Cartesian coordinates in space (Sect. 12.1).

- Overview of Multivariable Calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
- Equation of a sphere.
Overview of Multivariable Calculus

Mth 132, Calculus I: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.
Mth 133, Calculus II: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.
Overview of Multivariable Calculus

Mth 132, Calculus I: \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) \), differential calculus.
Mth 133, Calculus II: \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) \), integral calculus.
Mth 234, Multivariable Calculus:

\[
\begin{align*}
    f : \mathbb{R}^2 &\rightarrow \mathbb{R}, \quad f(x, y) \\
    f : \mathbb{R}^3 &\rightarrow \mathbb{R}, \quad f(x, y, z)
\end{align*}
\] scalar-valued.

\[
\begin{align*}
    r : \mathbb{R} &\rightarrow \mathbb{R}^3, \quad r(t) = \langle x(t), y(t), z(t) \rangle
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\] vector-valued.
Overview of Multivariable Calculus

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$$

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  r &: \mathbb{R} \to \mathbb{R}^3, \\
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  \quad &\quad \quad \text{vector-valued.}
\end{align*}
$$

We study how to differentiate and integrate such functions.
The functions of Multivariable Calculus

Example

▶ An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature $T$ of a plane surface, say a table. Each point $(x, y)$ on the table is associated with a number, its temperature $T(x, y)$.
The functions of Multivariable Calculus

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▶ An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature $T$ of an object, say a room. Each point $(x, y, z)$ in the room is associated with a number, its temperature $T(x, y, z)$. 
The functions of Multivariable Calculus

Example

▶ An example of a scalar-valued function of two variables, \( T : \mathbb{R}^2 \rightarrow \mathbb{R} \) is the temperature \( T \) of a plane surface, say a table. Each point \((x, y)\) on the table is associated with a number, its temperature \( T(x, y) \).

▶ An example of a scalar-valued function of three variables, \( T : \mathbb{R}^3 \rightarrow \mathbb{R} \) is the temperature \( T \) of an object, say a room. Each point \((x, y, z)\) in the room is associated with a number, its temperature \( T(x, y, z) \).

▶ An example of a vector-valued function of one variable, \( r : \mathbb{R} \rightarrow \mathbb{R}^3 \), is the position function in time of a particle moving in space, say a fly in a room. Each time \( t \) is associated with the position vector \( r(t) \) of the fly in the room.
Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
- **Cartesian coordinates in space.**
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
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Cartesian coordinates.

**Cartesian coordinates on $\mathbb{R}^2$:** Every point on a plane is labeled by an ordered pair $(x, y)$ by the rule given in the figure.
Cartesian coordinates.

**Cartesian coordinates on \( \mathbb{R}^2 \):** Every point on a plane is labeled by an ordered pair \((x, y)\) by the rule given in the figure.

**Cartesian coordinates in \( \mathbb{R}^3 \):** Every point in space is labeled by an ordered triple \((x, y, z)\) by the rule given in the figure.
Cartesian coordinates.

Example

Sketch the set \( S = \{ x \geq 0, \ y \geq 0, \ z = 0 \} \subset \mathbb{R}^3 \).
Cartesian coordinates.

Example
Sketch the set \( S = \{ x \geq 0, \ y \geq 0, \ z = 0 \} \subset \mathbb{R}^3 \).

Solution:
Cartesian coordinates.

Example

Sketch the set $S = \{0 \leq x \leq 1, \ -1 \leq y \leq 2, \ z = 1\} \subset \mathbb{R}^3$. 
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Sketch the set $S = \{0 \leq x \leq 1, \ -1 \leq y \leq 2, \ z = 1\} \subset \mathbb{R}^3$.
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Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
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Right and left handed Cartesian coordinates.

Definition
A Cartesian coordinate system is called *right-handed* (rh) iff it can be rotated into the coordinate system in the figure.

No rotation transforms a rh into a lh system.
Right and left handed Cartesian coordinates.

