MATH 234 EXAM #4

Name ....Solution....................................

Work all 4 problems. Show ALL your work. No calculators may be used on this test. The total available points is 50.

1. (12 points) Consider the vector field

$$\mathbf{F} = (z\cos(xz))\mathbf{i} + e^y\mathbf{j} + (x\cos(xz))\mathbf{k}.$$

(a) Compute the curl of $\mathbf{F}$. Is $\mathbf{F}$ conservative?

curl $\vec{F} = 0 \iff \vec{F}$ is conservative (using $\mathbb{R}^3$ is simply-conn)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\cos(xz) & e^y & x\cos(xz) \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y(x\cos(xz)) - \partial/\partial z(e^y)\right) - \vec{j}\left(\partial/\partial x(x\cos(xz)) - \partial/\partial z(z\cos(xz))\right)$$

$$+ \vec{k}\left(\partial/\partial x(e^y) - \partial/\partial y(z\cos(xz))\right)$$

$$= \vec{i}(0) - \vec{j}\left(\cos(xz) - zx\sin(xz) - \cos(xz) + xz\sin(xz)\right) + \vec{k}(0)$$

$= 0$ $\therefore$ $\vec{F}$ is conservative

(b) If $\mathbf{F}$ is conservative, find a potential function, f, for $\mathbf{F}$.

If $\vec{F} = \nabla f = \partial/\partial x f\,\vec{i} + \partial/\partial y f\,\vec{j} + \partial/\partial z f\,\vec{k}$

then $\partial f/\partial x = z\cos(xz)$

integrate wrt x to get: $f = \int^x z\cos(xz)\,dx = \sin(xz) + g(y,z)$

then

$$e^y = \frac{\partial f}{\partial y} = \partial/\partial y \sin(xz) + \partial/\partial y\, g(y,z)$$

so $\dfrac{\partial g}{\partial y} = e^y$ integrate wrt y to get $g(y,z) = e^y + h(z)$

then

$$x\cos(xz) = \frac{\partial f}{\partial z} = \partial/\partial z \sin(xz) + \partial/\partial z(e^y) + \partial/\partial z\, h(z)$$

$$= x\cos(xz) + \partial h/\partial z$$

hence $\partial h/\partial z = 0$ h = constant and

$f(x,y,z) = \sin(xz) + e^y$ is a potential function.

(c) Use the potential function f to find the work done along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.

work done $= \int_C \vec{F} \cdot \vec{T} \, ds = f(\vec{r}(1)) - f(\vec{r}(0))$

here $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$

$\vec{r}(1) = 1\vec{i} + 1\vec{j} + 1\vec{k}$

$f(\vec{r}(1)) = \sin(1) + e^1$

$\vec{r}(0) = 0\vec{i} + 0\vec{j} + 0\vec{k}$

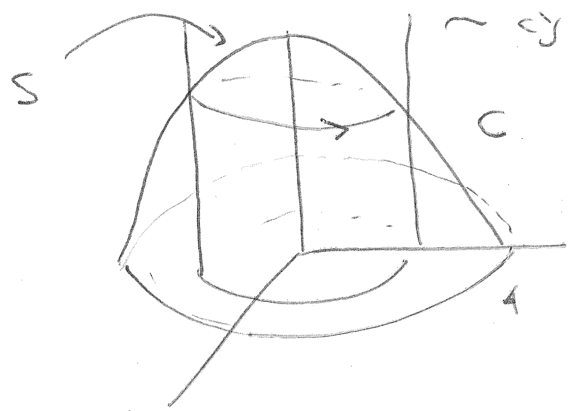
$f(\vec{r}(0)) = \sin(0) + e^0 = 1$

$\therefore$ work done $= \sin(1) + e - 1$

2. (10 points) Use Stokes theorem to find the circulation of the vector field

$$\mathbf{F} = xy^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$$

along the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above.



~ cylinder

C intersection of hemisphere & cylinder

S spherical cap

$\vec{n}$ unit normal (upward) to S

Stokes' theorem

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^3 & 1 & z \end{vmatrix} = \vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial 1}{\partial z}\right) - \vec{j}\left(\frac{\partial z}{\partial x} - \frac{\partial}{\partial z}(xy^3)\right) + \vec{k}\left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(xy^3)\right)$$

$$= -3xy^2\,\vec{k}$$

S level set $f(x,y,z) = x^2 + y^2 + z^2 = 16$

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \qquad |\nabla f| = 2(x^2+y^2+z^2)^{1/2}$$

$$\vec{n} = \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\vec{k}|}\,dx\,dy = \frac{2(x^2+y^2+z^2)^{1/2}}{2z}\,dx\,dy$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_{\substack{\text{shadow}\\ \text{region}}} \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \iint_{x^2+y^2\leq 4} -3xy^2\,dx\,dy$$

$$= \int_{0}^{2\pi} \int_{r=0}^{r=2} -3 r \cos\theta \, r^2 \sin^2\theta \, r \, dr \, d\theta$$

$$= -3 \int_{0}^{2\pi} \int_{0}^{2} r^4 \cos\theta \sin^2\theta \, dr \, d\theta$$

$$= -3 \cdot \frac{2^5}{5} \int_{0}^{2\pi} \cos\theta \sin^2\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= -3 \left(\frac{2^5}{5}\right) \int_{0}^{0} u^2 \, du$$

