Cartesian coordinates in space (12.1)

- Overview of vector calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
- Equation of a sphere.

Overview of Multivariable calculus

Mth 132 Calculus I: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Mth 133 Calculus II: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Mth 234 Multivariable Calculus:

$$\left.\begin{array}{rr} f : \mathbb{R}^2 \to \mathbb{R}, & f(x,y) \\ f : \mathbb{R}^3 \to \mathbb{R}, & f(x,y,z) \end{array}\right\} \quad \text{scalar-valued.}$$

$$\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3, \quad \boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle \quad \} \quad \text{vector-valued.}$$

We study how to differentiate and integrate such functions.

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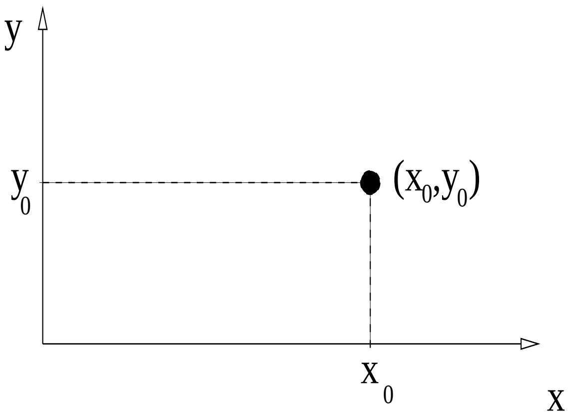
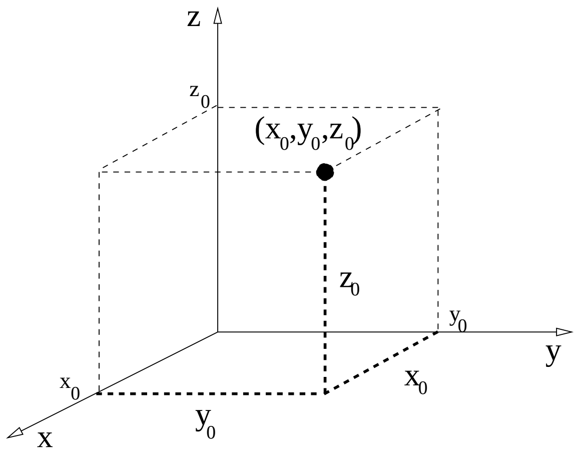
Review: Cartesian coordinates on the plane ($\mathbb{R}^2$)

Every point on a plane is labeled by an ordered pair (x, y).



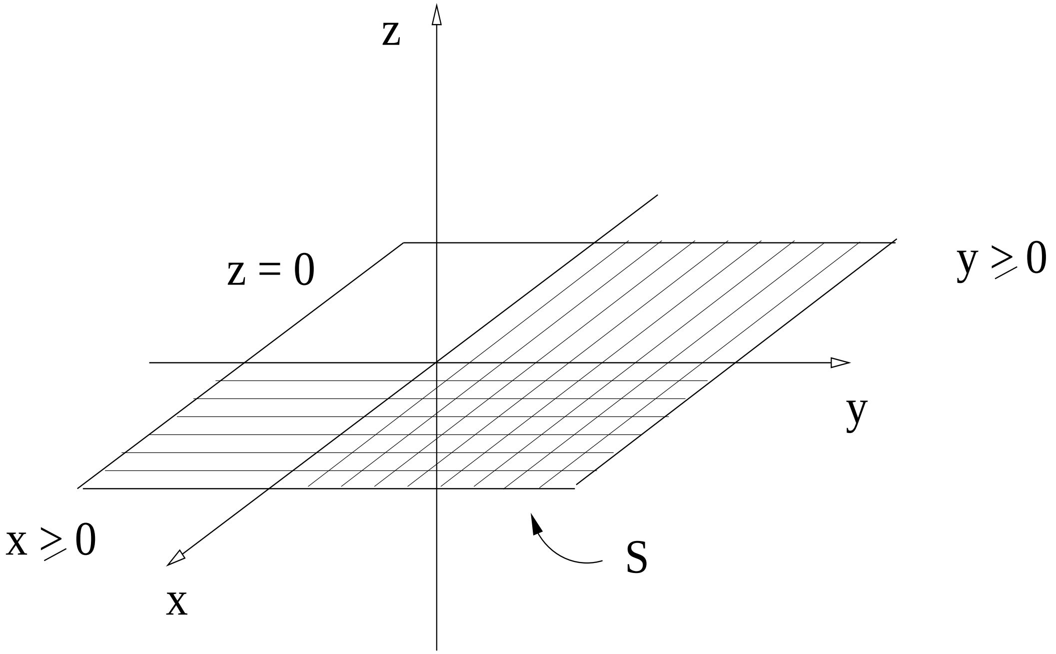


Cartesian coordinates in space ($\mathbb{R}^3$)

Every point in space is labeled by an ordered triple (x, y, z).

Example: Find the set $S = \{x \geqslant 0,\ y \geqslant 0,\ z = 0\} \subset \mathbb{R}^3$.

SOLUTION:

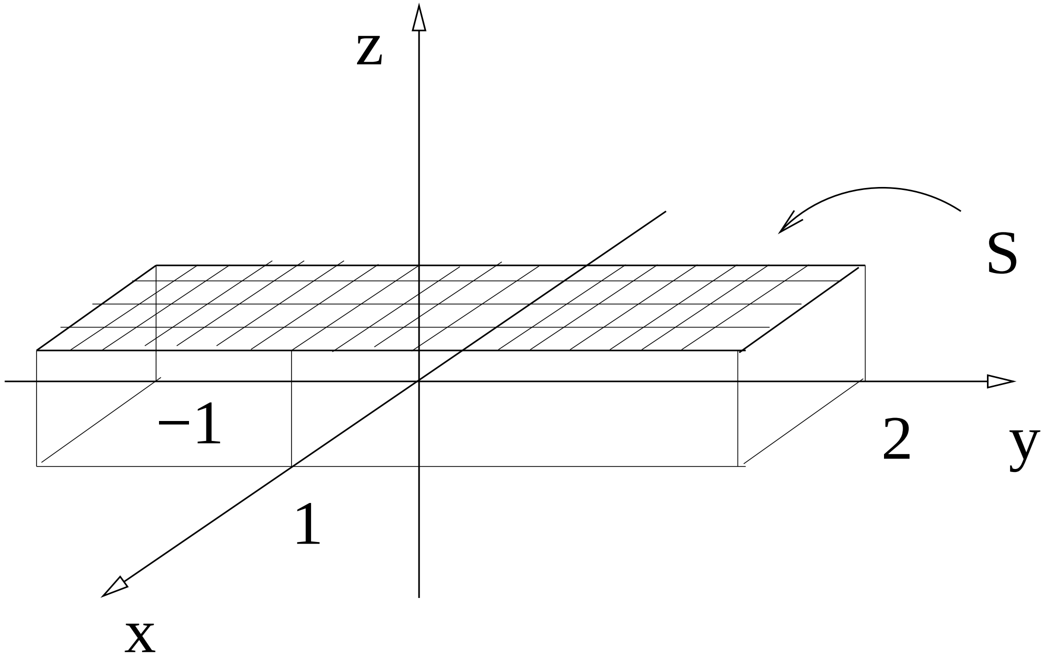


$\triangleleft$

Example: Find the set $S \subset \mathbb{R}^3$ given by

$$S = \{0 \leqslant x \leqslant 1, \ -1 \leqslant y \leqslant 2, \ z = 1\}.$$

Solution:



