Cartesian coordinates in space (Sect. 12.1).

- Overview of Multivariable Calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- Distance formula between two points in space.
- Equation of a sphere.

Overview of Multivariable Calculus

Mth 132, Calculus I: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Mth 133, Calculus II: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Mth 234, Multivariable Calculus:

$$\left.\begin{array}{ll} f : \mathbb{R}^2 \to \mathbb{R}, & f(x,y) \\ f : \mathbb{R}^3 \to \mathbb{R}, & f(x,y,z) \end{array}\right\} \quad \text{scalar-valued.}$$

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle \quad \} \quad \text{vector-valued.}$$

We study how to differentiate and integrate such functions.

The functions of Multivariable Calculus

Example

- An example of a scalar-valued function of two variables, $T : \mathbb{R}^2 \to \mathbb{R}$ is the temperature T of a plane surface, say a table. Each point (x, y) on the table is associated with a number, its temperature $T(x, y)$.
- An example of a scalar-valued function of three variables, $T : \mathbb{R}^3 \to \mathbb{R}$ is the temperature T of an object, say a room. Each point (x, y, z) in the room is associated with a number, its temperature $T(x, y, z)$.
- An example of a vector-valued function of one variable, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, is the position function in time of a particle moving in space, say a fly in a room. Each time t is associated with the position vector $\mathbf{r}(t)$ of the fly in the room.

$\triangleleft$

Cartesian coordinates in space (Sect. 12.1).

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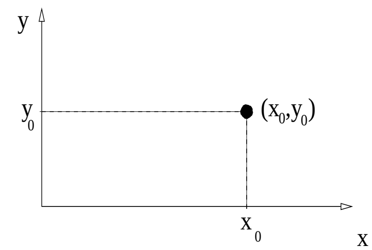
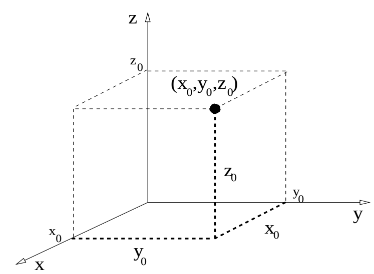
Cartesian coordinates.

Cartesian coordinates on $\mathbb{R}^2$: Every point on a plane is labeled by an ordered pair (x, y) by the rule given in the figure.



Cartesian coordinates in $\mathbb{R}^3$: Every point in space is labeled by an ordered triple (x, y, z) by the rule given in the figure.

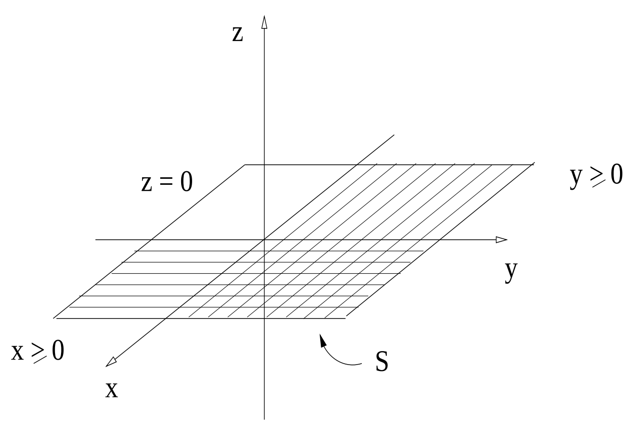


Cartesian coordinates.

Example

Sketch the set $S = \{x \geqslant 0,\ y \geqslant 0,\ z = 0\} \subset \mathbb{R}^3$.

