = 6\pi \frac{1}{20}$, that is, $I = \frac{3\pi}{10}$.

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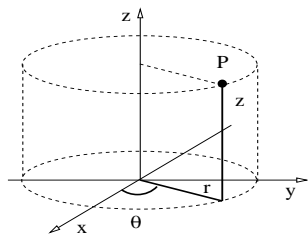
Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in spherical coordinates.
 - ▶ Review: Cylindrical coordinates.
 - ▶ Spherical coordinates in space.
 - ▶ Triple integral in spherical coordinates.

Cylindrical coordinates in space.

Definition

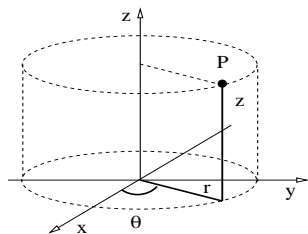
The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture.



Cylindrical coordinates in space.

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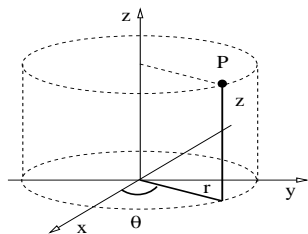


Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z .

Cylindrical coordinates in space.

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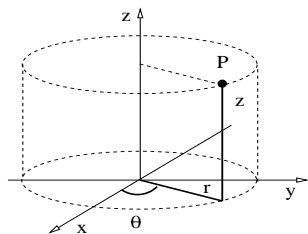
Theorem (Cartesian-cylindrical transformations)

The Cartesian coordinates of a point $P = (r, \theta, z)$ in the first quadrant are given by $x = r \cos(\theta)$, $y = r \sin(\theta)$, and $z = z$.

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The cylindrical coordinates of a point $P = (x, y, z)$ in the first quadrant are given by $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.

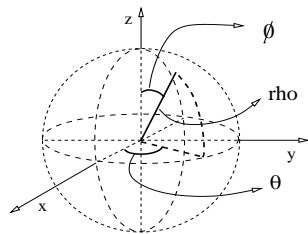
Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in spherical coordinates.
 - ▶ Review: Cylindrical coordinates.
 - ▶ **Spherical coordinates in space.**
 - ▶ Triple integral in spherical coordinates.

Spherical coordinates in \mathbb{R}^3

Definition

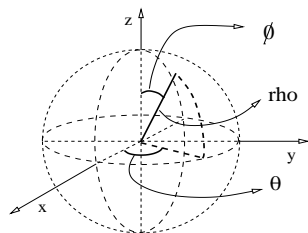
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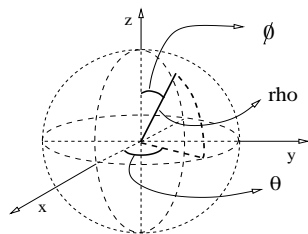
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The spherical coordinates of $P = (x, y, z)$ in the first quadrant are $\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\left(\frac{y}{x}\right)$, and $\phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$.

Spherical coordinates in \mathbb{R}^3

Example

Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

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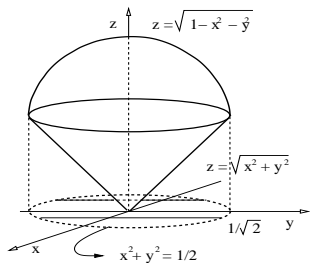
Solution: ($x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$.)

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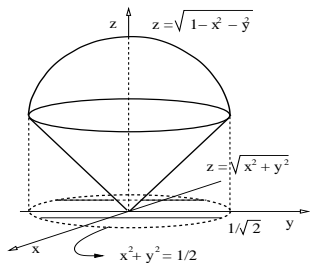
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Spherical coordinates in \mathbb{R}^3

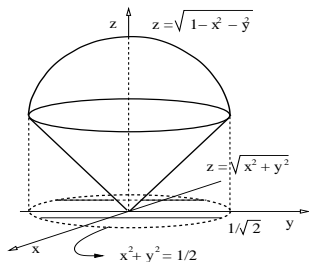
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The bottom surface is the cone:

$$\begin{aligned}\rho \cos(\phi) &= \sqrt{\rho^2 \sin^2(\phi)} \\ \cos(\phi) &= \sin(\phi),\end{aligned}$$



Spherical coordinates in \mathbb{R}^3

Example

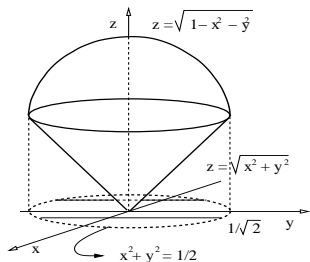
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so the cone is $\phi = \frac{\pi}{4}$.