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Cartesian coordinates in space (12.1)

- Overview of vector calculus.
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- **Right-handed, left-handed Cartesian coordinates.**
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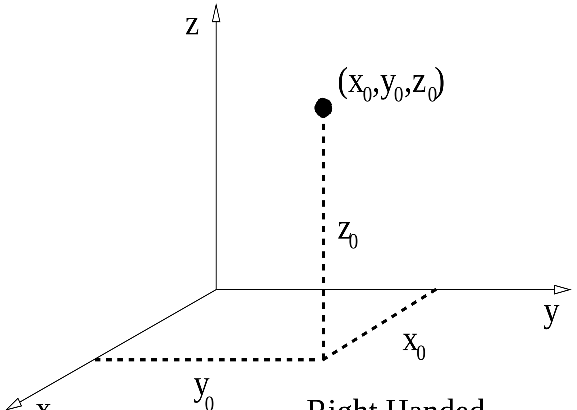
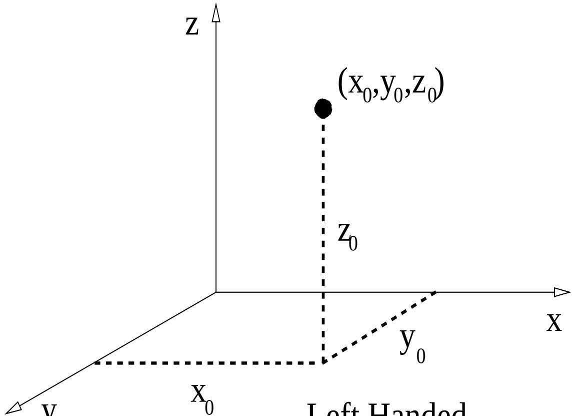
There are two types of Cartesian coordinate systems except by rotations: Right-handed (RH) and Left-handed (LH)



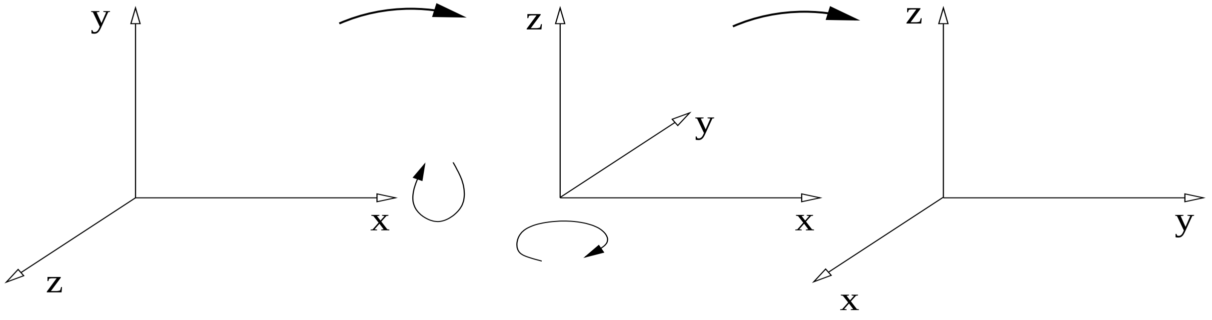
Right Handed



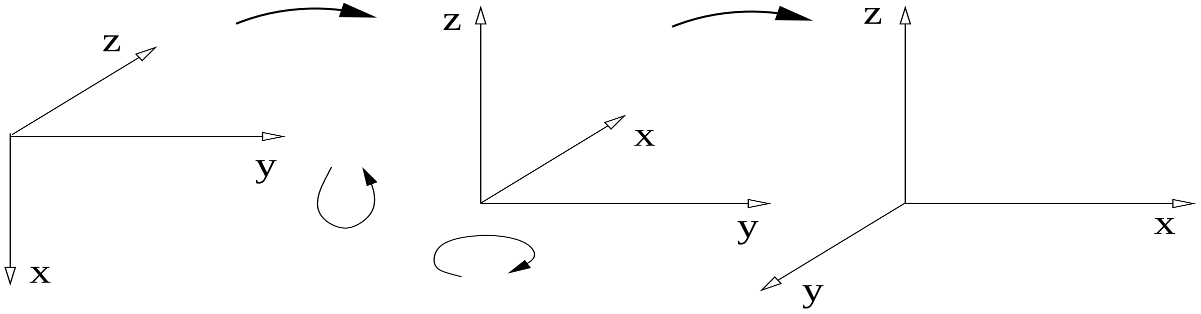
Left Handed

No rotation transforms one into the other

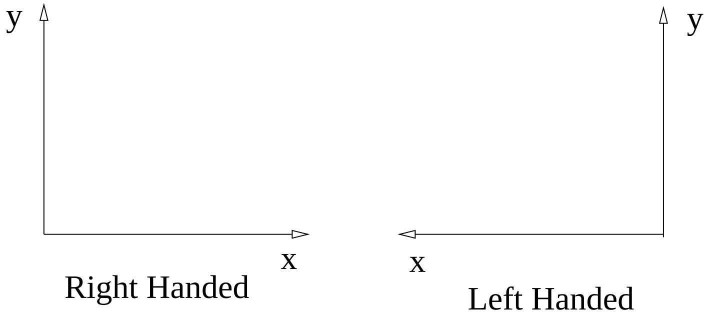
This coordinate system is right handed



This coordinate system is left handed



Remark: The same classification occurs in Cartesian coordinates on the plane.



- In $\mathbb{R}^3$ we will define the **cross product** of vectors.
- This product has different results in RH or LH Cartesian coordinates.
- There is no cross product in $\mathbb{R}^2$.

In class we use RH Cartesian coordinate systems

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- Equation of a sphere.