Solution:



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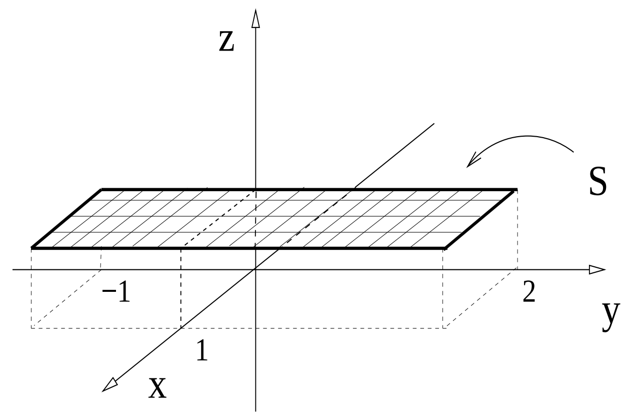
Cartesian coordinates.

Example

Sketch the set $S = \{0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ z = 1\} \subset \mathbb{R}^3$.

Solution:



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Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
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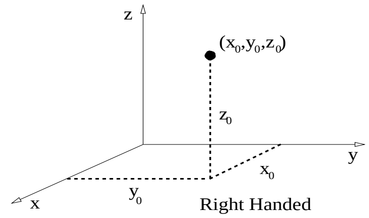
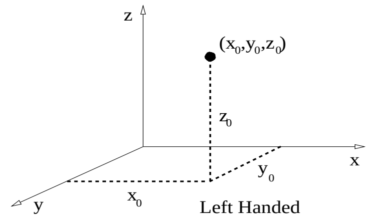
Right and left handed Cartesian coordinates.

Definition
A Cartesian coordinate system is called *right-handed* (rh) iff it can be rotated into the coordinate system in the figure.



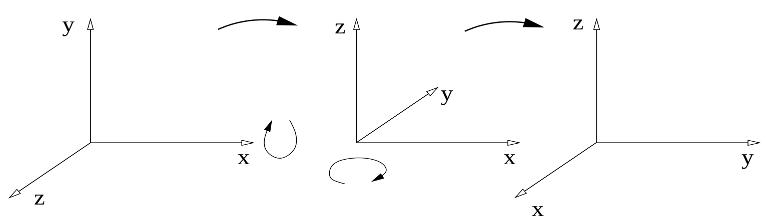
Definition
A Cartesian coordinate system is called *left-handed* (lh) iff it can be rotated into the coordinate system in the figure.



No rotation transforms a rh into a lh system.

Right and left handed Cartesian coordinates.

Example
This coordinate system is right-handed.



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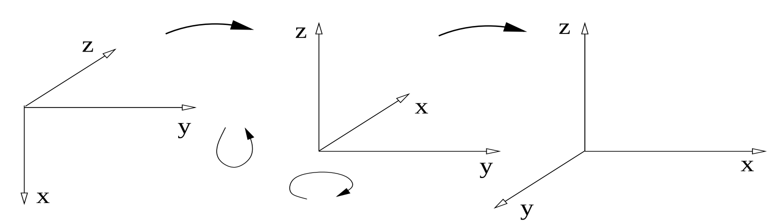
Example
This coordinate system is left handed.



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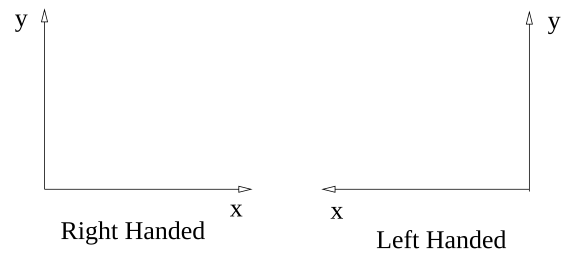
Right and left handed Cartesian coordinates

Remark: The same classification occurs in $\mathbb{R}^2$:



This classification is needed because:

- In $\mathbb{R}^3$ we will define the cross product of vectors, and this product has different results in rh or lh Cartesian coordinates.
- There is no cross product in $\mathbb{R}^2$.

In class we use rh Cartesian coordinates.

Cartesian coordinates in space (Sect. 12.1).

- Overview of vector calculus.
- Cartesian coordinates in space.
- Right-handed, left-handed Cartesian coordinates.
- **Distance formula between two points in space.**
- Equation of a sphere.

