Name:

TEST 4

answers

No Calculators

1:
2:
3:
4:
5:
6:

1 (14 points) Find the outward flux of the field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$ across the surface of the region bounded by the paraboloid $x^2 + y^2 + z = 1$ and the plane $z = 0$.

Answer: Since $\nabla \cdot \mathbf{F} = 0$ the Divergence Theorem (p. 1212) implies

$$\int\int_{S_0} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\int\int_D \nabla \cdot \mathbf{F}\, dV = 0.$$

2 (18 points) Find the outward flux of the field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$ across the part of the surface of the paraboloid $x^2 + y^2 + z = 1$ that lies in the half-space $z \geq 0$.

Answer: The surface S is given implictly by $g(x, y, z) = x^2 + y^2 + z = 1$ over the region $R : x^2 + y^2 \leq 1$ in $z = 0$ plane. See p. 1187, 1188:

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\int_R \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{k}|}\, dA = \int\int_R \frac{(-y)2x + x(2y) + 1}{1}\, dA = \int\int_R dA = \pi.$$

3 (18 points) Find the outward flux of the field $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j}$ across the circle $x^2 + y^2 = 2x$.

Answer: The circle $(x - 1)^2 + y^2 = 1$ has radius 1, let R denotes its inside. Let $M = x$, $N = y^2$. Green's Theorem (p.1172) implies

$$\oint \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_R (M_x + N_y) dA = \int\int_R (1 + 2y) dA = \int\int_R dA = \pi.$$

We used symmetry to get $\int\int_R y\, dA = 0$ and the fact that R has area π.

To do the double integral directly, use the polar coordinates in $x^2 + y^2 = 2x$ to obtain $r = 2\cos\theta$ on the circle, $-\pi/2 \leq \theta \leq \pi/2$. Hence

$$\int\int_R (1 + 2y) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (1 + 2r\sin\theta) r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} (2\cos^2\theta + \frac{16}{3}\cos^3\theta\sin\theta) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta + \frac{16}{3}\cos^3\theta\sin\theta) d\theta = \theta + \frac{1}{2}\sin 2\theta - \frac{4}{3}\cos^4\theta \Big|_{-\pi/2}^{\pi/2} = \pi.$$

4 (18 points) Let $\mathbf{F}(x, y, z) = y\mathbf{i}$. Use Stoke's Theorem to evaluate

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and $\mathbf{n}$ is the outward normal to S.

Answer: See picture 16.60. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $t : 0 \to 2\pi$ is a parametrization of the boundary C of S. The Stokes's Theorem (p. 1203) implies

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C y\, dx = -\int_0^{2\pi} \sin^2 t\, dt = \frac{-1}{2}\int_0^{2\pi} (1 - \cos 2t) dt = -\pi.$$

5 (18 points) Find the potential function f for

$$\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - z - e^x \sin y)\mathbf{j} + (xy - y + 1/z)\mathbf{k}.$$

Answer: See p.1165
$f_x = e^x \cos y + yz$ hence $f = e^x \cos y + xyz + g(y, z)$ for some g.
$f_y = -e^x \sin y + xz + g_y = xz - z - e^x \sin y$ hence $g_y = -z$, $g = -yz + h(z)$.
$f_z = xy - y + h'(z) = xy - y + 1/z$ hence $h' = 1/z$, $h = \ln z$.
Therefore $f = e^x \cos y + xyz - yz + \ln z$.

6 (14 points) Let $\mathbf{F}$ be as in problem 5. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where the curve C is given by $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + (2 + \sin t)\mathbf{k}$, $0 \le t \le \pi$.

Answer: See p. 1162

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi, \pi, 2) - f(0, 0, 2) = -e^\pi - 1 + 2\pi^2 - 2\pi.$$