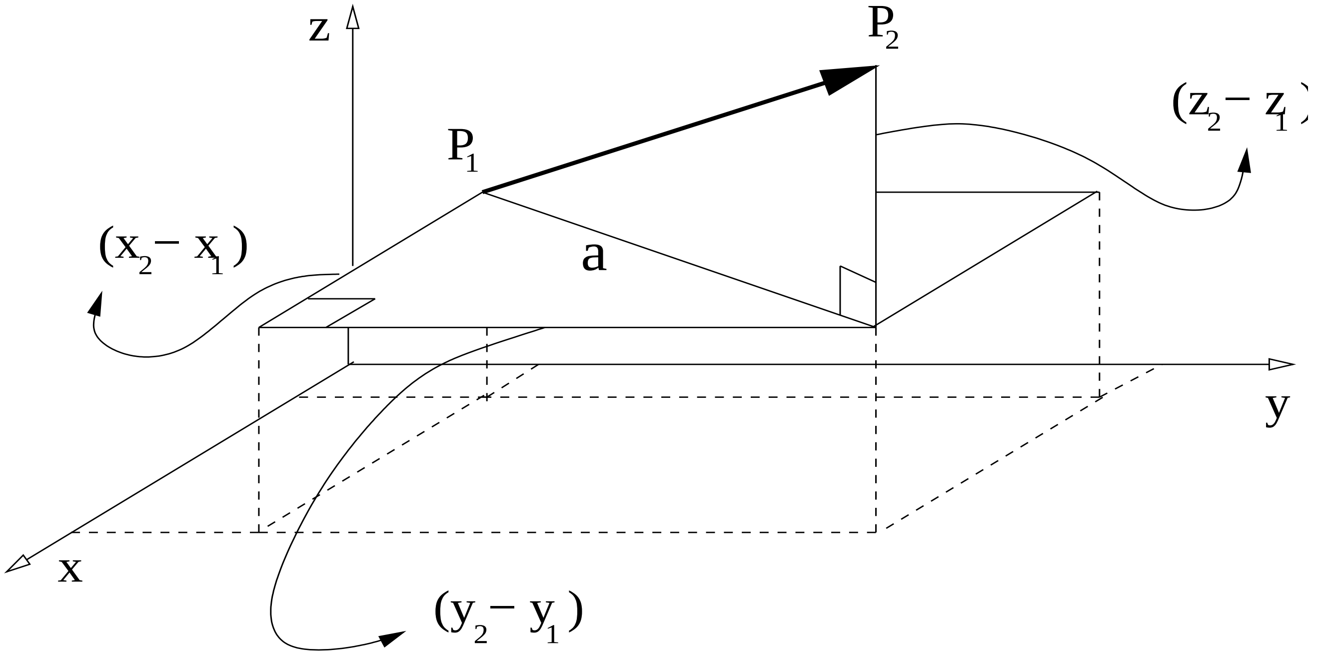
Distance formula between two points in space

Theorem 1 *The distance* $\left|P_1P_2\right|$ *between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$\left|P_1P_2\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The distance between points in space is crucial to define the idea of limit to functions in space

Proof: Pythagoras Theorem.



$$\left|P_1P_2\right|^2 = a^2 + (z_2 - z_1)^2,$$

$$a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

□

Example: Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Solution:

$$\begin{aligned} \left|P_1P_2\right| &= \sqrt{(3-1)^2 + (2-2)^2 + (1-3)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \quad \Rightarrow \quad \boxed{\left|P_1P_2\right| = 2\sqrt{2}}. \end{aligned}$$

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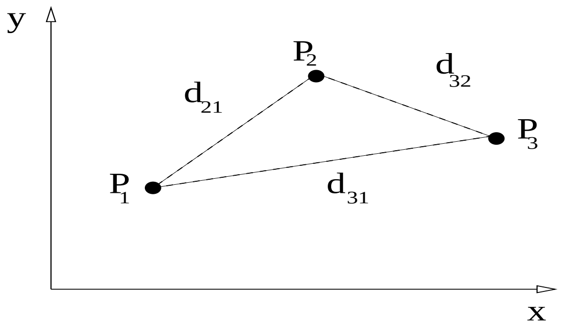
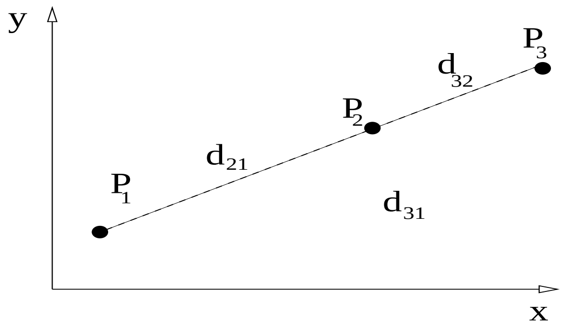
Example: Use the distance formula to determine whether three points in space are collinear.

SOLUTION:





$d_{21} + d_{32} > d_{31}$

Not collinear,

$d_{21} + d_{32} = d_{31}$

collinear.

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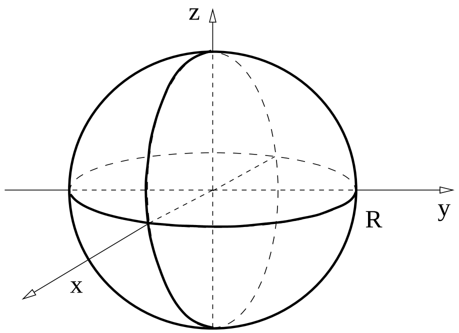
- **Equation of a sphere.**

A sphere is a set of points at fixed distance from a center

Definition 1 *A sphere centered at* $P_0 = (x_0, y_0, z_0)$ *of radius* R *is*

$$S = \Big\{ P = (x, y, z) \ : \ \big| P_0 P \big| = R \Big\}.$$



That is, $(x, y, z) \in S$ iff (if and only if)

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2}\,.$$

A Ball is a set of points contained in a sphere

Definition 2 *A ball centered at* $P_0 = (x_0, y_0, z_0)$ *of radius* R *is*

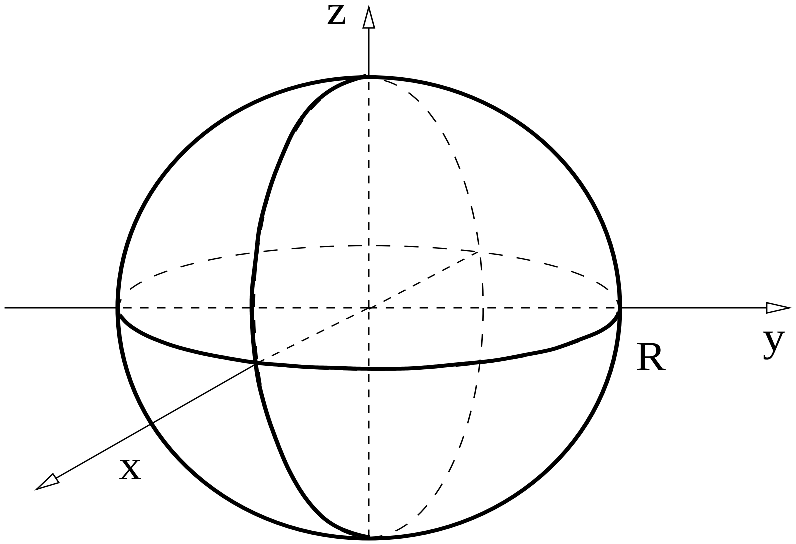
$$B = \Big\{ P = (x, y, z) \ : \ \Big| P_0 P \Big| < R \Big\}.$$

That is, $(x, y, z) \in B$ iff

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < R^2}.$$

Example: Plot a sphere centered at $P_0 = (0, 0, 0)$ of radius $R > 0$.

