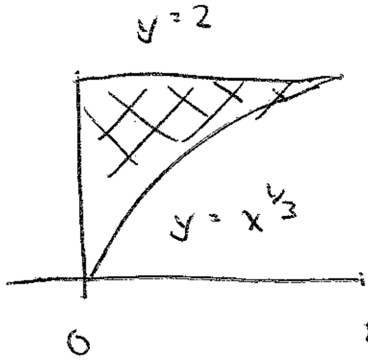
MATH 234 EXAM #3

Name ....Solutions....................

Work all 5 problems. Show ALL your work. No calculators may be used on this test. The total available points is 50.

1. (9 points) Sketch the region of integration and evaluate:

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dydx}{y^4+1}.$$



or $x = y^3$

$$\text{integral} = \int_0^2 \left(\int_0^{y^3} \frac{1}{y^4+1}\, dx \right) dy$$

$$= \int_0^2 \frac{y^3}{y^4+1}\, dy$$

$u = y^4+1$

$du = 4y^3\, dy$

$y^3 dy = \frac{1}{4} du$

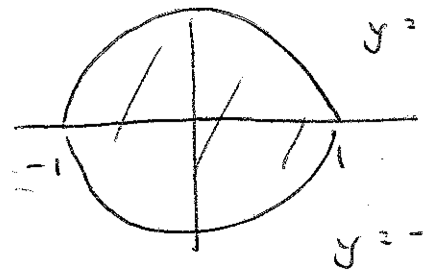
$$= \int_1^{17} \frac{1}{u} \frac{1}{4}\, du$$

$$= \frac{1}{4} \ln u \Big|_1^{17} = \frac{1}{4} \ln(17)$$

2. (9 points) Evaluate the following integral by changing to polar coordinates:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{(1+x^2+y^2)^2}.$$

region:



$y = \sqrt{1-x^2}$

or $x^2 + y^2 \leq 1$

$y = -\sqrt{1-x^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$(1+x^2+y^2)^2 = (1+r^2)^2$

integral $= \displaystyle\int_0^{2\pi}\int_0^1 \frac{2}{(1+r^2)^2}\, r\,dr\,d\theta$

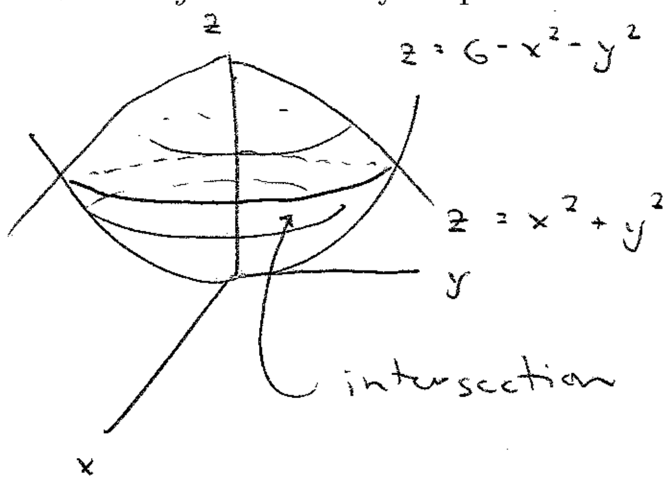
$= \displaystyle\int_0^{2\pi}\left(\int_1^2 \frac{du}{u^2}\right) d\theta$ $\quad u = 1 + r^2$, $du = 2r\,dr$

$= \displaystyle\int_0^{2\pi}\left(-\frac{1}{u}\Big|_1^2\right) d\theta$

$= \displaystyle\int_0^{2\pi}\left(-\frac{1}{2}+1\right) d\theta$

$= 2\pi \cdot \frac{1}{2}$

3. (10 points) Find the volume of the region bounded above by the paraboloid $z = 6 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$.



$6 - x^2 - y^2 = x^2 + y^2$

$6 = 2(x^2 + y^2)$

$x^2 + y^2 = 3$

shadow region in xy-plane : $x^2 + y^2 \le 3$

use cylindrical coord. so $x^2 + y^2 \le 3$ is $r^2 \le 3$

or $r \le \sqrt{3}$

$$\text{volume} = \int_0^{2\pi} \left(\int_0^{\sqrt{3}} \left(\int_{z=r^2}^{z=6-r^2} dz \right) r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{\sqrt{3}} (6 - r^2 - r^2)\, r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{\sqrt{3}} (6 - 2r^2)\, r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{6r^2}{2} - \frac{2r^4}{4} \right) \Bigg|_0^{\sqrt{3}} d\theta$$