Distance formula between two points in space.

Theorem

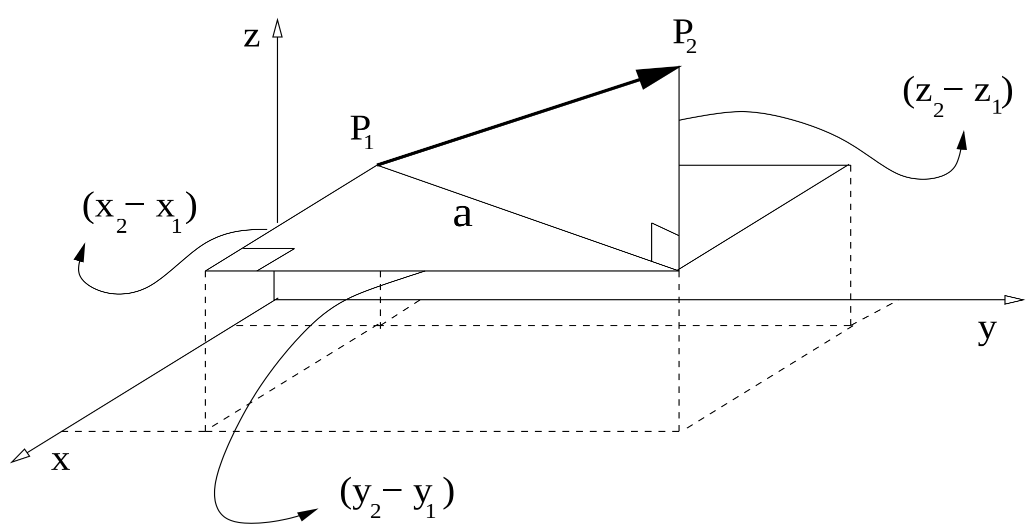
The distance $|P_1P_2|$ *between the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The distance between points in space is crucial to define the idea of limit to functions in space.

Proof.

Pythagoras Theorem.



$$|P_1P_2|^2 = a^2 + (z_2 - z_1)^2, \qquad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

□

Distance formula between two points in space

Example

Find the distance between $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$.

Solution:

$$\begin{aligned} |P_1 P_2| &= \sqrt{(3-1)^2 + (2-2)^2 + (1-3)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \quad \Rightarrow \quad |P_1 P_2| = 2\sqrt{2}. \end{aligned}$$

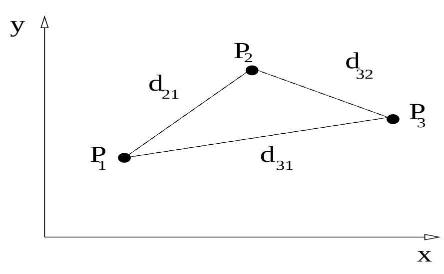
◁

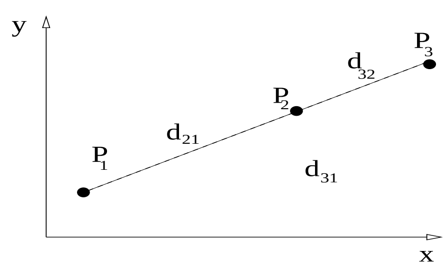
Distance formula between two points in space

Example

Use the distance formula to determine whether three points in space are collinear.

Solution:





$d_{21} + d_{32} > d_{31}$ — Not collinear,

$d_{21} + d_{32} = d_{31}$ — Collinear.

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Cartesian coordinates in space (12.1)

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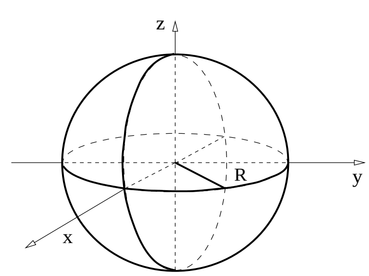
- **Equation of a sphere.**

A sphere is a set of points at fixed distance from a center.