SOLUTION:



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Example: Plot the sphere $x^2 + y^2 + z^2 + 4y = 0$

SOLUTION: **Technique: Complete the square.**

$$\begin{aligned} 0 &= x^2 + y^2 + 4y + z^2 \\ &= x^2 \left[y^2 + 2\left(\frac{4}{2}\right) y + \left(\frac{4}{2}\right)^2 \right] - \left(\frac{4}{2}\right)^2 + z^2 \\ &= x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 - 4. \end{aligned}$$

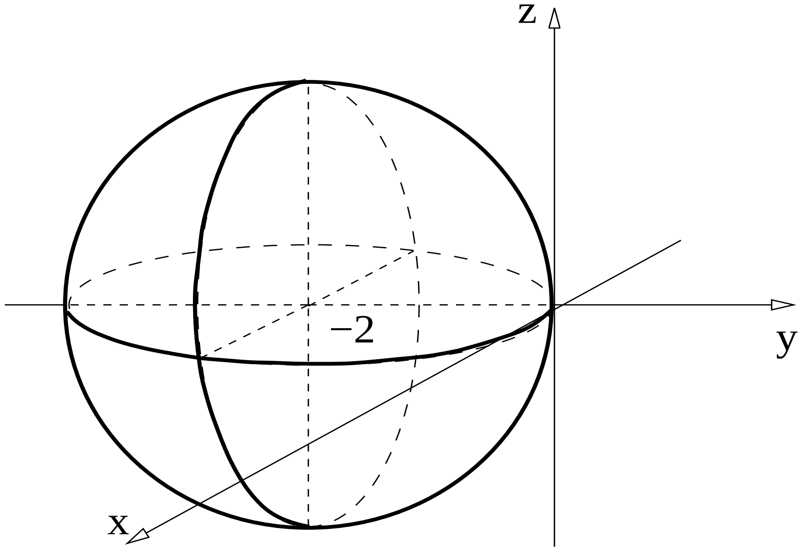
$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2.$$

Example: Plot the sphere $x^2 + y^2 + z^2 + 4y = 0$

Since

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2,$$

we conclude that $P_0 = (0, -2, 0)$ and $R = 2$, therefore,



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Exercise:

- Given constants a, b, c, and $d \in \mathbb{R}$, show that

$$x^2 + y^2 + z^2 - 2a\,x - 2b\,y - 2c\,z = d$$

is the equation of a sphere iff

$$d > -(a^2 + b^2 + c^2). \qquad (1)$$

- Furthermore, show that if Eq. (1) is satisfied, then the expressions for the center P_0 and the radius R of the sphere are given by

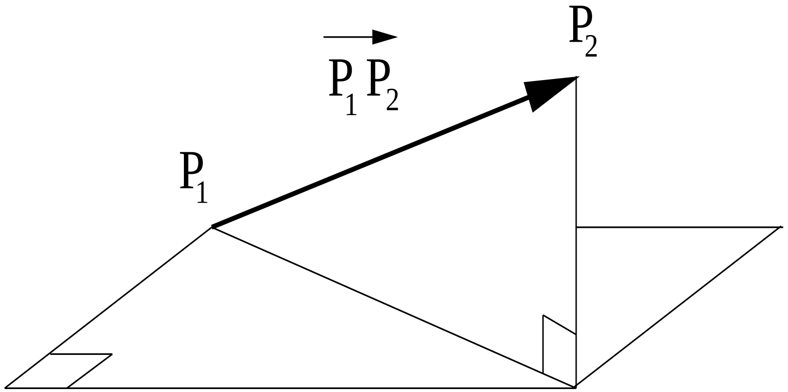
$$P_0 = (a, b, c), \qquad R = \sqrt{d + (a^2 + b^2 + c^2)}.$$

Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

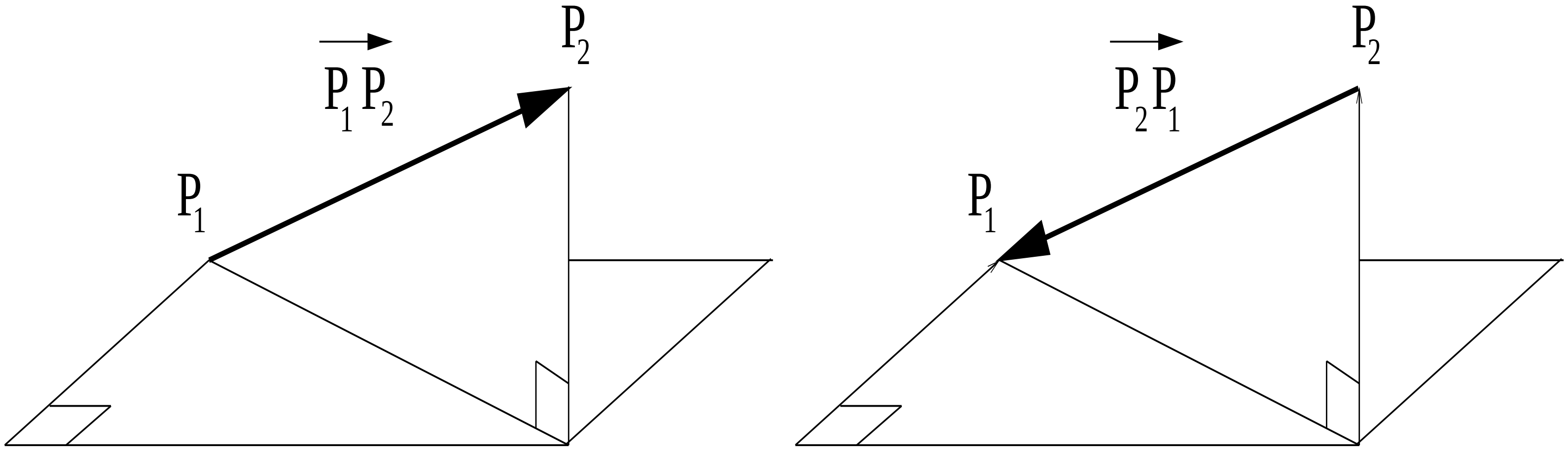
A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment

Definition 3 *A* **vector** *in* $\mathbb{R}^n$*, with* $n = 2, 3$*, is an ordered pair of points in* $\mathbb{R}^n$*, denoted as* $\overrightarrow{P_1P_2}$*, where* P_1*,* $P_2 \in \mathbb{R}^n$*. The point* P_1 *is called the* **initial point** *and* P_2 *is called the* **terminal point***.*



- A vector is drawn by an arrow pointing to the terminal point.
- A vector is denoted not only by $\overrightarrow{P_1P_2}$ but also by an arrow over a letter, like $\vec{v}$, or by a boldface letter, like $\boldsymbol{v}$.

The order of the points determines the direction.



The vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.