**Definition**
A Cartesian coordinate system is called *right-handed* (rh) iff it can be rotated into the coordinate system in the figure.

**Definition**
A Cartesian coordinate system is called *left-handed* (lh) iff it can be rotated into the coordinate system in the figure.

No rotation transforms a rh into a lh system.
Right and left handed Cartesian coordinates.

Example
This coordinate system is right-handed.
Right and left handed Cartesian coordinates.

Example
This coordinate system is right-handed.

Example
This coordinate system is left handed.
Right and left handed Cartesian coordinates

Remark: The same classification occurs in $\mathbb{R}^2$:

In $\mathbb{R}^3$ we will define the cross product of vectors, and this product has different results in rh or lh Cartesian coordinates.

There is no cross product in $\mathbb{R}^2$.

In class we use rh Cartesian coordinates.
Right and left handed Cartesian coordinates

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Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- **Distance formula between two points in space.**
- Equation of a sphere.
Distance formula between two points in space.

Theorem
The distance $|P_1P_2|$ between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$  

The distance between points in space is crucial to define the idea of limit to functions in space.
Proof.

Pythagoras Theorem.

\[ |P_1P_2|^2 = a^2 + (z_2 - z_1)^2, \quad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \]
Distance formula between two points in space

Example
Find the distance between \( P_1 = (1, 2, 3) \) and \( P_2 = (3, 2, 1) \).
Distance formula between two points in space

Example
Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Solution:

$$|P_1P_2| = \sqrt{(3 - 1)^2 + (2 - 2)^2 + (1 - 3)^2}$$
$$= \sqrt{4 + 4}$$
$$= \sqrt{8} \quad \Rightarrow \quad |P_1P_2| = 2\sqrt{2}.$$
Distance formula between two points in space

Example

Use the distance formula to determine whether three points in space are collinear.
Distance formula between two points in space

Example

Use the distance formula to determine whether three points in space are collinear.

Solution:

\[ d_{21} + d_{32} > d_{31} \]

Not collinear,

\[ d_{21} + d_{32} = d_{31} \]

Collinear.
Cartesian coordinates in space (12.1)

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- Distance formula between two points in space.
- **Equation of a sphere.**
A sphere is a set of points at fixed distance from a center.

**Definition**

A *sphere* centered at \( P_0 = (x_0, y_0, z_0) \) of radius \( R \) is the set

\[
S = \{ P = (x, y, z) : |P_0P| = R \}.
\]
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S = \{ P = (x, y, z) : |P_0 P| = R \}.
\]

**Remark:** The point \((x, y, z)\) belongs to the sphere \( S \) iff holds

\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.
\]

("iff" means "if and only iff.")
An open ball is a set of points contained in a sphere.

**Definition**

An *open ball* centered at \( P_0 = (x_0, y_0, z_0) \) of radius \( R \) is the set

\[
B = \{ P = (x, y, z) : |P_0P| < R \}.
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An open ball is a set of points contained in a sphere.

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**Remark:** The point \((x, y, z)\) belongs to the open ball \(B\) iff holds

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(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < R^2.
\]
Example
Plot a sphere centered at $P_0 = (0, 0, 0)$ of radius $R > 0$. 
Example
Plot a sphere centered at \( P_0 = (0, 0, 0) \) of radius \( R > 0 \).

Solution:
Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$. 

Solution:

Complete the square.

$0 = x^2 + y^2 + 4y + z^2 = x^2 + (y^2 + 2(4))^2 + z^2 = x^2 + (y + 2)^2 + z^2 = 4^2$.

$x^2 + y^2 + 4y + z^2 = 0$ $\iff$ $x^2 + (y + 2)^2 + z^2 = 4^2$. 