Definition

A *sphere* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$S = \{P = (x, y, z) \ : \ |P_0 P| = R\}.$$



Remark: The point (x, y, z) belongs to the sphere S iff holds

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

("iff" means "if and only iff.")

An open ball is a set of points contained in a sphere.

Definition

An *open ball* centered at $P_0 = (x_0, y_0, z_0)$ of radius R is the set

$$B = \{P = (x, y, z) \ : \ |P_0 P| < R\}.$$

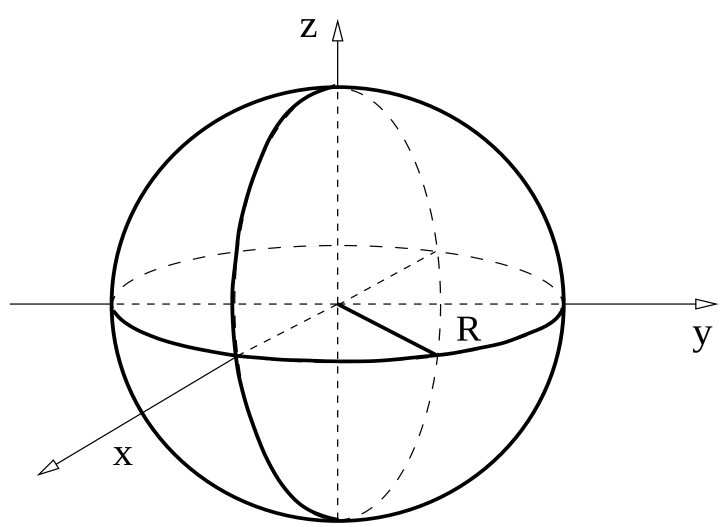
Remark: The point (x, y, z) belongs to the open ball B iff holds

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < R^2.$$

Example

Plot a sphere centered at $P_0 = (0, 0, 0)$ of radius $R > 0$.

Solution:



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Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Complete the square.

$$\begin{aligned} 0 &= x^2 + y^2 + 4y + z^2 \\ &= x^2 + \left[y^2 + 2\left(\frac{4}{2}\right) y + \left(\frac{4}{2}\right)^2 \right] - \left(\frac{4}{2}\right)^2 + z^2 \\ &= x^2 + \left(y + \frac{4}{2} \right)^2 + z^2 - 4. \end{aligned}$$

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2.$$

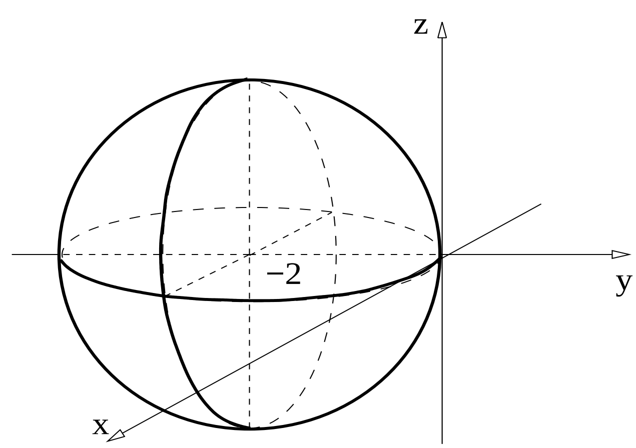
Example

Graph the sphere $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Since

$$x^2 + y^2 + 4y + z^2 = 0 \quad \Leftrightarrow \quad x^2 + (y+2)^2 + z^2 = 2^2,$$

we conclude that $P_0 = (0, -2, 0)$ and $R = 2$, therefore,



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Exercise

- Given constants a, b, c, and $d \in \mathbb{R}$, show that

$$x^2 + y^2 + z^2 - 2a\,x - 2b\,y - 2c\,z = d$$

is the equation of a sphere iff holds

$$d > -(a^2 + b^2 + c^2). \tag{1}$$

- Furthermore, show that if Eq. (1) is satisfied, then the expressions for the center P_0 and the radius R of the sphere are given by