Vectors on a plane and in space (12.2)

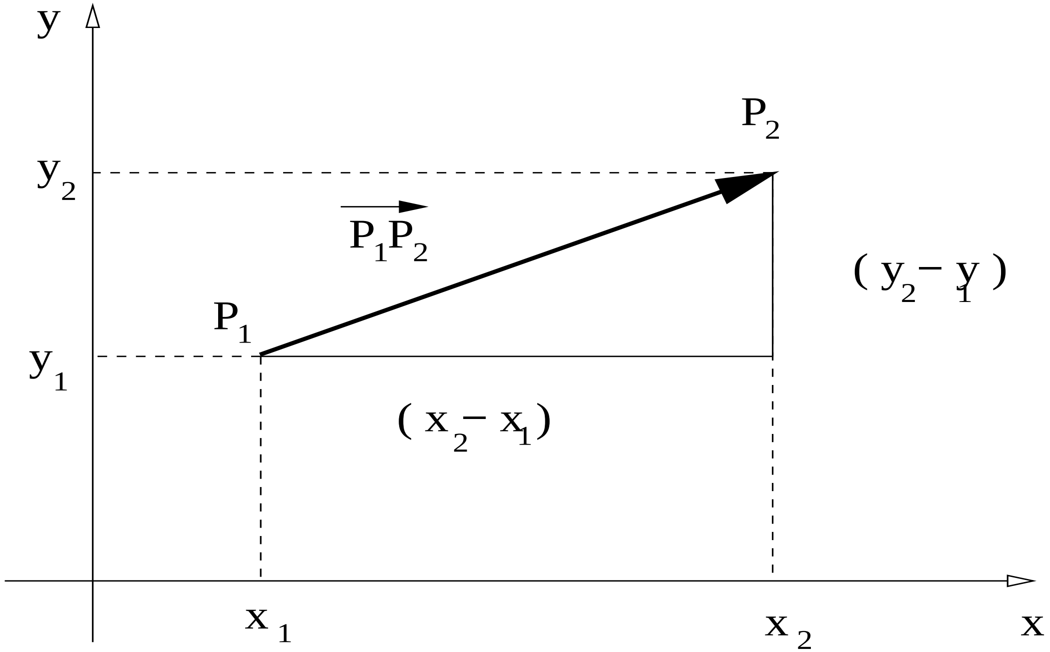
- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Vector components in Cartesian coordinates.**
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

Components of a vector in Cartesian coordinates

Theorem 2 *Given the points* $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, *the vector* $\overrightarrow{P_1P_2}$ *determines a unique ordered pair denoted as follows,*

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates.



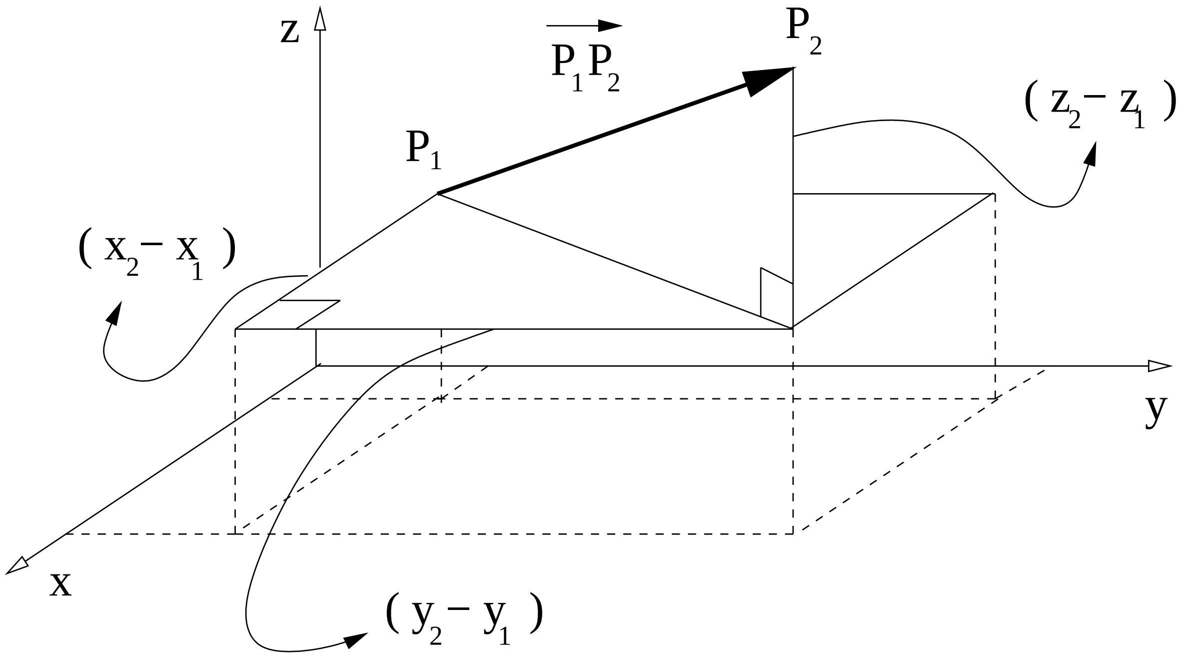
□

A similar result holds for vectors in space.

Theorem 3 *Given the points* $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, *the vector* $\overrightarrow{P_1P_2}$ *determines a unique ordered triple denoted as follows,*

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.$$

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates.



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Example: Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

SOLUTION:

$$\overrightarrow{P_1P_2} = \langle (3-1), (1-(-2)), (2-3) \rangle \quad \Rightarrow \quad \boxed{\overrightarrow{P_1P_2} = \langle 2, 3, -1 \rangle}.$$

$\lhd$

Example: Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

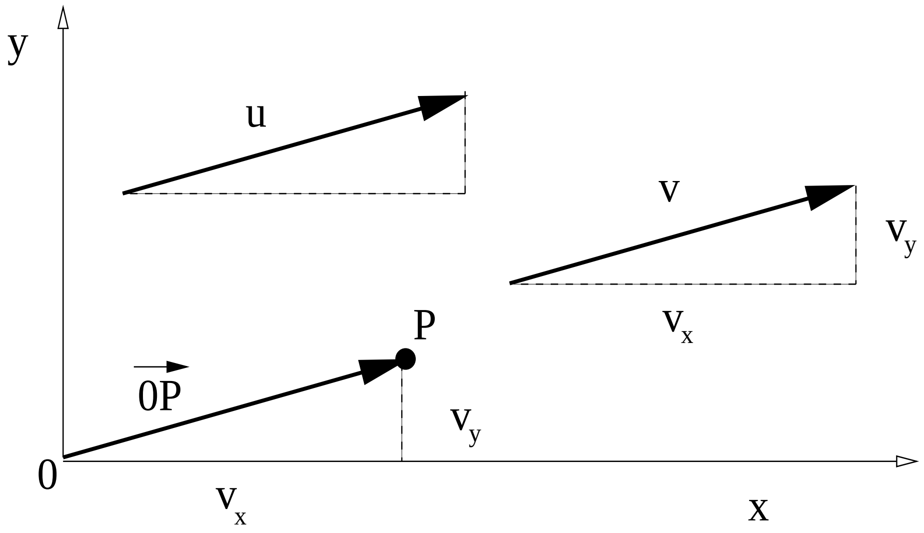
SOLUTION:

$$\overrightarrow{P_3P_4} = \langle (5-3), (4-1), (3-4) \rangle \quad \Rightarrow \quad \boxed{\overrightarrow{P_3P_4} = \langle 2, 3, -1 \rangle}.$$

$\lhd$

$\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ have the same components although they are different vectors.

