Cartesian coordinates in space (Sect. 12.1).

- Overview of Multivariable Calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
- Equation of a sphere.

Overview of Multivariable Calculus

Mth 132, Calculus I: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Mth 133, Calculus II: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Overview of Multivariable Calculus

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Mth 234, Multivariable Calculus:

$$\left.\begin{array}{ll} f : \mathbb{R}^2 \to \mathbb{R}, & f(x,y) \\ f : \mathbb{R}^3 \to \mathbb{R}, & f(x,y,z) \end{array}\right\} \quad \text{scalar-valued.}$$

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \quad \} \quad \text{vector-valued.}$$

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We study how to differentiate and integrate such functions.

The functions of Multivariable Calculus

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.

The functions of Multivariable Calculus

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- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of an object, say a room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.

The functions of Multivariable Calculus

Example

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- An example of a vector-valued function of one variable, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, is the position function in time of a particle moving in space, say a fly in a room. Each time t is associated with the position vector $\mathbf{r}(t)$ of the fly in the room.

$\triangleleft$

Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
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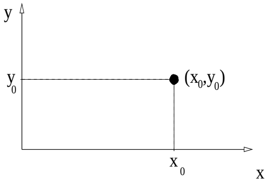
Cartesian coordinates.

Cartesian coordinates on $\mathbb{R}^2$: Every point on a plane is labeled by an ordered pair (x, y) by the rule given in the figure.



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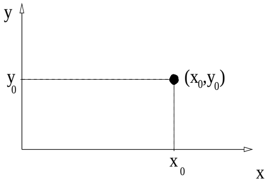
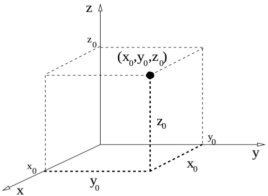


Cartesian coordinates in $\mathbb{R}^3$: Every point in space is labeled by an ordered triple (x, y, z) by the rule given in the figure.



Cartesian coordinates.

Example

Sketch the set $S = \{x \geqslant 0,\ y \geqslant 0,\ z = 0\} \subset \mathbb{R}^3$.

Cartesian coordinates.

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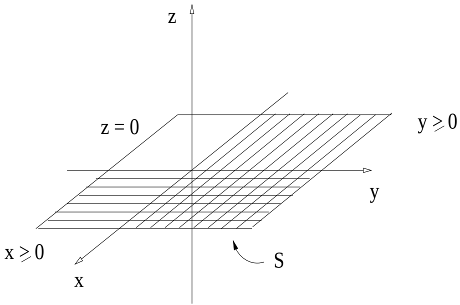
Solution:



Cartesian coordinates.

Example

Sketch the set $S = \{0 \leqslant x \leqslant 1, \; -1 \leqslant y \leqslant 2, \; z = 1\} \subset \mathbb{R}^3$.

Cartesian coordinates.

Example

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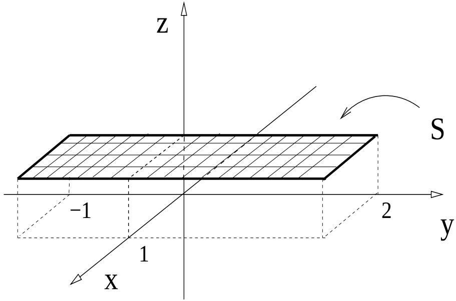

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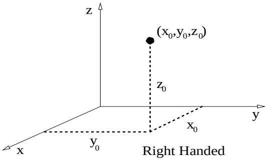
Right and left handed Cartesian coordinates.

Definition

A Cartesian coordinate system is called *right-handed* (rh) iff it can be rotated into the coordinate system in the figure.



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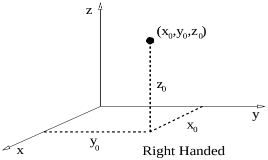
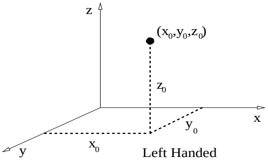


Definition

A Cartesian coordinate system is called *left-handed* (lh) iff it can be rotated into the coordinate system in the figure.