$$P_0 = (a, b, c), \qquad R = \sqrt{d + (a^2 + b^2 + c^2)}.$$

◁

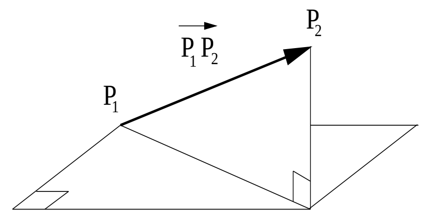
Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition

A *vector* in $\mathbb{R}^n$, with $n = 2, 3$, is an ordered pair of points in $\mathbb{R}^n$, denoted as $\overrightarrow{P_1P_2}$, where P_1, $P_2 \in \mathbb{R}^n$. The point P_1 is called the *initial point* and P_2 is called the *terminal point*.

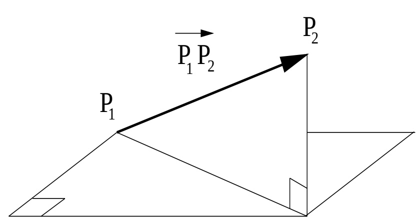


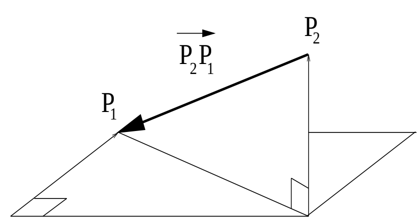
Remarks:

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an oriented line segment.
- A vector is drawn by an arrow pointing to the terminal point.
- A vector is denoted not only by $\overrightarrow{P_1P_2}$ but also by an arrow over a letter, like $\vec{v}$, or by a boldface letter, like $\mathbf{v}$.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Remark: The order of the points determines the direction. For example, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ have opposite directions.





Remark: By 1850 it was realized that different physical phenomena were described using a new concept at that time, called a vector. A vector was more than a number in the sense that it was needed more than a single number to specify it. Phenomena described using vectors included velocities, accelerations, forces, rotations, electric phenomena, magnetic phenomena, and heat transfer.

Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Vector components in Cartesian coordinates.**
- Magnitude of a vector and unit vectors.
- Addition and scalar multiplication.

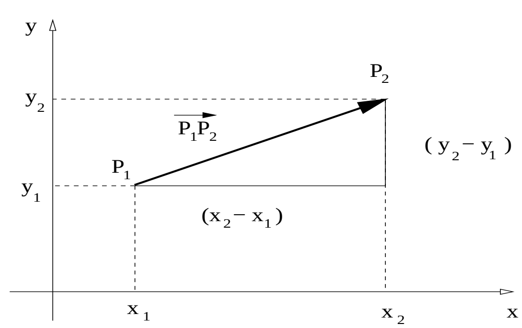
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, *the vector* $\overrightarrow{P_1P_2}$ *determines a unique ordered pair denoted as follows,*

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1) \rangle.$$

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Remark: A similar result holds for vectors in space.

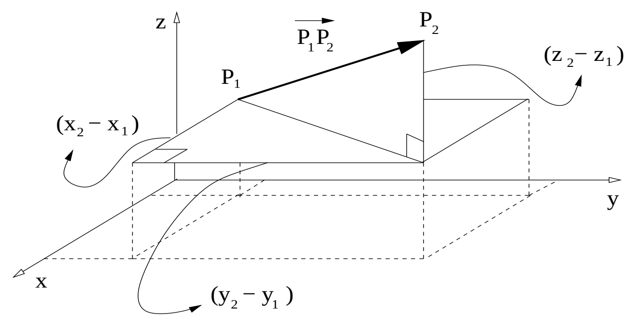
Components of a vector in Cartesian coordinates

Theorem

Given the points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the vector $\overrightarrow{P_1P_2}$ determines a unique ordered triple denoted as follows,

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle.$$

Proof: Draw the vector $\overrightarrow{P_1P_2}$ in Cartesian coordinates. □



Components of a vector in Cartesian coordinates

Example

Find the components of a vector with initial point $P_1 = (1, -2, 3)$ and terminal point $P_2 = (3, 1, 2)$.