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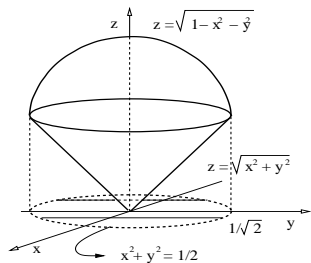
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so the cone is $\phi = \frac{\pi}{4}$.

Hence: $R = \left\{ (\rho, \phi, \theta) : \theta \in [0, 2\pi], \phi \in \left[0, \frac{\pi}{4}\right], \rho \in [0, 1] \right\}$.



Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- ▶ Integration in spherical coordinates.
 - ▶ Review: Cylindrical coordinates.
 - ▶ Spherical coordinates in space.
 - ▶ **Triple integral in spherical coordinates.**

Triple integral in spherical coordinates.

Theorem

If the function $f : R \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in spherical coordinates as follows,

$$\iiint_R f \, dv = \iiint_R f(\rho, \phi, \theta) \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.$$

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- ▶ Spherical coordinates are useful when the integration region R is described in a simple way using spherical coordinates.
- ▶ Notice the extra factor $\rho^2 \sin(\phi)$ on the right-hand side.

Triple integral in spherical coordinates.

Example

Find the volume of a sphere of radius R .

Triple integral in spherical coordinates.

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$$\begin{aligned} V &= \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin(\phi) d\rho d\phi d\theta, \\ &= \left[\int_0^{2\pi} d\theta \right] \left[\int_0^\pi \sin(\phi) d\phi \right] \left[\int_0^R \rho^2 d\rho \right], \\ &= 2\pi \left[-\cos(\phi) \Big|_0^\pi \right] \frac{R^3}{3}, \\ &= 2\pi \left[-\cos(\pi) + \cos(0) \right] \frac{R^3}{3}; \end{aligned}$$

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hence: $V = \frac{4}{3}\pi R^3$.



Triple integral in spherical coordinates.

Example

Use spherical coordinates to find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.

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Use spherical coordinates to find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.

Solution: $R = \left\{ (\rho, \phi, \theta) : \theta \in [0, 2\pi], \phi \in \left[0, \frac{\pi}{4}\right], \rho \in [0, 1] \right\}$.

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The calculation is simple, the region is a simple section of a sphere.

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The calculation is simple, the region is a simple section of a sphere.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi) d\rho d\phi d\theta, \\ &= \left[\int_0^{2\pi} d\theta \right] \left[\int_0^{\pi/4} \sin(\phi) d\phi \right] \left[\int_0^1 \rho^2 d\rho \right], \\ &= 2\pi \left[-\cos(\phi) \Big|_0^{\pi/4} \right] \left(\frac{\rho^3}{3} \Big|_0^1 \right), \\ &= 2\pi \left[-\frac{\sqrt{2}}{2} + 1 \right] \frac{1}{3} \quad \Rightarrow \quad V = \frac{\pi}{3}(2 - \sqrt{2}). \end{aligned}$$



Triple integral in spherical coordinates.

Example

Find the integral of $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ in the region $R = \{x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1\}$ using spherical coordinates.

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Solution: $R = \left\{ \theta \in \left[0, \frac{\pi}{2}\right], \phi \in \left[0, \frac{\pi}{2}\right], \rho \in [0, 1] \right\}$. Hence,

$$\begin{aligned} \iiint_R f \, dv &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{\rho^3} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta, \\ &= \left[\int_0^{\pi/2} d\theta \right] \left[\int_0^{\pi/2} \sin(\phi) \, d\phi \right] \left[\int_0^1 e^{\rho^3} \rho^2 \, d\rho \right]. \end{aligned}$$

Triple integral in spherical coordinates.

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Triple integral in spherical coordinates.

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} y \sqrt{x^2 + y^2 + z^2} dz dy dx.$$

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- ▶ Limits in x : $|x| \leq 2$;
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the positive side of the disk
 $x^2 + y^2 \leq 4$.

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Triple integral in spherical coordinates.