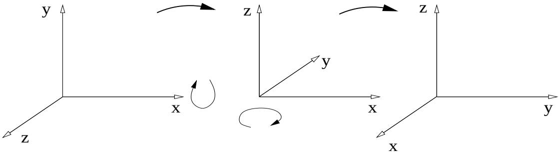


No rotation transforms a rh into a lh system.

Right and left handed Cartesian coordinates.

Example

This coordinate system is right-handed.



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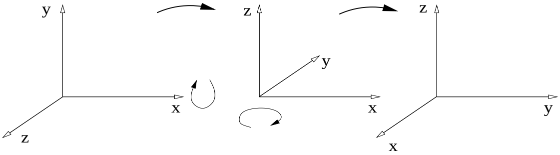
This coordinate system is right-handed.



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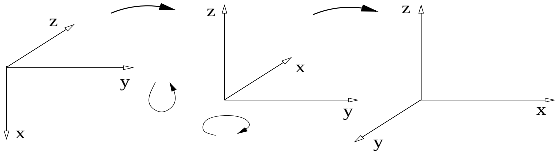
Example

This coordinate system is left handed.



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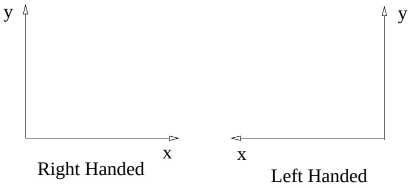
Remark: The same classification occurs in $\mathbb{R}^2$:



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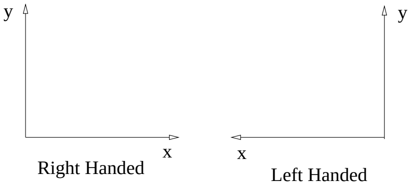

This classification is needed because:

- In $\mathbb{R}^3$ we will define the cross product of vectors, and this product has different results in rh or lh Cartesian coordinates.

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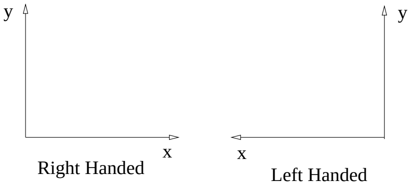

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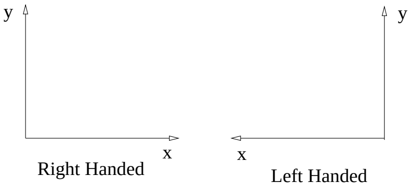

This classification is needed because:

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In class we use rh Cartesian coordinates.

Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
- Cartesian coordinates in space.
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- **Distance formula between two points in space.**
- Equation of a sphere.

Distance formula between two points in space.

Theorem

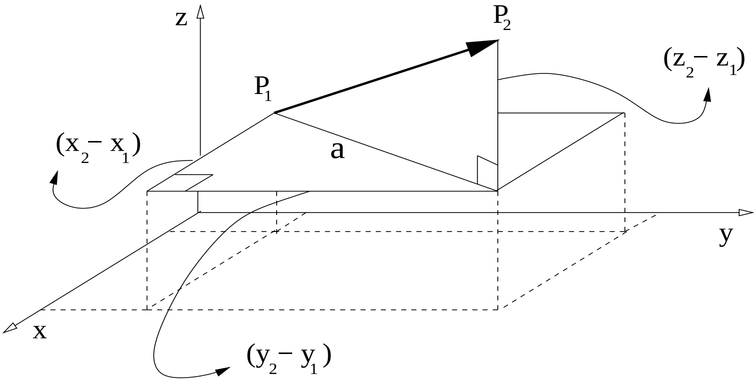
The distance $|P_1P_2|$ *between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The distance between points in space is crucial to define the idea of limit to functions in space.

Proof.