Example
Graph the sphere \( x^2 + y^2 + z^2 + 4y = 0 \).
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Solution: Complete the square.

\[
0 = x^2 + y^2 + 4y + z^2 \\
= x^2 + \left[ y^2 + 2 \left( \frac{4}{2} \right) y + \left( \frac{4}{2} \right)^2 \right] - \left( \frac{4}{2} \right)^2 + z^2 \\
= x^2 + \left( y + \frac{4}{2} \right)^2 + z^2 - 4.
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\]

\[
x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y + 2)^2 + z^2 = 2^2.
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Example
Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

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Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Since

$$x^2 + y^2 + 4y + z^2 = 0 \iff x^2 + (y + 2)^2 + z^2 = 2^2,$$

we conclude that $P_0 = (0, -2, 0)$ and $R = 2$, therefore,
Exercise

► Given constants $a$, $b$, $c$, and $d \in \mathbb{R}$, show that

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

is the equation of a sphere iff holds

$$d > -(a^2 + b^2 + c^2). \quad (1)$$

► Furthermore, show that if Eq. (1) is satisfied, then the expressions for the center $P_0$ and the radius $R$ of the sphere are given by

$$P_0 = (a, b, c), \quad R = \sqrt{d + (a^2 + b^2 + c^2)}.$$
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition

A vector in $\mathbb{R}^n$, with $n = 2, 3$, is an ordered pair of points in $\mathbb{R}^n$, denoted as $\overrightarrow{P_1P_2}$, where $P_1, P_2 \in \mathbb{R}^n$. The point $P_1$ is called the initial point and $P_2$ is called the terminal point.
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Remarks:
- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.
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**Remarks:**

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.
- A vector is drawn by an arrow pointing to the terminal point.
- A vector is denoted not only by $\overrightarrow{P_1P_2}$ but also by an arrow over a letter, like $\vec{v}$, or by a boldface letter, like $\mathbf{v}$. 
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Remark:** The order of the points determines the direction. For example, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Remark: The order of the points determines the direction. For example, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.

Remark: By 1850 it was realized that different physical phenomena were described using a new concept at that time, called a vector. A vector was more than a number in the sense that it was needed more than a single number to specify it. Phenomena described using vectors included velocities, accelerations, forces, rotations, electric phenomena, magnetic phenomena, and heat transfer.
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Vector components in Cartesian coordinates.**
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.
Components of a vector in Cartesian coordinates

Theorem

Given the points \( P_1 = (x_1, y_1) \), \( P_2 = (x_2, y_2) \) \( \in \mathbb{R}^2 \), the vector \( \overrightarrow{P_1P_2} \) determines a unique ordered pair denoted as follows,

\[
\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1) \rangle.
\]
Components of a vector in Cartesian coordinates

Theorem

Given the points \( P_1 = (x_1, y_1) \), \( P_2 = (x_2, y_2) \in \mathbb{R}^2 \), the vector \( \overrightarrow{P_1P_2} \) determines a unique ordered pair denoted as follows,

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Proof: Draw the vector \( \overrightarrow{P_1P_2} \) in Cartesian coordinates. \( \square \)
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Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered pair denoted as follows,

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$ 

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates.

Remark: A similar result holds for vectors in space.
Components of a vector in Cartesian coordinates

**Theorem**

Given the points $P_1 = (x_1, y_1, z_1), P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered triple denoted as follows,

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.$$
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered triple denoted as follows,

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Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □
Components of a vector in Cartesian coordinates

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Given the points \( P_1 = (x_1, y_1, z_1) \), \( P_2 = (x_2, y_2, z_2) \) ∈ \( \mathbb{R}^3 \), the vector \( \overrightarrow{P_1P_2} \) determines a unique ordered triple denoted as follows,

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\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.
\]

Proof: Draw the vector \( \overrightarrow{P_1P_2} \) in Cartesian coordinates.

\[ \square \]
Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point \( P_1 = (1, -2, 3) \) and terminal point \( P_2 = (3, 1, 2) \).
Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.
Solution:

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\overrightarrow{P_1P_2} = \langle (3 - 1), (1 - (-2)), (2 - 3) \rangle
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Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.
Solution:
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\overrightarrow{P_1P_2} = \langle (3 - 1), (1 - (-2)), (2 - 3) \rangle \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle 2, 3, -1 \rangle.
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Remark: $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ have the same components although they are different vectors.
Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point \( P_1 = (1, -2, 3) \) and terminal point \( P_2 = (3, 1, 2) \).