Solution:

$$\overrightarrow{P_1P_2} = \langle (3 - 1), (1 - (-2)), (2 - 3) \rangle \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle 2, 3, -1 \rangle.$$

Example

Find the components of a vector with initial point $P_3 = (3, 1, 4)$ and terminal point $P_4 = (5, 4, 3)$.

Solution:

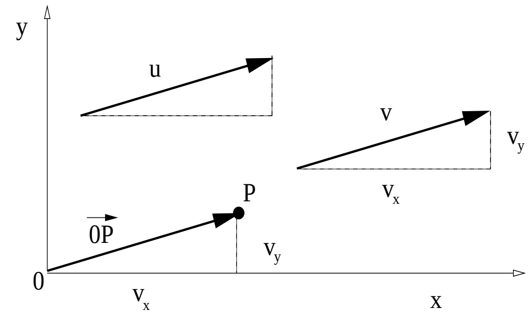
$$\overrightarrow{P_3P_4} = \langle (5 - 3), (4 - 1), (3 - 4) \rangle \quad \Rightarrow \quad \overrightarrow{P_3P_4} = \langle 2, 3, -1 \rangle.$$

Remark: $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ have the same components although they are different vectors.

Components of a vector in Cartesian coordinates

Remark:
The vector components do not determine a unique vector.

The vectors **u**, **v** and $\overrightarrow{0P}$ have the same components but they are all different, since they have different initial and terminal points.

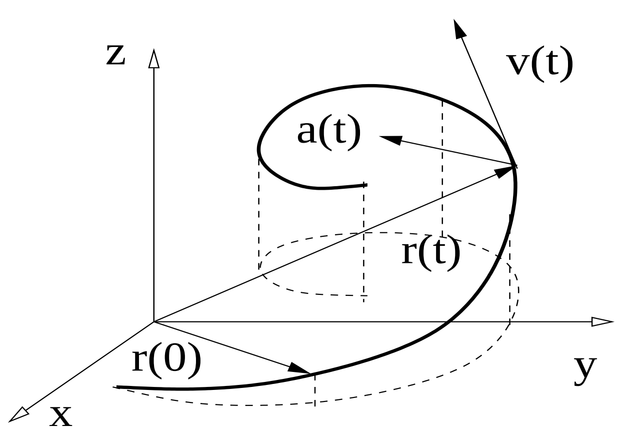


Definition
Given a vector $\overrightarrow{P_1 P_2} = \langle v_x, v_y \rangle$, the *standard position* vector is the vector $\overrightarrow{0P}$, where the point $0 = (0, 0)$ is the origin of the Cartesian coordinates and the point $P = (v_x, v_y)$.

Components of a vector in Cartesian coordinates

Remark: Vectors are used to describe motion of particles.

The position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ at the time t of a moving particle are described by vectors in space.



Vectors on a plane and in space (12.2)

- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Vector components in Cartesian coordinates.
- **Magnitude of a vector and unit vectors.**
- Addition and scalar multiplication.

Magnitude of a vector and unit vectors.

Definition

The *magnitude* or *length* of a vector $\overrightarrow{P_1P_2}$ is the distance from the initial point to the terminal point.

- If the vector $\overrightarrow{P_1P_2}$ has components

$$\overrightarrow{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle,$$

  then its magnitude, denoted as $|\overrightarrow{P_1P_2}|$, is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- If the vector $\mathbf{v}$ has components $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, then its magnitude, denoted as $|\mathbf{v}|$, is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Magnitude of a vector and unit vectors.

Example

Find the length of a vector with initial point $P_1 = (1, 2, 3)$ and terminal point $P_2 = (4, 3, 2)$.