Pythagoras Theorem.



$$|P_1P_2|^2 = a^2 + (z_2 - z_1)^2, \qquad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

□

Distance formula between two points in space

Example

Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Distance formula between two points in space

Example

Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Solution:

$$
\begin{aligned}
|P_1 P_2| &= \sqrt{(3-1)^2 + (2-2)^2 + (1-3)^2} \\
&= \sqrt{4+4} \\
&= \sqrt{8} \quad \Rightarrow \quad |P_1 P_2| = 2\sqrt{2}.
\end{aligned}
$$

$\triangleleft$

Distance formula between two points in space

Example

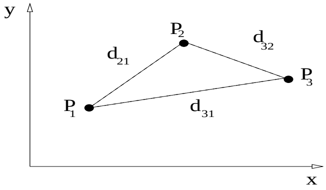
Use the distance formula to determine whether three points in space are collinear.

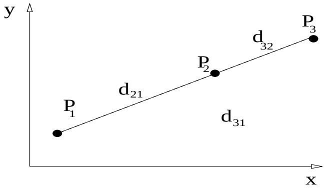
Distance formula between two points in space

Example

Use the distance formula to determine whether three points in space are collinear.

Solution:





$$d_{21} + d_{32} > d_{31}$$

Not collinear,

$$d_{21} + d_{32} = d_{31}$$

Collinear.

Cartesian coordinates in space (12.1)

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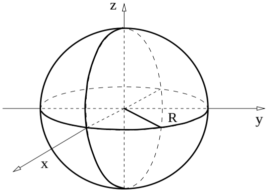
- **Equation of a sphere.**

A sphere is a set of points at fixed distance from a center.

Definition

A *sphere* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$S = \big\{ P = (x, y, z) \ : \ \big|P_0 P\big| = R \big\}.$$



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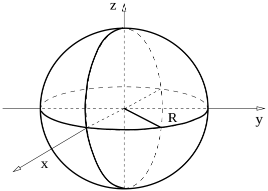

Remark: The point (x, y, z) belongs to the sphere S iff holds

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

("iff" means "if and only iff.")

An open ball is a set of points contained in a sphere.

Definition

An *open ball* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$B = \{P = (x, y, z) \; : \; |P_0 P| < R\}.$$

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Remark: The point (x, y, z) belongs to the open ball B iff holds

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < R^2.$$

Example

Plot a sphere centered at $P_0 = (0, 0, 0)$ of radius $R > 0$.

Example

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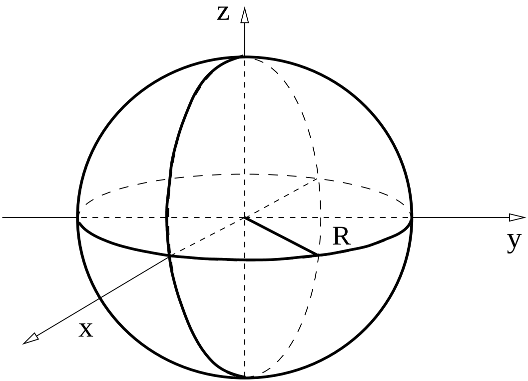
Solution:



◁

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

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Solution: Complete the square.

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$$\begin{aligned} 0 &= x^2 + y^2 + 4y + z^2 \\ &= x^2 + \left[y^2 + 2\left(\frac{4}{2}\right)y + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + z^2 \\ &= x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 - 4. \end{aligned}$$

Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

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\end{aligned}
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$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2.$$

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Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

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Example

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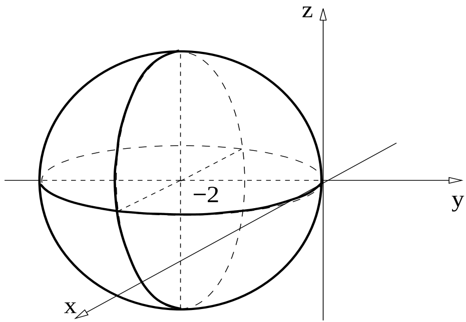
Solution: Since