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Example
Find the components of a vector with initial point \( P_3 = (3, 1, 4) \) and terminal point \( P_4 = (5, 4, 3) \).
Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

Solution:

$$\vec{P_1P_2} = \langle (3 - 1), (1 - (-2)), (2 - 3) \rangle \Rightarrow \vec{P_1P_2} = \langle 2, 3, -1 \rangle.$$

Example
Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

Solution:

$$\vec{P_3P_4} = \langle (5 - 3), (4 - 1), (3 - 4) \rangle$$
Components of a vector in Cartesian coordinates

Example
Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.
Solution:

$$
\vec{P_1P_2} = \langle (3 - 1), (1 - (-2)), (2 - 3) \rangle \Rightarrow \vec{P_1P_2} = \langle 2, 3, -1 \rangle.
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Remark:
$\vec{P_1P_2}$ and $\vec{P_3P_4}$ have the same components although they are different vectors.
Components of a vector in Cartesian coordinates

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Remark: \( \overrightarrow{P_1P_2} \) and \( \overrightarrow{P_3P_4} \) have the same components although they are different vectors.
Components of a vector in Cartesian coordinates

Remark:
The vector components do not determine a unique vector.
Components of a vector in Cartesian coordinates

**Remark:**
The vector components do not determine a unique vector.

The vectors \( \mathbf{u}, \mathbf{v} \) and \( \vec{0P} \) have the same components but they are all different, since they have different initial and terminal points.
Components of a vector in Cartesian coordinates

**Remark:**
The vector components do not determine a unique vector.

The vectors $\mathbf{u}$, $\mathbf{v}$ and $\overrightarrow{OP}$ have the same components but they are all different, since they have different initial and terminal points.

**Definition**
Given a vector $\overrightarrow{P_1P_2} = \langle v_x, v_y \rangle$, the *standard position* vector is the vector $\overrightarrow{OP}$, where the point $0 = (0, 0)$ is the origin of the Cartesian coordinates and the point $P = (v_x, v_y)$. 
Remark: Vectors are used to describe motion of particles.

The position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ at the time $t$ of a moving particle are described by vectors in space.
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- **Magnitude of a vector and unit vectors.**
- Addition and scalar multiplication.
Magnitude of a vector and unit vectors.

Definition
The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.
Magnitude of a vector and unit vectors.

Definition
The *magnitude* or *length* of a vector \( \overrightarrow{P_1P_2} \) is the distance from the initial point to the terminal point.

- If the vector \( \overrightarrow{P_1P_2} \) has components
  \[
  \overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,
  \]
  then its magnitude, denoted as \( |\overrightarrow{P_1P_2}| \), is given by
  \[
  |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
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Magnitude of a vector and unit vectors.

Definition
The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.

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$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,$$

then its magnitude, denoted as $|\overrightarrow{P_1P_2}|$, is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
Magnitude of a vector and unit vectors.

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Magnitude of a vector and unit vectors.

**Example**
Find the length of a vector with initial point $P_1 = (1, 2, 3)$ and terminal point $P_2 = (4, 3, 2)$. 

Solution:
First find the component of the vector $\overrightarrow{P_1P_2}$, that is, $\overrightarrow{P_1P_2} = \langle 4 - 1, 3 - 2, 2 - 3 \rangle$. 

$\overrightarrow{P_1P_2} = \langle 3, 1, -1 \rangle$. 

Therefore, its length is $|\overrightarrow{P_1P_2}| = \sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{11}$. 

Example
If the vector $v$ represents the velocity of a moving particle, then its length $|v|$ represents the speed of the particle.
Magnitude of a vector and unit vectors.

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If the vector $\mathbf{v}$ represents the velocity of a moving particle, then its length $|\mathbf{v}|$ represents the speed of the particle.
Magnitude of a vector and unit vectors.