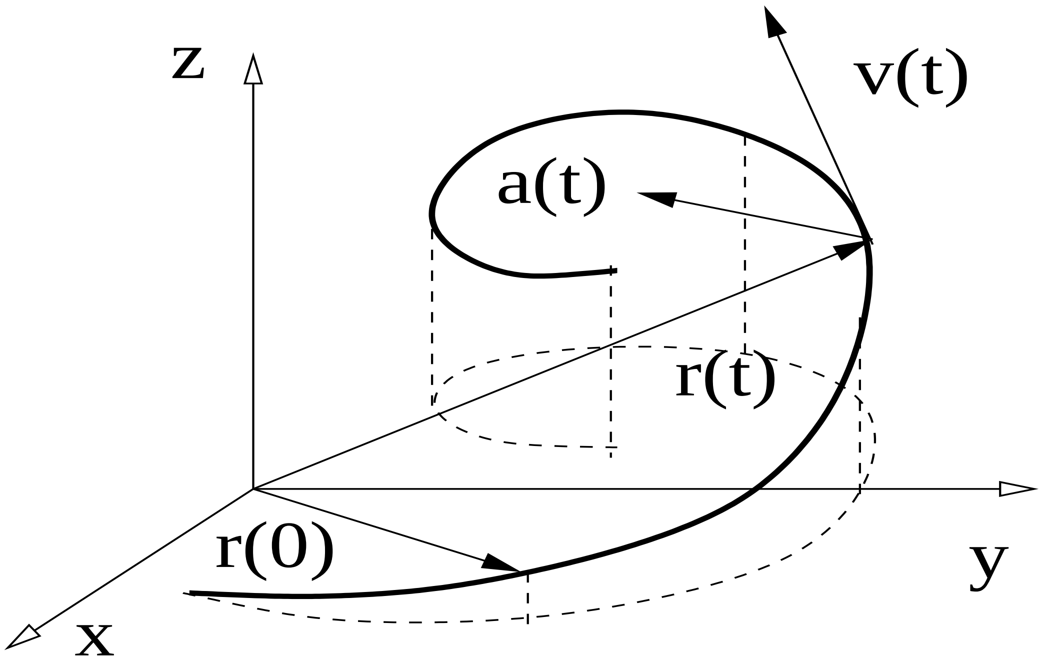
The vector components do not determine a unique vector.



The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\overrightarrow{0P}$ have the same components but they are all different, since they have different initial and terminal points.

Definition 4 *Given a vector* $\overrightarrow{P_1P_2} = \langle v_x, v_y \rangle$, *the* **standard position** *vector is the vector* $\overrightarrow{0P}$, *where* $0 = (0,0)$ *is the origin of the Cartesian coordinates and* $P = (v_x, v_y)$.

Vectors are used to describe motion of particles.



The position $\boldsymbol{r}(t)$, velocity $\boldsymbol{v}(t)$, and acceleration $\boldsymbol{a}(t)$ at the time t of a moving particle are described by vectors in space.

Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- **Magnitude of a vector and unit vectors.**
- Addition and scalar multiplication.

The length of a vector is the distance between its initial and terminal points

Definition 5 *The* **magnitude** *or* **length** *of a vector* $\overrightarrow{P_1P_2}$ *is the distance from the initial point to the terminal point.*

- If the vector $\overrightarrow{P_1P_2}$ has components

  $$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,$$

  then its magnitude, denoted as $\left|\overrightarrow{P_1P_2}\right|$, is given by

  $$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- If the vector $\boldsymbol{v}$ has components $\boldsymbol{v} = \langle v_x, v_y, v_z \rangle$, then its magnitude, denoted as $|\boldsymbol{v}|$ is given by

  $$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Example: Find the length of a vector with initial point $P_1 = (1, 2, 3)$ and terminal point $P_2 = (4, 3, 2)$.

Solution: First find the component of the vector $\overrightarrow{P_1P_2}$, that is,

$$\overrightarrow{P_1P_2} = \langle (4-1), (3-2), (2-3) \rangle \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle 3, 1, -1 \rangle.$$

Therefore, its length is

$$|\overrightarrow{P_1P_2}| = \sqrt{3^2 + 1^2 + (-1)^2} \quad \Rightarrow \quad \boxed{|\overrightarrow{P_1P_2}| = \sqrt{11}}.$$

$\triangleleft$

Example: If the vector $\boldsymbol{v}$ represents the velocity of a moving particle, then its length $|\boldsymbol{v}|$ represents the speed of the particle. $\triangleleft$

Unit vectors have length one.

Definition 6 *A vector* $\boldsymbol{v}$ *is called a* **unit** *vector iff* $|\boldsymbol{v}| = 1$.

Example: Show that $\boldsymbol{v} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$ is a unit vector.

Solution:

$$\begin{aligned} |\boldsymbol{v}| &= \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} \\ &= \sqrt{\frac{14}{14}} \quad \Rightarrow \quad \boxed{|\boldsymbol{v}| = 1}. \end{aligned}$$

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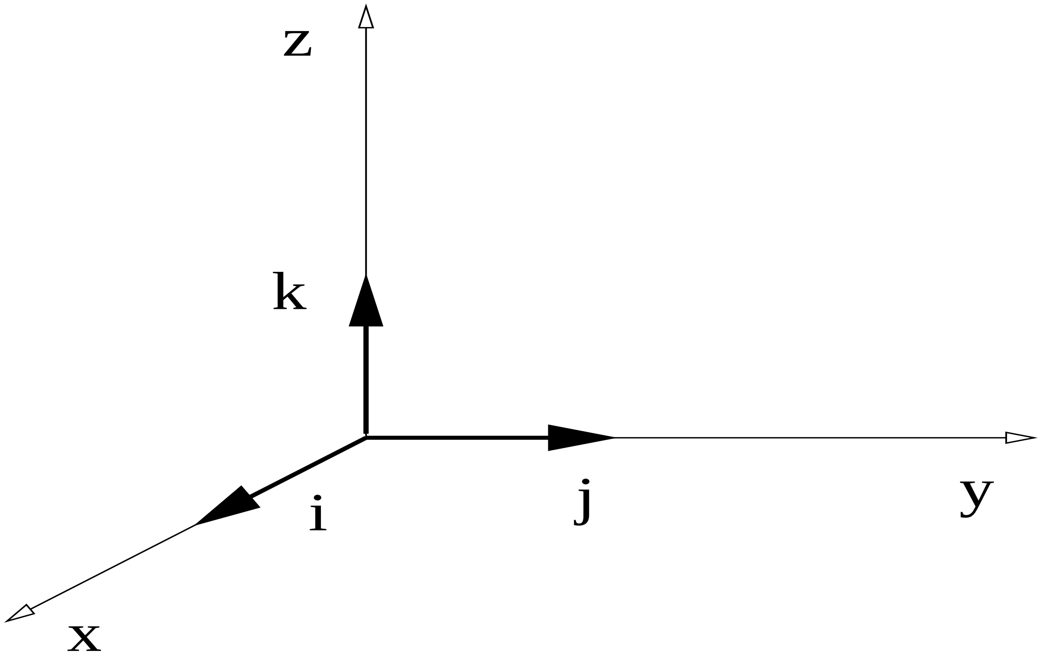
The following vectors are examples of unit vectors.