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2,$$

we conclude that $P_0 = (0, -2, 0)$ and $R = 2$, therefore,



Exercise

- Given constants a, b, c, and $d \in \mathbb{R}$, show that

$$x^2 + y^2 + z^2 - 2a\,x - 2b\,y - 2c\,z = d$$

is the equation of a sphere iff holds

$$d > -(a^2 + b^2 + c^2). \tag{1}$$

- Furthermore, show that if Eq. (1) is satisfied, then the expressions for the center P_0 and the radius R of the sphere are given by

$$P_0 = (a, b, c), \qquad R = \sqrt{d + (a^2 + b^2 + c^2)}.$$

◁

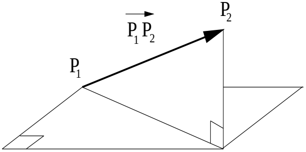
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition

A *vector* in $\mathbb{R}^n$, with $n = 2, 3$, is an ordered pair of points in $\mathbb{R}^n$, denoted as $\overrightarrow{P_1 P_2}$, where P_1, $P_2 \in \mathbb{R}^n$. The point P_1 is called the *initial point* and P_2 is called the *terminal point*.



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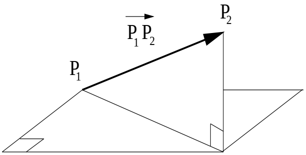

Remarks:

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.

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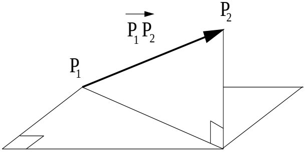

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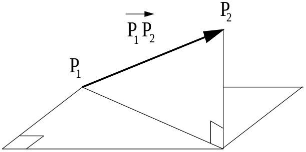
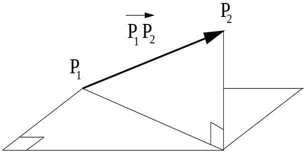


Remarks:

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.
- A vector is drawn by an arrow pointing to the terminal point.
- A vector is denoted not only by $\overrightarrow{P_1 P_2}$ but also by an arrow over a letter, like $\vec{v}$, or by a boldface letter, like $\mathbf{v}$.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Remark: The order of the points determines the direction. For example, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.



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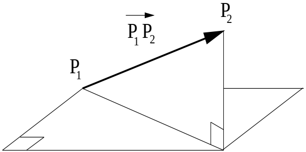

Remark: By 1850 it was realized that different physical phenomena were described using a new concept at that time, called a vector. A vector was more than a number in the sense that it was needed more than a single number to specify it. Phenomena described using vectors included velocities, accelerations, forces, rotations, electric phenomena, magnetic phenomena, and heat transfer.

Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Vector components in Cartesian coordinates.**
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, *the vector* $\overrightarrow{P_1 P_2}$ *determines a unique ordered pair denoted as follows,*

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Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. $\square$

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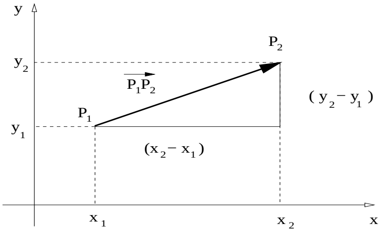
Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



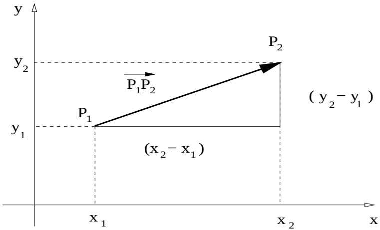
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered pair denoted as follows,

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Remark: A similar result holds for vectors in space.

Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, *the vector* $\overrightarrow{P_1 P_2}$ *determines a unique ordered triple denoted as follows,*

$$\overrightarrow{P_1 P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.$$

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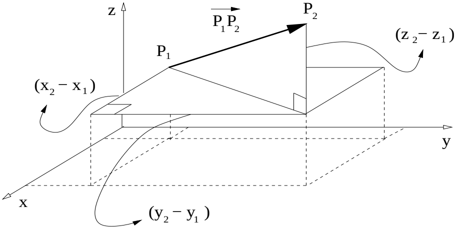