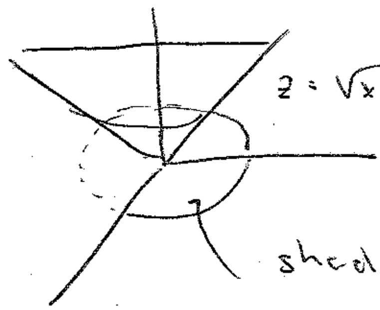
$$= \int_0^{2\pi} \left(9 - \frac{1}{2} 9 \right) d\theta = 2\pi \cdot \frac{1}{2} \cdot 9$$

4. (10 points) Convert the following integral to spherical coordinates and then evaluate the new integral:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dzdydx.$$

Region:

$z = 1 \qquad 1 = \rho\cos\phi \qquad \rho = \frac{1}{\cos\phi}$



$z = \sqrt{x^2+y^2}, \quad \phi = \pi/4$

shadow region $x^2+y^2=1$

integral $= \displaystyle\int_0^{2\pi}\int_0^{\pi/4}\left(\int_0^{1/\cos\phi} \rho^2 \sin\phi \, d\rho\right) d\phi \, d\theta$

$= \displaystyle\int_0^{2\pi}\left(\int_0^{\pi/4} \frac{1}{3}\frac{\sin\phi}{\cos^3\phi} \, d\phi\right) d\theta$

$= \displaystyle\int_0^{2\pi}\int_1^{\sqrt{2}/2}\left(-\frac{1}{3}\, du/u^3\right) d\theta \qquad u = \cos\phi, \quad du = -\sin\phi \, d\phi$

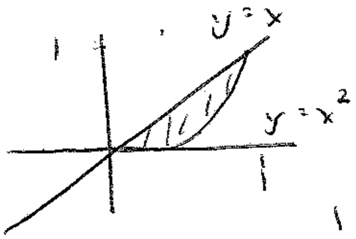
$= \displaystyle\int_0^{2\pi} \frac{1}{3}\left(\frac{u^{-2}}{-2}\right)\Big|_{\sqrt{2}/2}^{1} d\theta$

$= \displaystyle\int_0^{2\pi} \frac{1}{3}\left(-\frac{1}{2} + \frac{1}{2}\,\frac{4}{2}\right) d\theta$

$= \frac{2\pi}{3}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

5. (12 points) Use triple integrals to solve the following problems. Leave your answers in the form of iterated integrals. DO NOT EVALUATE. This applies to parts a, b, and c.

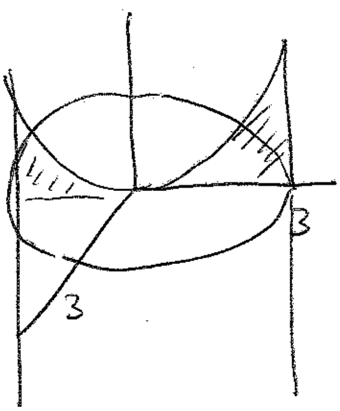
(a) Consider the cylinder parallel to the z-axis formed by $y = x^2$ and $y = x$ $(0 \leq x \leq 1)$. The planes $z = x + y$ and $z = y - x$ intersect this cylinder bounding a solid region D. Find the volume of D.



shadow region

$$\mathrm{vol}(D) = \int_0^1 \left(\int_{y=x^2}^{y=x} \left(\int_{z=y-x}^{z=x+y} dz \right) dy \right) dx$$

(b) Use cylindrical coordinates to compute the volume above the xy-plane bounded by $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 9$.

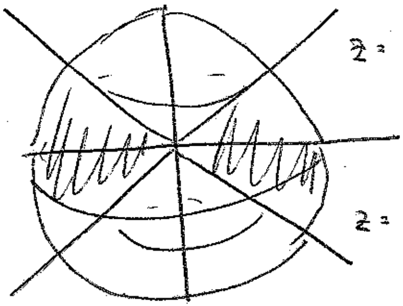


$x^2 + y^2 = 9$, $r^2 = 9$, $r = 3$

$z = x^2 + y^2$, $z = r^2$

$$\mathrm{vol} = \int_0^{2\pi} \left(\int_0^3 \left(\int_{z=0}^{z=r^2} dz \right) r\,dr \right) d\theta$$

(c) Use spherical coordinates to compute the volume of the portion of the solid sphere $x^2 + y^2 + z^2 \leq 10$ that lies between the surfaces $z = \sqrt{x^2 + y^2}$ and $z = -\sqrt{x^2 + y^2}$.



$z = \sqrt{x^2+y^2}$ $\quad \phi = \frac{\pi}{4}$

$x^2 + y^2 + z^2 = 10$

$z = -\sqrt{x^2+y^2}, \quad \phi = \frac{3\pi}{4}$

$\rho^2 = 10$

$\rho = \sqrt{10}$

$$\text{volume} = \int_0^{2\pi} \left(\int_{\pi/4}^{3\pi/4} \left(\int_0^{\rho = \sqrt{10}} \rho^2 \sin\phi \, d\rho \right) d\phi \right) d\theta$$