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle, \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle.$$



These vectors will be very useful to write any other vector.

Vectors on a plane and in space (12.2)

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- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- **Addition and scalar multiplication.**

Vectors can be added and multiplied by scalars.

Definition 7 *Given the vectors* $\boldsymbol{v} = \langle v_x, v_y, v_z \rangle$, $\boldsymbol{w} = \langle w_x, w_y, w_z \rangle$ *in* $\mathbb{R}^3$, *and a number* $a \in \mathbb{R}$, *then the the* **addition** *of* $(\boldsymbol{v} + \boldsymbol{w})$ *and the* **scalar multiplication** $(a\boldsymbol{v})$ *are given by*

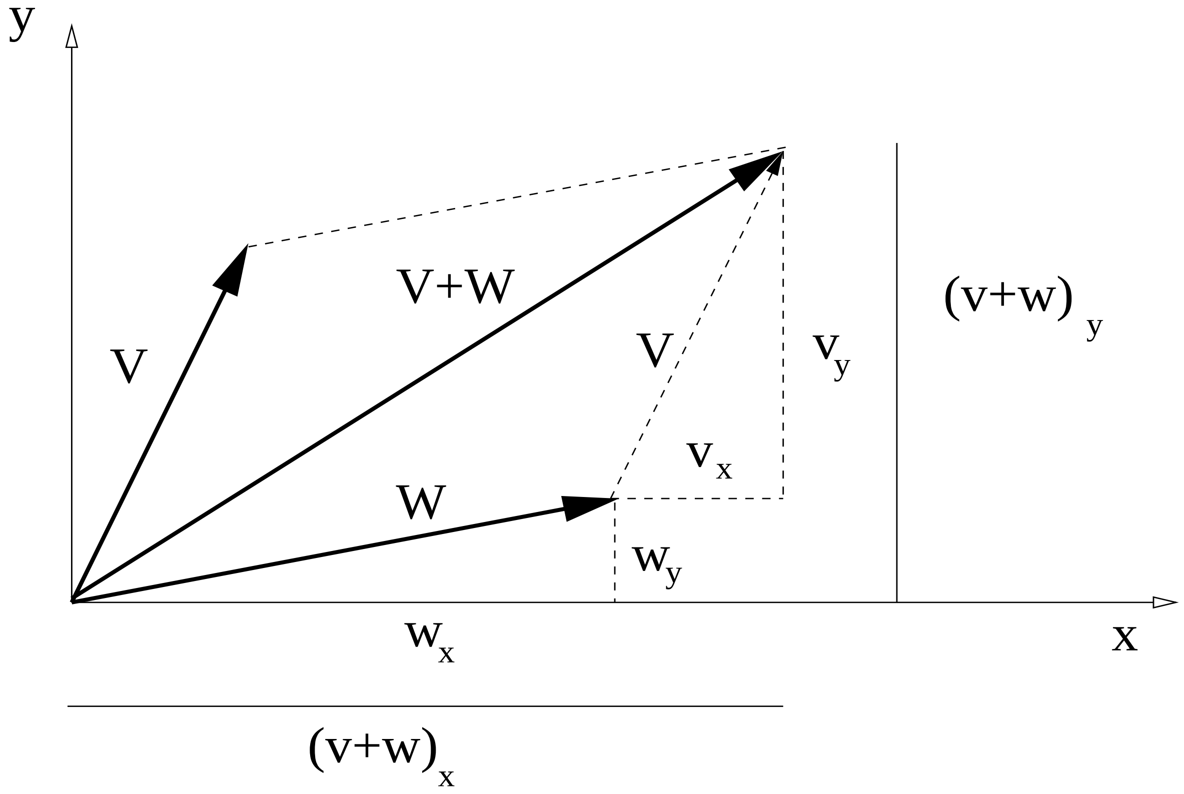
$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle, \\ a\mathbf{v} &= \langle av_x, av_y, av_z \rangle. \end{aligned}$$

The vector $-\boldsymbol{v} = (-1)\boldsymbol{v}$ *is called the* **opposite** *of vector* $\boldsymbol{v}$*.*

Remark: The difference of two vectors is the addition of one vector and the opposite of the other vector, that is, $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-1)\boldsymbol{w}$. In components we obtain the formula

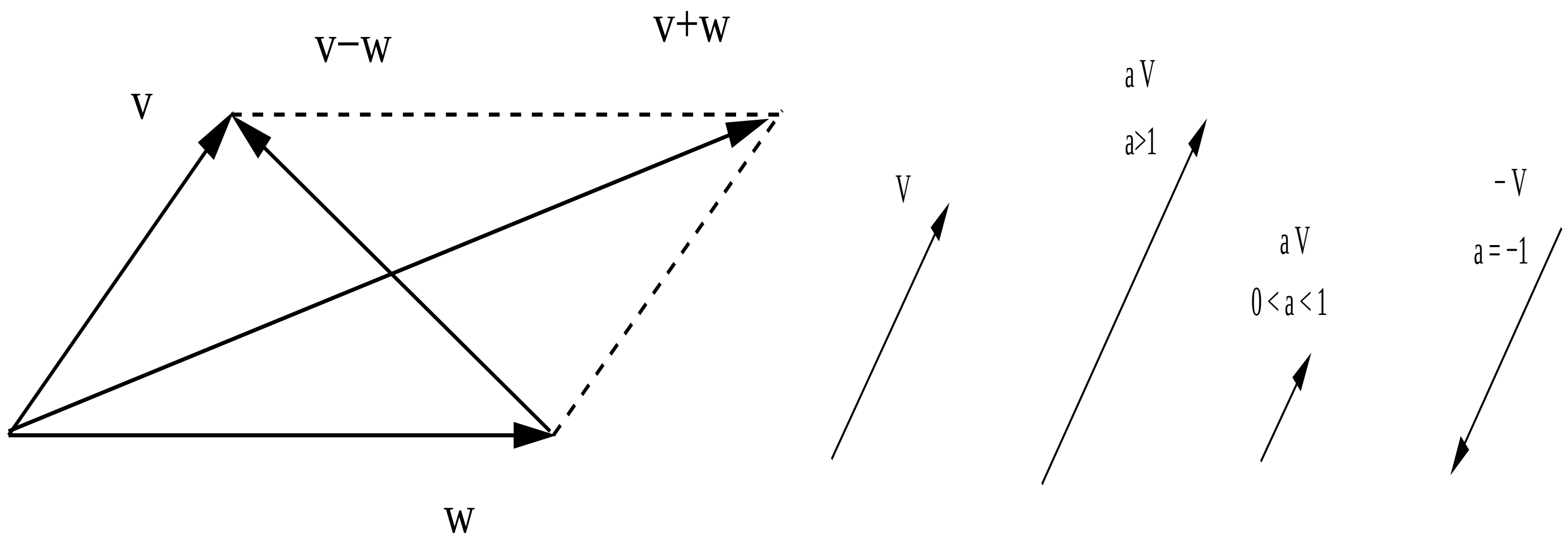
$$\boldsymbol{v} - \boldsymbol{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle.$$

The addition of two vectors is equivalent to the parallelogram law



The vector $(\boldsymbol{v}+\boldsymbol{w})$ is the diagonal of the parallelogram formed by vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ when they are in their standard position.