Components of a vector in Cartesian coordinates

Example

Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

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Example

Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

Components of a vector in Cartesian coordinates

Example

Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

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Example

Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

Solution:

$$\overrightarrow{P_3 P_4} = \langle (5-3), (4-1), (3-4) \rangle$$

Components of a vector in Cartesian coordinates

Example

Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

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Components of a vector in Cartesian coordinates

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Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

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$$\overrightarrow{P_3 P_4} = \langle (5-3), (4-1), (3-4) \rangle \quad \Rightarrow \quad \overrightarrow{P_3 P_4} = \langle 2, 3, -1 \rangle.$$

Remark: $\overrightarrow{P_1 P_2}$ and $\overrightarrow{P_3 P_4}$ have the same components although they are different vectors.

Components of a vector in Cartesian coordinates

Remark:

The vector components do not determine a unique vector.

Components of a vector in Cartesian coordinates

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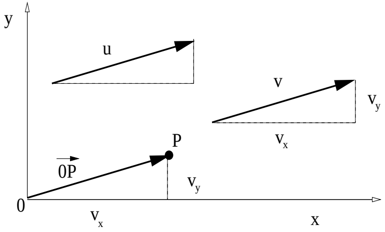
The vectors **u**, **v** and $\overrightarrow{0P}$ have the same components but they are all different, since they have different initial and terminal points.



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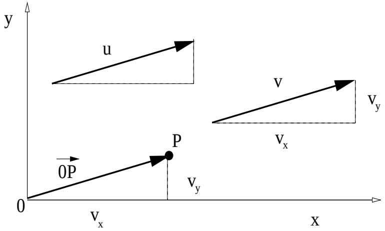
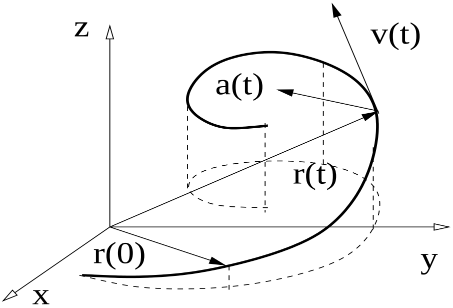


Definition
Given a vector $\overrightarrow{P_1 P_2} = \langle v_x, v_y \rangle$, the *standard position* vector is the vector $\overrightarrow{0P}$, where the point $0 = (0, 0)$ is the origin of the Cartesian coordinates and the point $P = (v_x, v_y)$.

Components of a vector in Cartesian coordinates

Remark: Vectors are used to describe motion of particles.

The position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ at the time t of a moving particle are described by vectors in space.



Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- **Magnitude of a vector and unit vectors.**
- Addition and scalar multiplication.

Magnitude of a vector and unit vectors.

Definition

The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.

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then its magnitude, denoted as $\left|\overrightarrow{P_1P_2}\right|$, is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

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- If the vector $\mathbf{v}$ has components $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, then its magnitude, denoted as $|\mathbf{v}|$, is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

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Find the length of a vector with initial point $P_1 = (1, 2, 3)$ and terminal point $P_2 = (4, 3, 2)$.

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Example

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Magnitude of a vector and unit vectors.

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Example

If the vector $\mathbf{v}$ represents the velocity of a moving particle, then its length $|\mathbf{v}|$ represents the speed of the particle. $\lhd$

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A vector $\mathbf{v}$ is a *unit vector* iff $\mathbf{v}$ has length one, that is, $|\mathbf{v}| = 1$.

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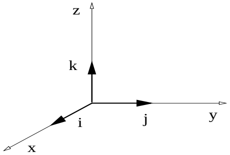
Example

The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are useful to express any other vector in $\mathbb{R}^3$.



Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
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- **Addition and scalar multiplication.**

Addition and scalar multiplication.

Definition

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
$$a\mathbf{v} = \langle a v_x, a v_y, a v_z \rangle.$$

Addition and scalar multiplication.

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$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
$$a\mathbf{v} = \langle av_x, av_y, av_z \rangle.$$

Remarks:

- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the *opposite* of vector $\mathbf{v}$.