Definition
A vector $\mathbf{v}$ is a unit vector iff $\mathbf{v}$ has length one, that is, $|\mathbf{v}| = 1$. 

Example
Show that $\mathbf{v} = \langle 1, \sqrt{14}, 2, \sqrt{14}, 3, \sqrt{14} \rangle$ is a unit vector.

Solution:
$|\mathbf{v}| = \sqrt{1^2 + 14 + 4 + 9 + 9 + 14} = \sqrt{14} \Rightarrow |\mathbf{v}| = 1$.

Example
The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are useful to express any other vector in $\mathbb{R}^3$. 
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A vector \( \mathbf{v} \) is a \textit{unit vector} iff \( \mathbf{v} \) has length one, that is, \( |\mathbf{v}| = 1 \).

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Show that \( \mathbf{v} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle \) is a unit vector.

Solution:
\[
|\mathbf{v}| = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} = 1.
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Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- **Addition and scalar multiplication.**
Addition and scalar multiplication.

Definition
Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the vector addition, $\mathbf{v} + \mathbf{w}$, and the scalar multiplication, $a\mathbf{v}$, are given by

$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
$$a\mathbf{v} = \langle av_x, av_y, av_z \rangle.$$
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Remarks:
- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the opposite of vector $\mathbf{v}$. 
Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the vector addition, $\mathbf{v} + \mathbf{w}$, and the scalar multiplication, $a\mathbf{v}$, are given by

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- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the opposite of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector,
Addition and scalar multiplication.

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Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the vector addition, $\mathbf{v} + \mathbf{w}$, and the scalar multiplication, $a \mathbf{v}$, are given by

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Remarks:
- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the opposite of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector, that is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$. This equation in components is

$$\mathbf{v} - \mathbf{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle.$$
Addition and scalar multiplication.

Remark: The addition of two vectors is equivalent to the parallelogram law: The vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$ when they are in their standard position.
Addition and scalar multiplication.

**Remark:** The addition and difference of two vectors.

![Diagram of vector addition and subtraction](image)
Addition and scalar multiplication.

Remark: The addition and difference of two vectors.

Remark: The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$. 
Addition and scalar multiplication.

Example
Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.
Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$,
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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

$$\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle$$
Addition and scalar multiplication.

Example
Given the vectors \( \mathbf{v} = \langle 2, 3 \rangle \) and \( \mathbf{w} = \langle -1, 2 \rangle \), find the magnitude of the vectors \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \).

Solution: We first compute the components of \( \mathbf{v} + \mathbf{w} \), that is,

\[
\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.
\]
Addition and scalar multiplication.

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Given the vectors \( \mathbf{v} = \langle 2, 3 \rangle \) and \( \mathbf{w} = \langle -1, 2 \rangle \), find the magnitude of the vectors \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \).

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\]

Therefore, its magnitude is

\[
|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2}
\]
Addition and scalar multiplication.

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\]

Therefore, its magnitude is

\[
|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.
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Addition and scalar multiplication.

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Given the vectors \( \mathbf{v} = \langle 2, 3 \rangle \) and \( \mathbf{w} = \langle -1, 2 \rangle \), find the magnitude of the vectors \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \).

Solution: We first compute the components of \( \mathbf{v} + \mathbf{w} \), that is,

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\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.
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\]

A similar calculation can be done for \( \mathbf{v} - \mathbf{w} \),
Addition and scalar multiplication.

Example
Given the vectors $v = \langle 2, 3 \rangle$ and $w = \langle -1, 2 \rangle$, find the magnitude of the vectors $v + w$ and $v - w$.

Solution: We first compute the components of $v + w$, that is,

$$v + w = \langle (2 - 1), (3 + 2) \rangle \Rightarrow v + w = \langle 1, 5 \rangle.$$ 

Therefore, its magnitude is

$$|v + w| = \sqrt{1^2 + 5^2} \Rightarrow |v + w| = \sqrt{26}.$$ 

A similar calculation can be done for $v - w$, that is,

$$v - w = \langle (2 + 1), (3 - 2) \rangle.$$
Addition and scalar multiplication.