The addition and difference of two vectors.



The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$,

Example: Given the vectors $\boldsymbol{v} = \langle 2, 3\rangle$ and $\boldsymbol{w} = \langle -1, 2\rangle$, find the magnitude of the vectors $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$.

SOLUTION: We first compute the components of $\boldsymbol{v} + \boldsymbol{w}$, that is,

$$\boldsymbol{v} + \boldsymbol{w} = \langle (2-1), (3+2)\rangle \quad \Rightarrow \quad \boldsymbol{v} + \boldsymbol{w} = \langle 1, 5\rangle.$$

Therefore, its magnitude is

$$|\boldsymbol{v} + \boldsymbol{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad \boxed{|\boldsymbol{v} + \boldsymbol{w}| = \sqrt{26}}\,.$$

A similar calculation can be done for $\boldsymbol{v} - \boldsymbol{w}$, that is,

$$\boldsymbol{v} - \boldsymbol{w} = \langle (2+1), (3-2)\rangle \quad \Rightarrow \quad \boldsymbol{v} - \boldsymbol{w} = \langle 3, 1\rangle.$$

Therefore, its magnitude is

$$|\boldsymbol{v} - \boldsymbol{w}| = \sqrt{3^2 + 1^2} \quad \Rightarrow \quad \boxed{|\boldsymbol{v} - \boldsymbol{w}| = \sqrt{10}}\,.$$

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Unit vectors along any direction are simple to obtain.

Theorem 4 *If the vector* $\boldsymbol{v} \neq 0$, *then the vector* $\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$ *is a unit vector along the direction given by* $\boldsymbol{v}$.

Proof: We show the case where $\boldsymbol{v} \in \mathbb{R}^2$. If we write $\boldsymbol{v} = \langle v_x, v_y \rangle$, then $|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2}$. Therefore, we obtain

$$\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{|\boldsymbol{v}|}\langle v_x, v_y \rangle = \left\langle \frac{v_x}{|\boldsymbol{v}|}, \frac{v_y}{|\boldsymbol{v}|} \right\rangle.$$

This is a unit vector, since

$$\left|\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right| = \sqrt{\left[\frac{v_x}{|\boldsymbol{v}|}\right]^2 + \left[\frac{v_y}{|\boldsymbol{v}|}\right]^2} = \frac{1}{|\boldsymbol{v}|}\sqrt{v_x^2 + v_y^2} = \frac{|\boldsymbol{v}|}{|\boldsymbol{v}|} \quad \Rightarrow \quad \boxed{\left|\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right| = 1}.$$

$\square$

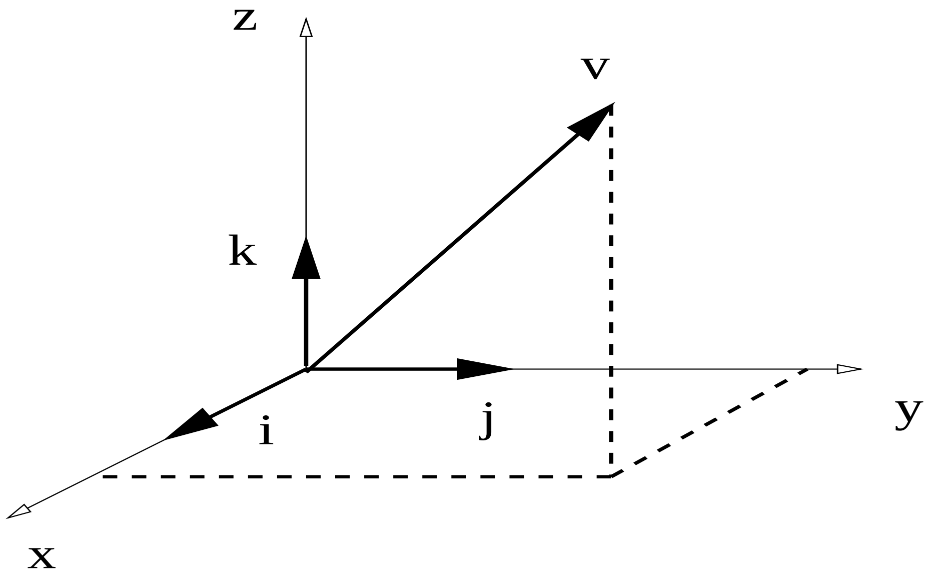
Just divide the vector by its own length.

The vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are useful to express any other vector

Recall: $\boldsymbol{i} = \langle 1, 0, 0\rangle, \quad \boldsymbol{j} = \langle 0, 1, 0\rangle, \quad \boldsymbol{k} = \langle 0, 0, 1\rangle.$

Theorem 5 *Every vector* $\boldsymbol{v} = \langle v_x, v_y, v_z\rangle$ *in* $\mathbb{R}^3$ *can be expressed in a unique way as a linear combination of vectors* $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ *as follows*

$$\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}.$$

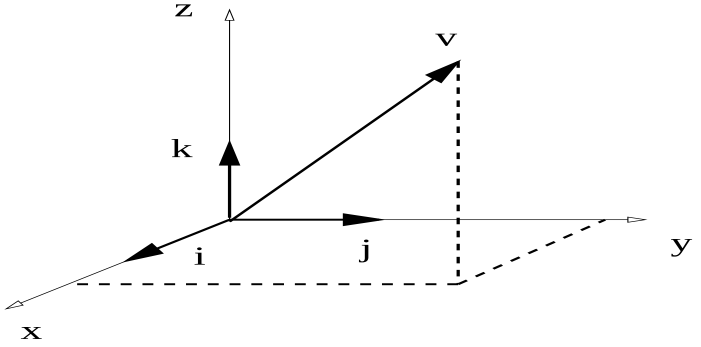


The vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are useful to express any other vector

Proof: Use the definitions of vector addition and scalar multiplication as follows,

$$\begin{aligned}
\boldsymbol{v} &= \langle v_x, v_y, v_z \rangle \\
&= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\
&= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\
&= v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}.
\end{aligned}$$

□



Example: Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$.

SOLUTION: We first compute the components of the vector $\boldsymbol{v} = \overrightarrow{P_1 P_2}$, that is,

$$\boldsymbol{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.$$

We therefore conclude that $\boxed{\boldsymbol{v} = -2\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}}$. $\triangleleft$

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Example: Find a unit vector $\boldsymbol{w}$ opposite to $\boldsymbol{v}$ found above.

SOLUTION: Since the length of $\boldsymbol{v}$ is given by

$$|\boldsymbol{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24},$$

we conclude that the solution vector $\boldsymbol{w}$ is given by

$$\boxed{\boldsymbol{w} = -\frac{1}{\sqrt{24}} \langle -2, 4, 2 \rangle}.$$

◁