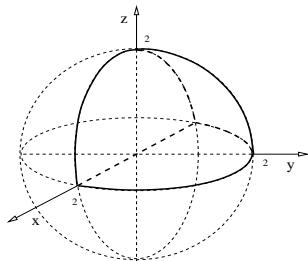
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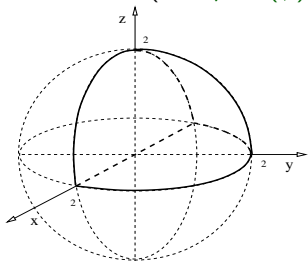
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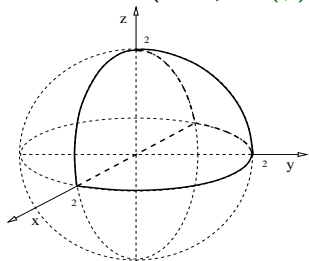
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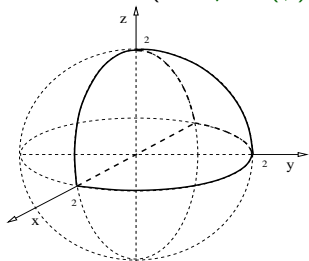
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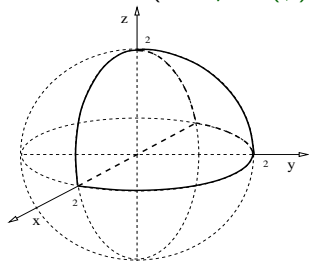
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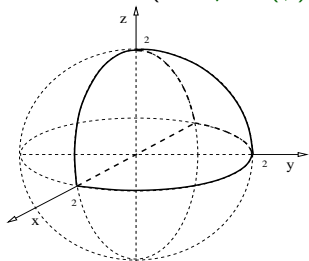
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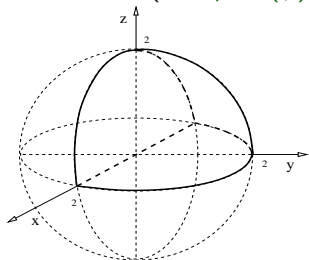
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$$\begin{aligned} I &= \left[\int_0^\pi \sin(\theta) d\theta \right] \left[\int_0^{\pi/2} \sin^2(\phi) d\phi \right] \left[\int_0^2 \rho^4 d\rho \right], \\ &= \left(-\cos(\theta) \Big|_0^\pi \right) \left[\int_0^{\pi/2} \frac{1}{2} (1 - \cos(2\phi)) d\phi \right] \left(\frac{\rho^5}{5} \Big|_0^2 \right), \\ &= 2 \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - \frac{1}{2} \left(\sin(2\phi) \Big|_0^{\pi/2} \right) \right] \frac{2^5}{5} \quad \Rightarrow \quad I = \frac{2^4 \pi}{5}. \end{aligned}$$

Triple integral in spherical coordinates.

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi) d\rho d\phi d\theta$.

Triple integral in spherical coordinates.

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi) d\rho d\phi d\theta$.

Solution: Recall: $\sec(\phi) = 1/\cos(\phi)$.

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Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi) d\rho d\phi d\theta$.

Solution: Recall: $\sec(\phi) = 1/\cos(\phi)$.

$$\begin{aligned} I &= 2\pi \int_0^{\pi/3} \left(\rho^3 \Big|_{\sec(\phi)}^2 \right) \sin(\phi) d\phi, \\ &= 2\pi \int_0^{\pi/3} \left(2^3 - \frac{1}{\cos^3(\phi)} \right) \sin(\phi) d\phi \end{aligned}$$

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$$I = 2\pi \left[2^3 \left(-\cos(\phi) \Big|_0^{\pi/3} \right) + \int_1^{1/2} \frac{du}{u^3} \right].$$

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$$I = 2\pi \left[2^3 \left(-\frac{1}{2} + 1 \right) - \int_{1/2}^1 u^{-3} du \right]$$

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$$I = 2\pi \left[2^3 \left(-\frac{1}{2} + 1 \right) - \int_{1/2}^1 u^{-3} du \right] = 2\pi \left[4 - \left(\frac{u^{-2}}{-2} \Big|_{1/2}^1 \right) \right],$$

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$$I = 2\pi \left[4 + \frac{1}{2} \left(u^{-2} \Big|_{1/2}^1 \right) \right]$$

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$$I = 2\pi \left[4 + \frac{1}{2} \left(u^{-2} \Big|_{1/2}^1 \right) \right] = 2\pi \left[4 + \frac{1}{2} (1 - 2^2) \right] = 2\pi \left[\frac{8}{2} - \frac{3}{2} \right]$$

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Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi) d\rho d\phi d\theta$.

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We conclude: $I = 5\pi$.