Solution: First find the component of the vector $\overrightarrow{P_1P_2}$, that is,

$$\overrightarrow{P_1P_2} = \langle (4-1), (3-2), (2-3) \rangle \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle 3, 1, -1 \rangle.$$

Therefore, its length is

$$|\overrightarrow{P_1P_2}| = \sqrt{3^2 + 1^2 + (-1)^2} \quad \Rightarrow \quad |\overrightarrow{P_1P_2}| = \sqrt{11}.$$

Example

If the vector **v** represents the velocity of a moving particle, then its length $|\mathbf{v}|$ represents the speed of the particle. ◁

Magnitude of a vector and unit vectors.

Definition

A vector **v** is a *unit vector* iff **v** has length one, that is, $|\mathbf{v}| = 1$.

Example

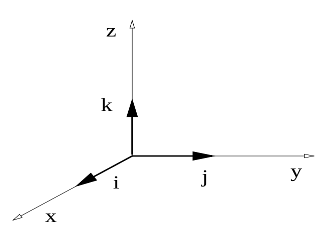
Show that $\mathbf{v} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$ is a unit vector.

Solution:

$$|\mathbf{v}| = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} \quad \Rightarrow \quad |\mathbf{v}| = 1.$$

Example

The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are useful to express any other vector in $\mathbb{R}^3$.



Vectors on a plane and in space (12.2)

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- **Addition and scalar multiplication.**

Addition and scalar multiplication.

Definition

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the *vector addition*, $\mathbf{v} + \mathbf{w}$, and the *scalar multiplication*, $a\mathbf{v}$, are given by

$$\mathbf{v} + \mathbf{w} = \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle,$$
$$a\mathbf{v} = \langle av_x, av_y, av_z \rangle.$$

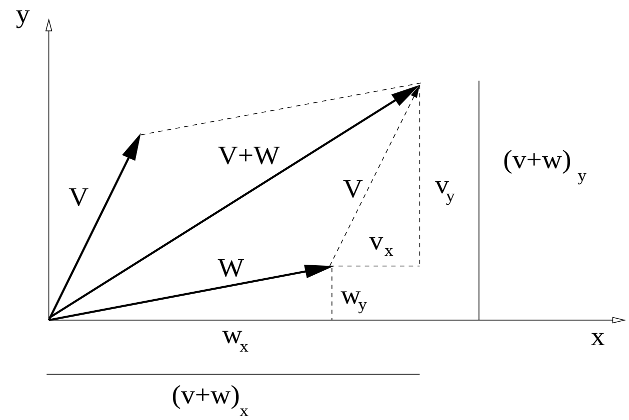
Remarks:

- The vector $-\mathbf{v} = (-1)\mathbf{v}$ is called the *opposite* of vector $\mathbf{v}$.
- The difference of two vectors is the addition of one vector and the opposite of the other vector, that is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$. This equation in components is

$$\mathbf{v} - \mathbf{w} = \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle.$$

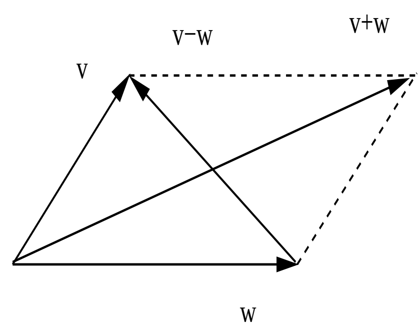
Addition and scalar multiplication.

Remark: The addition of two vectors is equivalent to the parallelogram law: The vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$ when they are in their standard position.

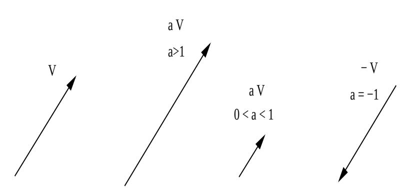


Addition and scalar multiplication.

Remark: The addition and difference of two vectors.