**Example**

Given the vectors \( \mathbf{v} = \langle 2, 3 \rangle \) and \( \mathbf{w} = \langle -1, 2 \rangle \), find the magnitude of the vectors \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \).

**Solution:** We first compute the components of \( \mathbf{v} + \mathbf{w} \), that is,

\[
\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.
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Therefore, its magnitude is

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|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.
\]

A similar calculation can be done for \( \mathbf{v} - \mathbf{w} \), that is,

\[
\mathbf{v} - \mathbf{w} = \langle (2 + 1), (3 - 2) \rangle \quad \Rightarrow \quad \mathbf{v} - \mathbf{w} = \langle 3, 1 \rangle.
\]
Addition and scalar multiplication.

Example
Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

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A similar calculation can be done for $\mathbf{v} - \mathbf{w}$, that is,

$$\mathbf{v} - \mathbf{w} = \langle (2 + 1), (3 - 2) \rangle \Rightarrow \mathbf{v} - \mathbf{w} = \langle 3, 1 \rangle.$$ 

Therefore, its magnitude is

$$|\mathbf{v} - \mathbf{w}| = \sqrt{3^2 + 1^2}.$$
Addition and scalar multiplication.

Example

Given the vectors \( \mathbf{v} = \langle 2, 3 \rangle \) and \( \mathbf{w} = \langle -1, 2 \rangle \), find the magnitude of the vectors \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \).

Solution: We first compute the components of \( \mathbf{v} + \mathbf{w} \), that is,

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\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.
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|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.
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A similar calculation can be done for \( \mathbf{v} - \mathbf{w} \), that is,

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\mathbf{v} - \mathbf{w} = \langle (2 + 1), (3 - 2) \rangle \quad \Rightarrow \quad \mathbf{v} - \mathbf{w} = \langle 3, 1 \rangle.
\]

Therefore, its magnitude is

\[
|\mathbf{v} - \mathbf{w}| = \sqrt{3^2 + 1^2} \quad \Rightarrow \quad |\mathbf{v} - \mathbf{w}| = \sqrt{10}.
\]
Addition and scalar multiplication.

**Theorem**

*If the vector $v \neq 0$, then the vector $u = \frac{v}{|v|}$ is a unit vector.*
Addition and scalar multiplication.

**Theorem**

*If the vector \( \mathbf{v} \neq \mathbf{0} \), then the vector \( \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \) is a unit vector.*

**Proof:** (Case \( \mathbf{v} \in \mathbb{R}^2 \) only).

\[
|\mathbf{u}| = \left| \frac{\mathbf{v}}{|\mathbf{v}|} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1
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Addition and scalar multiplication.

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If \( \mathbf{v} = \langle v_x, v_y \rangle \in \mathbb{R}^2 \), then \( |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \).
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If \( \mathbf{v} = \langle v_x, v_y \rangle \in \mathbb{R}^2 \), then \( |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \), and

\[
\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{v_x}{|\mathbf{v}|}, \frac{v_y}{|\mathbf{v}|} \right\rangle.
\]

This is a unit vector, since

\[
|\mathbf{u}| = \left| \frac{\mathbf{v}}{|\mathbf{v}|} \right| = \sqrt{\left( \frac{v_x}{|\mathbf{v}|} \right)^2 + \left( \frac{v_y}{|\mathbf{v}|} \right)^2} = \frac{1}{|\mathbf{v}|} \sqrt{v_x^2 + v_y^2}
\]
Addition and scalar multiplication.

