Name:

TEST 1

answers

No Calculators

1:
2:
3:
4:
5:

1 (48 points) A hummingbird starts at a feeder located at $F(0,0,10)$ (distances in feet) and flies straight toward a point $B(10,20,30)$ on a branch with a speed $3ft/s$. An observer is located at $O(5,10,10)$.

(a) (5) How long does it take the hummingbird to reach the branch?

$$\overrightarrow{FB} = < 10, 20, 20 >, \qquad |\overrightarrow{FB}| = \sqrt{100+400+400} = 30$$

$$\text{travel time } = \frac{|\overrightarrow{FB}|}{\text{speed}} = \frac{30}{3} = 10 \text{ seconds.}$$

(b) (5) What is the hummingbird's velocity vector?

$$\mathbf{v} = \text{ speed } \times \text{ unit direction vector } = 3\frac{1}{|\overrightarrow{FB}|}\overrightarrow{FB} = < 1, 2, 2 >$$

(c) (5) Write down the equation of the line that contains the hummingbird's path.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t \quad \text{where} \quad \mathbf{r}_0 = < 0, 0, 10 > \quad \text{is the starting point, hence}$$

$$x = t \qquad y = 2t \qquad z = 10 + 2t$$

(d) (5) Where is the hummingbird going to be in 2 seconds?

$$x = 2 \qquad y = 2 * 2 = 4 \qquad z = 10 + 2 * 2 = 14$$

e) (7) What is the projection of $\overrightarrow{FO}$ onto $\overrightarrow{FB}$?

$$\text{proj}_{\overrightarrow{FB}}\, \overrightarrow{FO} = \frac{\overrightarrow{FB} \cdot \overrightarrow{FO}}{|\overrightarrow{FB}|^2}\overrightarrow{FB} = \frac{25}{9} < 1, 2, 2 > = \left\langle \frac{25}{9}, \frac{50}{9}, \frac{50}{9} \right\rangle$$

f) (7) What is the equation of the plane that contains the triangle $\triangle FOB$?

$$\overrightarrow{FB} \times \overrightarrow{FO} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 20 & 20 \\ 5 & 10 & 0 \end{pmatrix} = < -200, 100, 0 > = \text{ normal to the plane}$$

$(0,0,10)$ is a point in the plane, hence

$$-200(x-0) + 100(y-0) + 0(z-10) = 0 \quad \text{or} \quad 2x - y = 0$$

g) (7) What is the area of the triangle $\triangle FOB$?

$$\text{area } = \frac{1}{2}|\overrightarrow{FB} \times \overrightarrow{FO}| = \frac{1}{2}\sqrt{200^2 + 100^2} = 50\sqrt{5}$$

h) (7) How close to the observer will the hummingbird get?

using g:

$$\text{distance } = \frac{|\overrightarrow{FB} \times \overrightarrow{FO}|}{|\overrightarrow{FB}|} = \frac{100\sqrt{5}}{30}$$

using e:

$$\text{distance } = \left|\overrightarrow{FO} - \text{proj}_{\overrightarrow{FB}}\, \overrightarrow{FO}\right| = \sqrt{\left(5 - \frac{25}{9}\right)^2 + \left(10 - \frac{50}{9}\right)^2 + \left(\frac{50}{9}\right)^2} = \frac{10\sqrt{5}}{3}$$

2 (24 points)

a) Find the parametric equation of the line through the point $P(1, 0, -1)$ and perpendicular to the plane $5x - 2y + 3z = 7$.

$$\text{normal to the plane } = \mathbf{v} = < 5, -2, 3 >$$

$$\text{line: } x = 1 + 5t \quad y = 0 - 2t \quad z = -1 + 3t$$

b) Find the point R where this line intersects the plane.

$$\text{finding } t \text{ such that } \quad 5(1 + 5t) - 2(-2t) + 3(-1 + 3t) = 7 \quad \text{gives } t = \frac{5}{38}$$

$$x = 1 + \frac{25}{38} = \frac{63}{38} \qquad y = -\frac{10}{38} \qquad z = -1 + \frac{15}{38} = -\frac{23}{38}$$

$$R = \left(\frac{63}{38}, -\frac{10}{38}, -\frac{23}{38}\right)$$

c) Find the distance of the point P from the plane.

using b

$$\text{distance } = |\overrightarrow{PR}| = \sqrt{\left(1 - \frac{63}{38}\right)^2 + \left(\frac{10}{38}\right)^2 + \left(-1 + \frac{23}{38}\right)^2} = \frac{5}{\sqrt{38}}$$

as in the book, p.886, choosing a point $Q = (1, -1, 0)$ on the plane gives

$$\text{distance } = \frac{|\overrightarrow{PQ} \cdot \mathbf{v}|}{|\mathbf{v}|} = \frac{5}{\sqrt{38}}$$

3 (18 points) Find an equation of the plane that contains the origin and the line

$$x = 1 + 2t, \qquad y = 1 - t, \qquad z = 2t, \qquad\qquad -\infty < t < \infty.$$

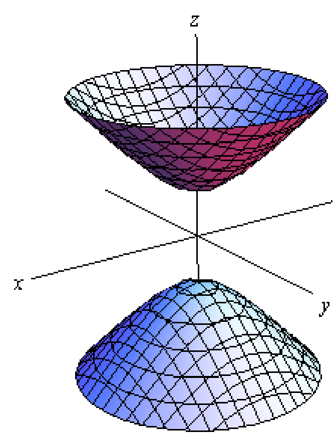
vector parallel to the line $\mathbf{v} =< 2, -1, 2 >$
point on the line ($t = 0$): $Q = (1, 1, 0)$, position vector $\mathbf{u} =< 1, 1, 0 >$.
normal to the plane:

$$\mathbf{v} \times \mathbf{u} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} =< -2, 2, 3 >$$

plane: $-2x + 2y + 3z = 0$

4 (10 points) Sketch and identify the surface $x^2 + y^2 - z^2 + 1 = 0$.

In the plane $z = c$ we have a circle $x^2 + y^2 = c^2 - 1$
radius $\sqrt{c^2 - 1}$, hence $|z|$ has to be ≥ 1.
So, we have hyperboloid of two sheets (p. 895)



5 (10 extra credit points if you get 90 or more points on problems 1-4 points) Find the projection of the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty$$

onto a plane which contains a point with a position vector $\mathbf{r}_1$ and has a normal $\mathbf{n}$. *Hint:* Start by following the first steps in problem 2.

Let $P(t)$ be a point at a fixed t on the line. The equation of the line perpendicular to the plane and passing through $P(t)$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} + s\mathbf{n}, \qquad -\infty < s < \infty.$$

It intersects the plane when

$$(\mathbf{r}_0 + t\mathbf{v} + s\mathbf{n} - \mathbf{r}_1) \cdot \mathbf{n} = 0$$

hence

$$s = -\frac{(\mathbf{r}_0 + t\mathbf{v} - \mathbf{r}_1) \cdot \mathbf{n}}{|\mathbf{n}|^2}.$$

Hence the projection of $P(t)$ on the plane is

$$\mathbf{r}_0 + t\mathbf{v} - \frac{(\mathbf{r}_0 + t\mathbf{v} - \mathbf{r}_1) \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n}$$

- which represents a line with a direction

$$\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n} = \mathbf{v} - \mathrm{proj}_{\mathbf{n}} \mathbf{v}$$

and passing through a point

$$\mathbf{r}_0 - \frac{(\mathbf{r}_0 - \mathbf{r}_1) \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n}.$$