Remark: The scalar multiplication stretches a vector if $a > 1$ and compresses the vector if $0 < a < 1$.



Addition and scalar multiplication.

Example

Given the vectors $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$, find the magnitude of the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

Solution: We first compute the components of $\mathbf{v} + \mathbf{w}$, that is,

$$\mathbf{v} + \mathbf{w} = \langle (2 - 1), (3 + 2) \rangle \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} = \langle 1, 5 \rangle.$$

Therefore, its magnitude is

$$|\mathbf{v} + \mathbf{w}| = \sqrt{1^2 + 5^2} \quad \Rightarrow \quad |\mathbf{v} + \mathbf{w}| = \sqrt{26}.$$

A similar calculation can be done for $\mathbf{v} - \mathbf{w}$, that is,

$$\mathbf{v} - \mathbf{w} = \langle (2 + 1), (3 - 2) \rangle \quad \Rightarrow \quad \mathbf{v} - \mathbf{w} = \langle 3, 1 \rangle.$$

Therefore, its magnitude is

$$|\mathbf{v} - \mathbf{w}| = \sqrt{3^2 + 1^2} \quad \Rightarrow \quad |\mathbf{v} - \mathbf{w}| = \sqrt{10}.$$

Addition and scalar multiplication.

Theorem

If the vector $\mathbf{v} \neq \mathbf{0}$, *then the vector* $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ *is a unit vector.*

Proof: (Case $\mathbf{v} \in \mathbb{R}^2$ only).
If $\mathbf{v} = \langle v_x, v_y \rangle \in \mathbb{R}^2$, then $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$, and

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \Big\langle \frac{v_x}{|\mathbf{v}|}, \frac{v_y}{|\mathbf{v}|} \Big\rangle.$$

This is a unit vector, since

$$|\mathbf{u}| = \Big| \frac{\mathbf{v}}{|\mathbf{v}|} \Big| = \sqrt{\Big(\frac{v_x}{|\mathbf{v}|}\Big)^2 + \Big(\frac{v_y}{|\mathbf{v}|}\Big)^2} = \frac{1}{|\mathbf{v}|}\sqrt{v_x^2 + v_y^2} = \frac{|\mathbf{v}|}{|\mathbf{v}|} = 1.$$

□

Addition and scalar multiplication.

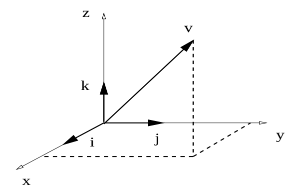
Theorem

Every vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *in* $\mathbb{R}^3$ *can be expressed in a unique way as a linear combination of vectors* $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$,*and* $\mathbf{k} = \langle 0, 0, 1 \rangle$ *as follows*

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

Proof: Use the definitions of vector addition and scalar multiplication as follows,

$$\begin{aligned}\mathbf{v} &= \langle v_x, v_y, v_z \rangle \\ &= \langle v_x, 0, 0 \rangle + \langle 0, v_y, 0 \rangle + \langle 0, 0, v_z \rangle \\ &= v_x \langle 1, 0, 0 \rangle + v_y \langle 0, 1, 0 \rangle + v_z \langle 0, 0, 1 \rangle \\ &= v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.\end{aligned}$$



□

Addition and scalar multiplication.

Example

Express the vector with initial and terminal points $P_1 = (1, 0, 3)$, $P_2 = (-1, 4, 5)$ in the form $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Solution: First compute the components of $\mathbf{v} = \overrightarrow{P_1 P_2}$, that is,

$$\mathbf{v} = \langle (-1 - 1), (4 - 0), (5 - 3) \rangle = \langle -2, 4, 2 \rangle.$$

Then, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. ◁

Example

Find a unit vector $\mathbf{w}$ opposite to $\mathbf{v}$ found above.

Solution: Since $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$, we conclude that $\mathbf{w} = -\frac{1}{\sqrt{24}} \langle -2, 4, 2 \rangle$. ◁