**Theorem**
If the vector $v \neq 0$, then the vector $u = \frac{v}{|v|}$ is a unit vector.

**Proof:** (Case $v \in \mathbb{R}^2$ only).
If $v = \langle v_x, v_y \rangle \in \mathbb{R}^2$, then $|v| = \sqrt{v_x^2 + v_y^2}$, and

$$u = \frac{v}{|v|} = \langle \frac{v_x}{|v|}, \frac{v_y}{|v|} \rangle.$$ 

This is a unit vector, since

$$|u| = \left| \frac{v}{|v|} \right| = \sqrt{\left( \frac{v_x}{|v|} \right)^2 + \left( \frac{v_y}{|v|} \right)^2} = \frac{1}{|v|} \sqrt{v_x^2 + v_y^2} = \frac{|v|}{|v|}.$$
Addition and scalar multiplication.

**Theorem**

*If the vector \( \mathbf{v} \neq \mathbf{0} \), then the vector \( \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \) is a unit vector.*

**Proof:** (Case \( \mathbf{v} \in \mathbb{R}^2 \) only).

If \( \mathbf{v} = \langle v_x, v_y \rangle \in \mathbb{R}^2 \), then

\[
|\mathbf{v}| = \sqrt{v_x^2 + v_y^2},
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\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{v_x}{|\mathbf{v}|}, \frac{v_y}{|\mathbf{v}|} \right\rangle.
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\]
Addition and scalar multiplication.

**Theorem**

*Every vector* \( \mathbf{v} = \langle v_x, v_y, v_z \rangle \) *in* \( \mathbb{R}^3 \) *can be expressed in a unique way as a linear combination of vectors* \( \mathbf{i} = \langle 1, 0, 0 \rangle \), \( \mathbf{j} = \langle 0, 1, 0 \rangle \), and \( \mathbf{k} = \langle 0, 0, 1 \rangle \) *as follows*

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\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.
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Proof: Use the definitions of vector addition and scalar multiplication as follows,
Addition and scalar multiplication.

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\]

**Proof:** Use the definitions of vector addition and scalar multiplication as follows,

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\mathbf{v} = \langle v_x, v_y, v_z \rangle \\
= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\
= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\
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\]
Addition and scalar multiplication.

Example

Express the vector with initial and terminal points \( P_1 = (1, 0, 3), \)
\( P_2 = (-1, 4, 5) \) in the form \( \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}. \)
Addition and scalar multiplication.

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Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1P_2}$,
Addition and scalar multiplication.

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Solution: First compute the components of \( \mathbf{v} = \overrightarrow{P_1P_2}, \) that is,

\[
\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle
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Solution: First compute the components of \( \mathbf{v} = \overrightarrow{P_1P_2} \), that is,

\[
\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.
\]
Addition and scalar multiplication.

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Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. 

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Then, \( \mathbf{v} = -2 \mathbf{i} + 4 \mathbf{j} + 2 \mathbf{k} \).

Example
Find a unit vector \( \mathbf{w} \) opposite to \( \mathbf{v} \) found above.
Addition and scalar multiplication.

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Example
Find a unit vector \( \mathbf{w} \) opposite to \( \mathbf{v} \) found above.

Solution: Since \( |\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} \)
Addition and scalar multiplication.

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Example
Find a unit vector \( \mathbf{w} \) opposite to \( \mathbf{v} \) found above.

Solution: Since \( |\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} \)
Addition and scalar multiplication.

Example
Express the vector with initial and terminal points \( P_1 = (1, 0, 3) \), \( P_2 = (-1, 4, 5) \) in the form \( \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \).

Solution: First compute the components of \( \mathbf{v} = \overrightarrow{P_1P_2} \), that is,

\[
\mathbf{v} = \langle -1 - 1, 4 - 0, 5 - 3 \rangle = \langle -2, 4, 2 \rangle.
\]

Then, \( \mathbf{v} = -2 \mathbf{i} + 4 \mathbf{j} + 2 \mathbf{k}. \)

Example
Find a unit vector \( \mathbf{w} \) opposite to \( \mathbf{v} \) found above.

Solution: Since \( |\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}, \)
Addition and scalar multiplication.

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Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1P_2}$, that is,

$$\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.$$  

Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. △

Example
Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$, we conclude that $\mathbf{w} = -\frac{1}{\sqrt{24}} \langle -2, 4, 2 \rangle$. △