$$= 0$$

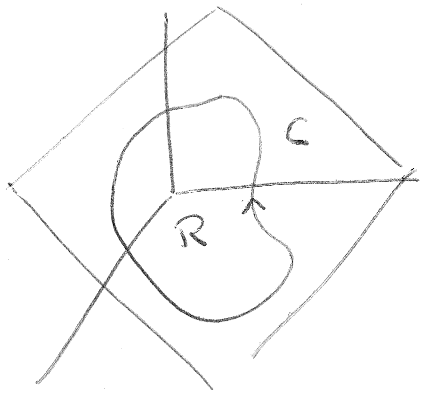
3. (10 points) Let C be a simple closed curve in the plane:

$$x + 3y + 2z = 4.$$

Show that

$$\int_C 3y dx + (3z + x) dy - (y + x) dz,$$

depends only on the area of the region enclosed by C and not on the position or shape of C. (Use either orientation of C).



Set $\vec{F} = 3y\vec{i} + (3z+x)\vec{j} - (y+x)\vec{k}$

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

R level set $f(x,y,z) = x + 3y + 2z = 4$

$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} dx dy = \frac{\sqrt{14}}{2} dx dy$

then $\oint_C \vec{F} \frac{d\vec{r}}{dt} dt = \oint_C 3y dx + (3z+x)dy - (y+x)dz$

$\overset{\text{Stokes Thm}}{=} \iint_R \text{curl } \vec{F} \cdot \vec{n} \, d\sigma$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & 3z+x & -(y+x) \end{vmatrix} = \vec{i}(-1-3) - \vec{j}(-1-0) + \vec{k}(1-3)$$

$$= -4\vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{n} = \frac{1\vec{i} + 3\vec{j} + 2\vec{k}}{\sqrt{14}}$$

$$\therefore \iint_R (-4\vec{i} + \vec{j} - 2\vec{k}) \cdot \frac{(1\vec{i} + 3\vec{j} + 2\vec{k})}{\sqrt{14}} \frac{\sqrt{14}}{2} dx dy$$

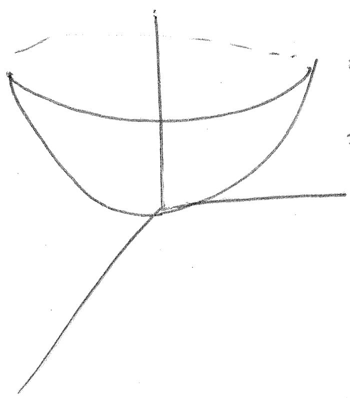
$$= \iint_R \frac{(-4 + 3 - 4)}{2} dx dy = -\frac{5}{2} \text{area}(R).$$

4. (18 points) Find the flux of the vector field

$$\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$$

outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$:

(a) By computing a surface integral.



level set $f(x,y,z) = z - x^2 - y^2 = 0$

$\nabla f = -2x\vec{i} - 2y\vec{j} + k$

$|\nabla f| = (4x^2 + 4y^2 + 1)^{1/2}$

flux out $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$

outward normal.

$\vec{F} = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$

$\vec{n} = -\frac{\nabla f}{|\nabla f|} = \frac{-1}{(4x^2 + 4y^2 + 1)^{1/2}}(-2x\vec{i} - 2y\vec{j} + 1\vec{k})$ outward normal

$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} = \frac{(4x^2 + 4y^2 + 1)^{1/2}}{1}$

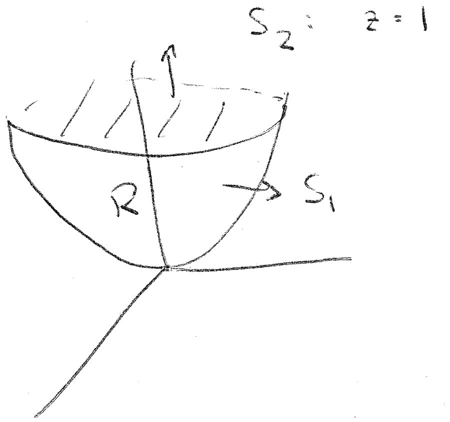
flux out

$$= \iint_{x^2+y^2 \le 1} (4x\vec{i} + 4y\vec{j} + 2\vec{k}) \cdot (2x\vec{i} + 2y\vec{j} - \vec{k})\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 1} (8x^2 + 8y^2 - 2)\, dx\, dy = \int_0^{2\pi} \int_0^1 (8r^2 - 2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{8}{4} r^4 - r^2 \Big|_{r=0}^{r=1} d\theta = 2\pi(2 - 1) = 2\pi$$

(b) By computing a volume integral. (Use the Divergence Theorem.)



S_2 unit normal $= \vec{k}$

shadow region $x^2+y^2 \le 1$

Divergence Theorem.

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_{R} \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$\operatorname{div}\vec{F} = 4 + 4 = 8$$

so $$\iiint_R 8\,dx\,dy\,dz = 8\int_0^{2\pi}\int_0^1\int_{z=r^2}^{z=1} dz\,r\,dr\,d\theta$$

$$= 8\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= 8\int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1\,d\theta$$

$$= 16\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = 4\pi$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{x^2+y^2\le 1} (4x\vec{i} + 4y\vec{j} + 2\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} 2\,dx\,dy = 2\text{ area(unit disk)} = 2\pi$$

Hence $$\iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma = 4\pi - 2\pi = 2\pi$$ (as shown above)