Addition and scalar multiplication.

Definition

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle, \\ a\mathbf{v} &= \langle av_x, av_y, av_z \rangle. \end{aligned}$$

Remarks:

- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the *opposite* of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector,

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Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

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Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
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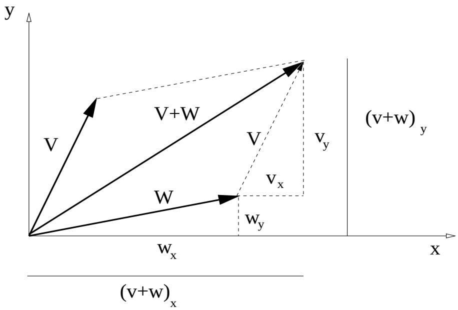
Remarks:

- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the *opposite* of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector, that is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$. This equation in components is

$$\mathbf{v} - \mathbf{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle.$$

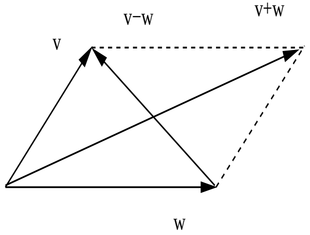
Addition and scalar multiplication.

Remark: The addition of two vectors is equivalent to the parallelogram law: The vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$ when they are in their standard position.



Addition and scalar multiplication.

Remark: The addition and difference of two vectors.



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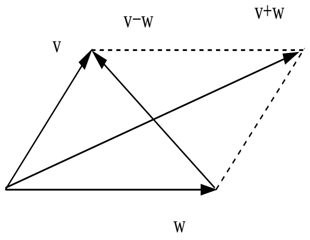
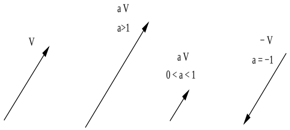


Remark: The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$.



Addition and scalar multiplication.

Example

Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$,

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Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

$$\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle$$

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

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$$\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.$$

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

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Therefore, its magnitude is

$$|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2}$$

Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

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Therefore, its magnitude is

$$|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.$$

Addition and scalar multiplication.

Example

Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

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A similar calculation can be done for $\mathbf{v} - \mathbf{w}$,

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Addition and scalar multiplication.

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Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

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$$\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.$$

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Therefore, its magnitude is

$$|\mathbf{v} - \mathbf{w}| = \sqrt{3^2 + 1^2} \quad \Rightarrow \quad |\mathbf{v} - \mathbf{w}| = \sqrt{10}.$$

Addition and scalar multiplication.

Theorem

If the vector $\mathbf{v} \neq \mathbf{0}$, *then the vector* $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ *is a unit vector.*

Addition and scalar multiplication.

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□

Addition and scalar multiplication.

Theorem

Every vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *in* $\mathbb{R}^3$ *can be expressed in a unique way as a linear combination of vectors* $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, *and* $\mathbf{k} = \langle 0, 0, 1 \rangle$ *as follows*

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

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Addition and scalar multiplication.

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Proof: Use the definitions of vector addition and scalar multiplication as follows,

$$\begin{aligned}
\mathbf{v} &= \langle v_x, v_y, v_z \rangle \\
&= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\
&= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\
&= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.
\end{aligned}$$

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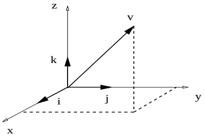

□

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

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Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$,

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Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle$$

Addition and scalar multiplication.

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Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.$$

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x\mathbf{i} + v_y\,\mathbf{j} + v_z\mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1-1), (4-0), (5-3) \rangle = \langle -2, 4, 2 \rangle.$$

Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. $\triangleleft$

Addition and scalar multiplication.

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Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

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Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2}$

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Example

Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4}$

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.$$

Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. $\triangleleft$

Example

Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$,

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x\mathbf{i} + v_y\,\mathbf{j} + v_z\mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

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Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. $\triangleleft$

Example

Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$, we conclude that $\mathbf{w} = -\dfrac{1}{\sqrt{24}}\langle -2, 4, 2 \rangle$. $\triangleleft$